Exercise 25.1

1. Solution:

Given,

$\vec{a} = \hat{i} + 3\hat{j} - 2\hat{k}$ and $\vec{b} = -\hat{i} + 3\hat{k}$

We know that, if

$\vec{a} = a_1\hat{i} + b_1\hat{j} + c_1\hat{k}$ and $\vec{b} = a_2\hat{i} + b_2\hat{j} + c_2\hat{k}$, then

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}$$

So, here

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 3 & -2 \\ -1 & 0 & 3 \end{vmatrix}$$

$$= \hat{i}(9-0) - \hat{j}(3-2) + \hat{k}(0+3)$$

$$\therefore \vec{a} \times \vec{b} = 9\hat{i} - \hat{j} + 3\hat{k}$$

Now,

$$|\vec{a} \times \vec{b}| = \sqrt{(9)^2 + (-1)^2 + (3)^2}$$

$$= \sqrt{81+1+9}$$

$$\therefore |\vec{a} \times \vec{b}| = \sqrt{91}$$

2.

(i) Solution:

$\vec{a} = 3\hat{i} + 4\hat{j}$ and $\vec{b} = \hat{i} + \hat{j} + \hat{k}$

We know that, if

$\vec{a} = a_1\hat{i} + b_1\hat{j} + c_1\hat{k}$ and $\vec{b} = a_2\hat{i} + b_2\hat{j} + c_2\hat{k}$, then

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}$$

So, here

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 4 & 0 \\ 1 & 1 & 1 \end{vmatrix}$$

$$= \hat{i}(4-0) - \hat{j}(3-0) + \hat{k}(3-4)$$

$\therefore\ \vec{a}\times\vec{b} = 4\hat{i} - 3\hat{j} - \hat{k}$

Now,

$$|\vec{a}\times\vec{b}| = \sqrt{(4)^2 + (-3)^2 + (-1)^2}$$
$$= \sqrt{16+9+1}$$
$$\therefore |\vec{a}\times\vec{b}| = \sqrt{26}$$

(ii) Solution:

$\vec{a} = 2\hat{i} + \hat{j}$ and $\vec{b} = \hat{i} + \hat{j} + \hat{k}$

We know that, if

$\vec{a} = a_1\hat{i} + b_1\hat{j} + c_1\hat{k}$ and $\vec{b} = a_2\hat{i} + b_2\hat{j} + c_2\hat{k}$, then

$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}$$

So, here

$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 0 & 1 \\ 1 & 1 & 1 \end{vmatrix}$$
$$= \hat{i}(0-1) - \hat{j}(2-1) + \hat{k}(2-0)$$
$$\therefore\ \vec{a}\times\vec{b} = -\hat{i} - \hat{j} + 2\hat{k}$$

Now,

$$|\vec{a}\times\vec{b}| = \sqrt{(-1)^2 + (-1)^2 + (2)^2}$$
$$= \sqrt{1+1+4}$$
$$\therefore |\vec{a}\times\vec{b}| = \sqrt{6}$$

3.
(i) Solution:

$\vec{a} = 4\hat{i} - \hat{j} + 3\hat{k}$ and $\vec{b} = -2\hat{i} + \hat{j} - 2\hat{k}$

Now,

A vector perpendicular to both $\vec{a}$ and $\vec{b} = \vec{a}\times\vec{b}$.

$$\vec{a}\times\vec{b} = \vec{c}\,(\text{say}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & -1 & 3 \\ -2 & 1 & -2 \end{vmatrix}$$

$$\vec{c} = \hat{i}(2-3) - \hat{j}(-8+6) + \hat{k}(4-2)$$

$\vec{c} = -\hat{i} + 2\hat{j} + 2\hat{k}$

We know that,

$\vec{c}$ is a vector perpendicular to both $\vec{a}$ and $\vec{b}$.

So, it's unit vector is given by

$$\hat{c} = \frac{\vec{c}}{|\vec{c}|}$$

$$= \frac{-\hat{i} + 2\hat{j} + 2\hat{k}}{\sqrt{(-1)^2 + (2)^2 + (2)^2}}$$

$$= \frac{-\hat{i} + 2\hat{j} + 2\hat{k}}{\sqrt{1 + 4 + 4}}$$

$$= \frac{1}{3}\left(-\hat{i} + 2\hat{j} + 2\hat{k}\right)$$

Therefore, unit vector perpendicular to both $\vec{a}$ and $\vec{b} = \frac{1}{3}\left(-\hat{i} + 2\hat{j} + 2\hat{k}\right)$.

(ii) Solution:

Given vectors are,

$\vec{a} = 2\hat{i} + \hat{j} + \hat{k}$ and $\vec{b} = \hat{i} + 2\hat{j} + \hat{k}$

Now,

A vector perpendicular to the plane containing the vector $\vec{a}$ and $\vec{b} = \vec{a} \times \vec{b}$.

$$\vec{a} \times \vec{b} = \vec{c}\,(\text{say}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & 1 \\ 1 & 2 & 1 \end{vmatrix}$$

$\vec{c} = \hat{i}\,(1 - 2) - \hat{j}\,(2 - 1) + \hat{k}\,(4 - 1)$

$\vec{c} = -\hat{i} - \hat{j} + 3\hat{k}$

We know that,

$\vec{c}$ is a vector perpendicular to both $\vec{a}$ and $\vec{b}$.

So, it's unit vector is given by

$$\hat{c} = \frac{\vec{c}}{|\vec{c}|}$$

$$= \frac{-\hat{i} - \hat{j} + 3\hat{k}}{\sqrt{(-1)^2 + (-1)^2 + (3)^2}}$$

$$= \frac{-\hat{i} - \hat{j} + 3\hat{k}}{\sqrt{1 + 1 + 9}}$$

$$= \frac{1}{\sqrt{11}}\left(-\hat{i} - \hat{j} + 3\hat{k}\right)$$

Therefore, unit vector perpendicular to the plane containing the vector both $\vec{a}$ and $\vec{b} = \frac{1}{\sqrt{11}}\left(-\hat{i} - \hat{j} + 3\hat{k}\right)$

4. Solution:

Given,

$\vec{a} = (3\hat{k} + 4\hat{j}) \times (\hat{i} + \hat{j} - \hat{k})$

We know that, if

$\vec{a} = a_1\hat{i} + b_1\hat{j} + c_1\hat{k}$ and $\vec{b} = a_2\hat{i} + b_2\hat{j} + c_2\hat{k}$, then

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}$$

So, here

$$\vec{a} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 4 & 3 \\ 1 & 1 & -1 \end{vmatrix}$$

$$= \hat{i}(-4-3) - \hat{j}(0-3) + \hat{k}(0-4)$$

$$\therefore \vec{a} = -7\hat{i} + 3\hat{j} - 4\hat{k}$$

Now,

$$|\vec{a}| = \sqrt{(-7)^2 + (3)^2 + (-4)^2}$$

$$= \sqrt{49 + 9 + 16}$$

$$\therefore |\vec{a}| = \sqrt{74}$$

5. Solution:

Given,

$\vec{a} = 4\hat{i} + 3\hat{j} + \hat{k}$ and $\vec{b} = \hat{i} - 2\hat{k}$

Now, the unit vector of $\hat{b}$ is given by

$$\hat{b} = \frac{\vec{b}}{|\vec{b}|}$$

$$= \frac{\hat{i} - 2\hat{k}}{\sqrt{(1)^2 + (-2)^2}}$$

$$= \frac{\hat{i} - 2\hat{k}}{\sqrt{1+4}}$$

$$= \frac{\hat{i} - 2\hat{k}}{\sqrt{5}}$$

So,

$$2\hat{b} = \frac{2}{\sqrt{5}}\hat{i} - \frac{4}{\sqrt{5}}\hat{k}$$

And, $\vec{a} = 4\hat{i} + 3\hat{j} + \hat{k}$

The cross product is given by

$$2\hat{b} \times \vec{a} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{2}{\sqrt{5}} & 0 & -\frac{4}{\sqrt{5}} \\ 4 & 3 & 1 \end{vmatrix}$$

$$= \hat{i}\left(0 + \frac{12}{\sqrt{5}}\right) - \hat{j}\left(\frac{2}{\sqrt{5}} + \frac{16}{\sqrt{5}}\right) + \hat{k}\left(\frac{6}{\sqrt{5}} - 0\right)$$

$$\therefore\ 2\hat{b} \times \vec{a} = \frac{12}{\sqrt{5}}\hat{i} - \frac{18}{\sqrt{5}}\hat{j} + \frac{6}{\sqrt{5}}\hat{k}$$

Now,

$$\left|2\hat{b} \times \vec{a}\right| = \sqrt{\left(\frac{12}{\sqrt{5}}\right)^2 + \left(-\frac{18}{\sqrt{5}}\right)^2 + \left(\frac{6}{\sqrt{5}}\right)^2}$$

$$= \sqrt{\frac{144}{5} + \frac{324}{5} + \frac{36}{5}}$$

$$\therefore\ \left|2\hat{b} \times \vec{a}\right| = \sqrt{\frac{504}{5}}$$

6. Solution:

Given,

$\vec{a} = 3\hat{i} - \hat{j} - 2\hat{k}$ and $\vec{b} = 2\hat{i} + 3\hat{j} + \hat{k}$

Required to find $(\vec{a} + 2\vec{b}) \times (2\vec{a} - \vec{b})$

Now,

$$\vec{a} + 2\vec{b} = \left(3\hat{i} - \hat{j} - 2\hat{k}\right) + 2\left(2\hat{i} + 3\hat{j} + \hat{k}\right)$$

$$= 3\hat{i} - \hat{j} - 2\hat{k} + 4\hat{i} + 6\hat{j} + 2\hat{k}$$

$$\Rightarrow \vec{a} + 2\vec{b} = 7\hat{i} + 5\hat{j}$$

And,

$$2\vec{a} - \vec{b} = 2\left(3\hat{i} - \hat{j} - 2\hat{k}\right) - \left(2\hat{i} + 3\hat{j} + \hat{k}\right)$$

$$= 6\hat{i} - 2\hat{j} - 4\hat{k} - 2\hat{i} - 3\hat{j} - \hat{k}$$

$$\Rightarrow 2\vec{a} - \vec{b} = 4\hat{i} - 5\hat{j} - 5\hat{k}$$

We know that if $\vec{a} = a_1\hat{i} + b_1\hat{j} + c_1\hat{k}$ and $\vec{b} = a_2\hat{i} + b_2\hat{j} + c_2\hat{k}$ then,

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}$$

Thus,

$$\left(\vec{a} + 2\vec{b}\right) \times \left(2\vec{a} - \vec{b}\right) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 7 & 5 & 0 \\ 4 & -5 & -5 \end{vmatrix}$$

$$= \hat{i}\left(-25 - 0\right) - \hat{j}\left(-35 - 0\right) + \hat{k}\left(-35 - 20\right)$$

$$= -25\hat{i} + 35\hat{j} - 55\hat{k}$$

$$\therefore \left(\vec{a} + 2\vec{b}\right) \times \left(2\vec{a} - \vec{b}\right) = -25\hat{i} + 35\hat{j} - 55\hat{k}$$