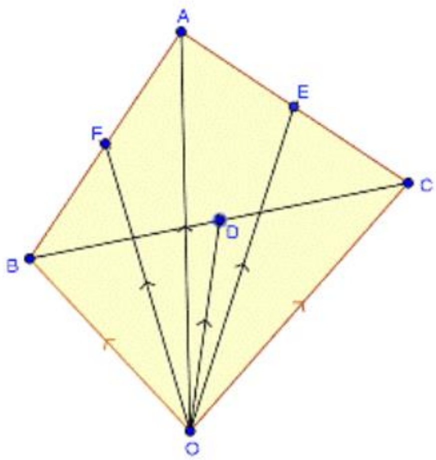
Exercise 23.4

1. Solution:



Given,

In ΔABC, D, E and F are the mid-points of the sides of BC, CA and AB respectively. And O is any point in space.

Let $\vec{a}, \vec{b}, \vec{c}, \vec{d}, \vec{e}, \vec{f}$ be the position vectors of points A, B, C, D, E and F with respect to O.

So,

$$\overrightarrow{OA} = \vec{a},\ \overrightarrow{OB} = \vec{b},\ \overrightarrow{OC} = \vec{c}$$

$$\overrightarrow{OD} = \vec{d},\ \overrightarrow{OE} = \vec{e},\ \overrightarrow{OF} = \vec{f}$$

$$\vec{d} = \frac{\vec{b} + \vec{c}}{2}$$

$$\vec{e} = \frac{\vec{a} + \vec{c}}{2} \qquad \text{[Using mid point formula]}$$

$$\vec{f} = \frac{\vec{a} + \vec{b}}{2}$$

Now,

$$\overrightarrow{OD} + \overrightarrow{OE} + \overrightarrow{OF} = \vec{d} + \vec{e} + \vec{f}$$

$$= \frac{\vec{b} + \vec{c}}{2} + \frac{\vec{a} + \vec{c}}{2} + \frac{\vec{a} + \vec{b}}{2}$$

$$= \frac{\vec{b} + \vec{c} + \vec{a} + \vec{c} + \vec{a} + \vec{b}}{2}$$

$$= \frac{2\left(\vec{a} + \vec{b} + \vec{c}\right)}{2}$$

$$= \vec{a} + \vec{b} + \vec{c}$$

$$= \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}$$

So,

$$\overrightarrow{OD} + \overrightarrow{OE} + \overrightarrow{OF} = \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}$$

2. Solution:

Required to prove: The sum of the three vectors determined by medians of a triangle directed from the vertices is zero.

Let ABC be a triangle such the position vector of A, B and C are $\vec{a}, \vec{b}$ and $\vec{c}$ respectively.
As AD, BE and CF are medians
Then, D, E and F are mid-points
So, we have

Position vector of $D = \frac{\vec{b}+\vec{c}}{2}$ [Using mid point formula]

Position vector of $E = \frac{\vec{c}+\vec{a}}{2}$ [Using mid point formula]

Position vector of $F = \frac{\vec{a}+\vec{b}}{2}$ [Using mid point formula]

Now,

$$\overrightarrow{AD}+\overrightarrow{BE}+\overrightarrow{CF}$$
$$=\left(\frac{\vec{b}+\vec{c}}{2}-\vec{a}\right)+\left(\frac{\vec{c}+\vec{a}}{2}-\vec{b}\right)+\left(\frac{\vec{a}+\vec{b}}{2}-\vec{c}\right)$$
$$=\frac{\vec{b}+\vec{c}-2\vec{a}}{2}+\frac{\vec{c}+\vec{a}-2\vec{b}}{2}+\frac{\vec{a}+\vec{b}-2\vec{c}}{2}$$
$$=\frac{\vec{b}+\vec{c}-2\vec{a}+\vec{c}+\vec{a}-2\vec{b}+\vec{a}+\vec{b}-2\vec{c}}{2}$$
$$=\frac{2\vec{b}+2\vec{c}+2\vec{a}-2\vec{b}-2\vec{a}-2\vec{c}}{2}$$
$$=\frac{\vec{0}}{2}$$
$$=\vec{0}$$
$$\therefore \overrightarrow{AD}+\overrightarrow{BE}+\overrightarrow{CF}=\vec{0}$$

3. Solution:

Given, ABCD is a parallelogram and P is the point of intersection of diagonals and O be the point of reference.

Using triangle law in ΔAOP,

$\overrightarrow{OP} + \overrightarrow{PA} = \overrightarrow{OA}$ (i)

Using triangle law in ΔOBP,

$\overrightarrow{OP} + \overrightarrow{PB} = \overrightarrow{OB}$ (ii)

Using triangle law in ΔOPC,

$\overrightarrow{OP} + \overrightarrow{PC} = \overrightarrow{OC}$ (iii)

Using triangle law in ΔOPD,

$\overrightarrow{OP} + \overrightarrow{PD} = \overrightarrow{OD}$ (iv)

Adding equation (i), (ii), (iii), and (iv),

$$\overrightarrow{OP} + \overrightarrow{PA} + \overrightarrow{OP} + \overrightarrow{PB} + \overrightarrow{OP} + \overrightarrow{PC} + \overrightarrow{OP} + \overrightarrow{PD} = \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} + \overrightarrow{OD}$$

$$4\overrightarrow{OP} + \overrightarrow{PA} + \overrightarrow{PB} + \overrightarrow{PC} + \overrightarrow{PD} = \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} + \overrightarrow{OD}$$

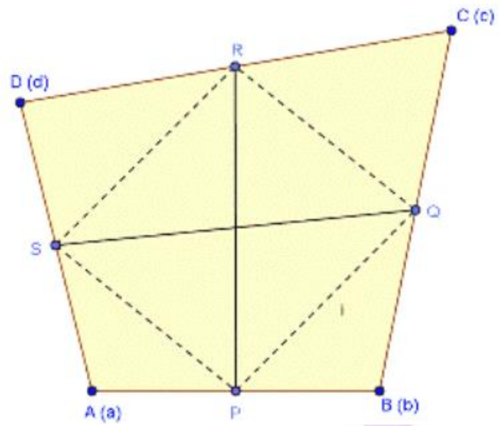
$$4\overrightarrow{OP} + \overrightarrow{PA} + \overrightarrow{PB} - \overrightarrow{PA} - \overrightarrow{PB} = \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} + \overrightarrow{OD}$$

[Since $\overrightarrow{PC} = -\overrightarrow{PA}$ and $\overrightarrow{PD} = -\overrightarrow{PB}$ as P is mid point of AC, BD]

$$4\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} + \overrightarrow{OD}$$

4. Solution:

Let's consider ABCD be a quadrilateral and P, Q, R and S be the mid-points of sides



Let ABCD be a quadrilateral and P, Q, R and S be the mid points of sides AB, BC, CD and DA respectively.

Let position vector of A, B, C and D be $\vec{a}, \vec{b}, \vec{c}$, and $\vec{d}$.

So, the position vector of P, Q, R and S are $\left(\frac{\vec{a}+\vec{b}}{2}\right)$, $\left(\frac{\vec{b}+\vec{c}}{2}\right)$, $\left(\frac{\vec{c}+\vec{d}}{2}\right)$ and $\left(\frac{\vec{d}+\vec{a}}{2}\right)$ respectively.

Now,

Position vector of $\overrightarrow{PQ}$ = Position vector of Q – Position vector of P

$$= \left(\frac{\vec{b}+\vec{c}}{2}\right) - \left(\frac{\vec{a}+\vec{b}}{2}\right)$$
$$= \frac{\vec{b}+\vec{c}-\vec{a}-\vec{b}}{2}$$
$$= \frac{\vec{c}-\vec{a}}{2} \qquad \text{...(i)}$$

Position vector of $\overrightarrow{SR}$ = Position vector of R – Position vector of S

$$= \left(\frac{\vec{c}+\vec{d}}{2}\right) - \left(\frac{\vec{a}+\vec{d}}{2}\right)$$
$$= \frac{\vec{c}+\vec{d}-\vec{a}-\vec{d}}{2}$$
$$= \frac{\vec{c}-\vec{a}}{2} \qquad \text{... (ii)}$$

Using (i) and (ii) , we have

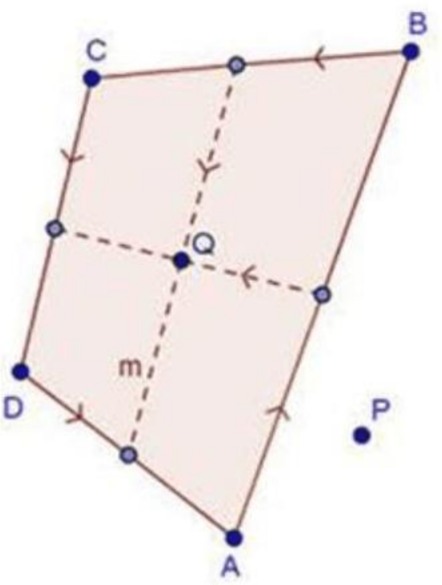
$\overrightarrow{PQ} = \overrightarrow{SR}$

Thus, PQRS is a parallelogram and hence, PR bisects QS.

[Since, diagonals of a parallelogram bisect each other]

Therefore, the line segment joining the mid point of opposite sides of a quadrilateral bisects each other.

5. Solution:



Let $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ be the position vectors of the points A, B, C and D respectively.

Then position vector of

mid point of $AB = \frac{\vec{a}+\vec{b}}{2}$

mid point of $BC = \frac{\vec{b}+\vec{c}}{2}$

mid point of $CD = \frac{\vec{c}+\vec{d}}{2}$

mid point of $DA = \frac{\vec{a}+\vec{d}}{2}$

Q is the mid point of the line joining the mid points of AB and CD.
Hence,

$$Q = \frac{\frac{\vec{a}+\vec{a}}{2}+\frac{\vec{c}+\vec{d}}{2}}{2}$$

$$= \frac{\vec{a}+\vec{b}+\vec{c}+\vec{d}}{4}$$

Let $\vec{p}$ be the position vector of P.
Then,

$$\overrightarrow{PA}+\overrightarrow{PB}+\overrightarrow{PC}+\overrightarrow{PD}$$

$$= \vec{a}-\vec{p}+\vec{b}-\vec{p}+\vec{c}-\vec{p}+\vec{d}-\vec{p}$$

$$= \left(\vec{a}+\vec{b}+\vec{c}+\vec{d}\right)-4\vec{p}$$

$$= 4\left(\frac{\vec{a}+\vec{b}+\vec{c}+\vec{d}}{4}-\vec{p}\right)$$

$$= 4\overrightarrow{PQ}$$