

EXERCISE 28.5

Q1.i

Solution:

We know that,

Shortest distance between lines $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu\vec{b}_2$ is given by

$$\text{S.D.} = \frac{|(\vec{a}_2 - \vec{a}_1) \cdot (\vec{b}_1 \times \vec{b}_2)|}{|\vec{b}_1 \times \vec{b}_2|} \dots\dots\dots (1)$$

Given:

The equation of lines is,

$$\vec{r} = 3\hat{i} + 8\hat{j} + 3\hat{k} + \lambda(3\hat{i} - \hat{j} + \hat{k})$$

$$\vec{r} = (-3\hat{i} - 7\hat{j} + 6\hat{k}) + \mu(-3\hat{i} + 2\hat{j} + 4\hat{k})$$

Where,

$$\vec{a}_1 = (3\hat{i} + 8\hat{j} + 3\hat{k}), \vec{b}_1 = 3\hat{i} - \hat{j} + \hat{k}$$

$$\vec{a}_2 = (-3\hat{i} - 7\hat{j} + 6\hat{k}), \vec{b}_2 = (-3\hat{i} + 2\hat{j} + 4\hat{k})$$

So now,

$$\begin{aligned} \vec{a}_2 - \vec{a}_1 &= (-3\hat{i} - 7\hat{j} + 6\hat{k}) - (3\hat{i} + 8\hat{j} + 3\hat{k}) \\ &= -3\hat{i} - 7\hat{j} + 6\hat{k} - 3\hat{i} - 8\hat{j} - 3\hat{k} \\ (\vec{a}_2 - \vec{a}_1) &= -6\hat{i} - 15\hat{j} + 3\hat{k} \end{aligned} \dots\dots\dots (2)$$

Let us solve for b, we get

$$\begin{aligned} (\vec{b}_1 \times \vec{b}_2) &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -1 & 1 \\ -3 & 2 & 4 \end{vmatrix} \\ &= \hat{i}(-4 - 2) - \hat{j}(12 + 3) + \hat{k}(6 - 3) \\ &= (-6\hat{i} - 15\hat{j} + 3\hat{k}) \end{aligned} \dots\dots\dots (3)$$

By solving (2) and (3), we get

$$\begin{aligned} (\vec{a}_2 - \vec{a}_1) \cdot (\vec{b}_1 \times \vec{b}_2) &= (-6\hat{i} - 15\hat{j} + 3\hat{k}) \cdot (-6\hat{i} - 15\hat{j} + 3\hat{k}) \\ &= (-6)(-6) + (-15)(-15) + (3)(3) \\ &= 36 + 225 + 9 \\ &= 270 \end{aligned}$$

$$\begin{aligned} |\vec{b}_1 \times \vec{b}_2| &= \sqrt{(-6)^2 + (-15)^2 + (3)^2} \\ &= \sqrt{36 + 225 + 9} \\ &= \sqrt{270} \end{aligned}$$

Now let us substitute the above obtained values in equation (1), to get the shortest distance between given lines,

$$\begin{aligned} \text{S.D.} &= \frac{270}{\sqrt{270}} \\ &= \sqrt{270} \\ &= 3\sqrt{30} \end{aligned}$$

Hence, the shortest distance is $3\sqrt{30}$ units.

ii.

Solution:

Given:

Equation of lines is,

$$\vec{r} = (3\hat{i} + 5\hat{j} + 7\hat{k}) + \lambda(\hat{i} - 2\hat{j} + 7\hat{k}) \text{ and}$$

$$\vec{r} = (-\hat{i} - \hat{j} - \hat{k}) + \mu(7\hat{i} - 6\hat{j} + \hat{k})$$

We know that,

Shortest distance between lines $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu\vec{b}_2$ is given by

$$\text{S.D.} = \frac{|(\vec{a}_2 - \vec{a}_1) \cdot (\vec{b}_1 \times \vec{b}_2)|}{|\vec{b}_1 \times \vec{b}_2|} \dots\dots\dots (1)$$

Where,

$$\vec{a}_1 = (3\hat{i} + 5\hat{j} + 7\hat{k}), \vec{b}_1 = (\hat{i} - 2\hat{j} + 7\hat{k})$$

$$\vec{a}_2 = (-\hat{i} - \hat{j} - \hat{k}), \vec{b}_2 = (7\hat{i} - 6\hat{j} + \hat{k})$$

So now,

$$\begin{aligned} \vec{a}_2 - \vec{a}_1 &= (-\hat{i} - \hat{j} - \hat{k}) - (3\hat{i} + 5\hat{j} + 7\hat{k}) \\ &= -\hat{i} - \hat{j} - \hat{k} - 3\hat{i} - 5\hat{j} - 7\hat{k} \\ &= -4\hat{i} - 6\hat{j} - 8\hat{k} = -2(2\hat{i} + 3\hat{j} + 4\hat{k}) \dots\dots\dots (2) \end{aligned}$$

Let us solve for b, we get

$$\begin{aligned} \vec{b}_1 \times \vec{b}_2 &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & 7 \\ 7 & -6 & 1 \end{vmatrix} \\ &= \hat{i}(-2 + 42) - \hat{j}(1 - 49) + \hat{k}(-6 + 14) \\ &= 40\hat{i} + 48\hat{j} + 8\hat{k} \\ &= 8(5\hat{i} + 6\hat{j} + \hat{k}) \end{aligned} \quad \dots\dots (3)$$

By solving (2) and (3), we get

$$\begin{aligned} (\vec{a}_2 - \vec{a}_1) \cdot (\vec{b}_1 \times \vec{b}_2) &= \{-2(2\hat{i} + 3\hat{j} + 4\hat{k})\} \cdot \{8(5\hat{i} + 6\hat{j} + \hat{k})\} \\ &= -16[(2)(5) + (3)(6) + (4)(1)] \\ &= -16[10 + 18 + 4] \\ &= -16 \times 32 \\ &= -512 \end{aligned}$$

$$\begin{aligned} |\vec{b}_1 \times \vec{b}_2| &= 8\sqrt{(5)^2 + (6)^2 + (1)^2} \\ &= 8\sqrt{25 + 36 + 1} \\ &= 8\sqrt{62} \end{aligned}$$

Now let us substitute the above obtained values in equation (1), to get the shortest distance between given lines,

$$\begin{aligned} \text{S.D.} &= \left| \frac{-512}{8\sqrt{62}} \right| \\ &= \frac{512}{\sqrt{3968}} \end{aligned}$$

Hence, the shortest distance is $512/\sqrt{3968}$ units.

iii.

Solution:

Given:

Equation of lines is,

$$\vec{r} = (\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda(2\hat{i} + 3\hat{j} + 4\hat{k}) \text{ and}$$

$$\vec{r} = (2\hat{i} + 4\hat{j} + 5\hat{k}) + \mu(3\hat{i} + 4\hat{j} + 5\hat{k})$$

We know that,

Shortest distance between lines $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu\vec{b}_2$ is given by

$$\text{S.D.} = \left| \frac{(\vec{a}_2 - \vec{a}_1) \cdot (\vec{b}_1 \times \vec{b}_2)}{|\vec{b}_1 \times \vec{b}_2|} \right| \quad \dots\dots (1)$$

Where,

$$\vec{a}_1 = (\hat{i} + 2\hat{j} + 3\hat{k}), \quad \vec{b}_1 = (2\hat{i} + 3\hat{j} + 4\hat{k})$$

$$\vec{a}_2 = (2\hat{i} + 4\hat{j} + 5\hat{k}), \quad \vec{b}_2 = (3\hat{i} + 4\hat{j} + 5\hat{k})$$

So now,

$$\begin{aligned} (\vec{a}_2 - \vec{a}_1) &= (2\hat{i} + 4\hat{j} + 5\hat{k}) - (\hat{i} + 2\hat{j} + 3\hat{k}) \\ &= 2\hat{i} + 4\hat{j} + 5\hat{k} - \hat{i} - 2\hat{j} - 3\hat{k} \\ (\vec{a}_2 - \vec{a}_1) &= \hat{i} + 2\hat{j} + 2\hat{k} \end{aligned} \quad \dots\dots\dots (2)$$

Let us solve for b, we get

$$\begin{aligned} (\vec{b}_1 \times \vec{b}_2) &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix} \\ &= \hat{i}(15 - 16) - \hat{j}(10 - 12) + \hat{k}(8 - 9) \\ (\vec{b}_1 \times \vec{b}_2) &= -\hat{i} + 2\hat{j} - \hat{k} \end{aligned} \quad \dots\dots\dots (3)$$

By solving (2) and (3), we get

$$\begin{aligned} (\vec{a}_2 - \vec{a}_1) (\vec{b}_1 \times \vec{b}_2) &= (\hat{i} + 2\hat{j} + 2\hat{k})(-\hat{i} + 2\hat{j} - \hat{k}) \\ &= (1)(-1) + (2)(2) + (2)(-1) \\ &= -1 + 4 - 2 \\ &= 1 \end{aligned}$$

$$\begin{aligned} |\vec{b}_1 \times \vec{b}_2| &= \sqrt{(-1)^2 + (2)^2 + (-1)^2} \\ &= \sqrt{1 + 4 + 1} \\ &= \sqrt{6} \end{aligned}$$

Now let us substitute the above obtained values in equation (1), to get the shortest distance between given lines,

$$S.D. = \left| \frac{1}{\sqrt{6}} \right|$$

Hence, the shortest distance is $1/\sqrt{6}$ units.

iv.

Solution:

Given:

The vector equations are

$$\vec{r} = (1-t)\hat{i} + (t-2)\hat{j} + (3-t)\hat{k}$$

$$\vec{r} = (s+1)\hat{i} + (2s-1)\hat{j} - (2s+1)\hat{k}$$

The above equations can be written as

$$\vec{r} = (i - 2j + 3k) + t(-i + j - k)$$

$$\vec{r} = (i - j - k) + s(i + 2j - 2k)$$

By using the formula,

Shortest distance is given by $\left| \frac{(\vec{a}_2 - \vec{a}_1) \cdot (\vec{b}_1 \times \vec{b}_2)}{|\vec{b}_1 \times \vec{b}_2|} \right|$

$$(\vec{b}_1 \times \vec{b}_2) = -3j - 3k$$

$$(\vec{a}_2 - \vec{a}_1) = j - 4k$$

$$(\vec{a}_2 - \vec{a}_1) \cdot (\vec{b}_1 \times \vec{b}_2) = 9$$

$$|\vec{b}_1 \times \vec{b}_2| = 3\sqrt{2}$$

Hence,

The shortest distance is $9/3\sqrt{2} = 3/\sqrt{2}$ units.

v.

Solution:

Given:

Equation of lines is,

$$\vec{r} = (\lambda - 1)\hat{i} + (\lambda + 1)\hat{j} - (1 + \lambda)\hat{k}$$

$$\vec{r} = (-\hat{i} + \hat{j} - \hat{k}) + \lambda(\hat{i} + \hat{j} - \hat{k}) \text{ and}$$

$$\vec{r} = (1 - \mu)\hat{i} + (2\mu - 1)\hat{j} + (\mu + 2)\hat{k}$$

$$\vec{r} = (\hat{i} - \hat{j} + 2\hat{k}) + \mu(-\hat{i} + 2\hat{j} + \hat{k})$$

We know that,

Shortest distance between lines $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu\vec{b}_2$ is given by

$$S.D. = \left| \frac{(\vec{a}_2 - \vec{a}_1) \cdot (\vec{b}_1 \times \vec{b}_2)}{|\vec{b}_1 \times \vec{b}_2|} \right| \dots\dots\dots (1)$$

Where,

$$\vec{a}_1 = (-\hat{i} + \hat{j} - \hat{k}), \vec{b}_1 = (\hat{i} + \hat{j} - \hat{k}) \text{ and}$$

$$\vec{a}_2 = (\hat{i} - \hat{j} + 2\hat{k}), \vec{b}_2 = (-\hat{i} + 2\hat{j} + \hat{k})$$

So now,

$$\begin{aligned}
 (\vec{a}_2 - \vec{a}_1) &= (\hat{i} - \hat{j} + 2\hat{k}) - (-\hat{i} + \hat{j} - \hat{k}) \\
 &= \hat{i} - \hat{j} + 2\hat{k} + \hat{i} - \hat{j} + \hat{k} \\
 (\vec{a}_2 - \vec{a}_1) &= 2\hat{i} - 2\hat{j} + 3\hat{k} \quad \dots\dots\dots (2)
 \end{aligned}$$

Let us solve for b, we get

$$\begin{aligned}
 (\vec{b}_1 \times \vec{b}_2) &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & -1 \\ -1 & 2 & 1 \end{vmatrix} \\
 &= \hat{i}(1+2) - \hat{j}(1-1) + \hat{k}(2+1) \\
 (\vec{b}_1 \times \vec{b}_2) &= 3\hat{i} + 3\hat{k}
 \end{aligned}$$

By solving (2) and (3), we get

$$\begin{aligned}
 (\vec{a}_2 - \vec{a}_1) \cdot (\vec{b}_1 \times \vec{b}_2) &= (2\hat{i} - 2\hat{j} + 3\hat{k}) \cdot (3\hat{i} + 3\hat{k}) \\
 &= (2)(3) + (-2)(0) + (3)(3) \\
 &= 6 + 0 + 9 \\
 &= 15
 \end{aligned}$$

$$\begin{aligned}
 |\vec{b}_1 \times \vec{b}_2| &= \sqrt{(3)^2 + (3)^2} \\
 &= \sqrt{18} \\
 &= 3\sqrt{2}
 \end{aligned}$$

Now let us substitute the above obtained values in equation (1), to get the shortest distance between given lines,

$$\begin{aligned}
 \text{S.D.} &= \left| \frac{15}{3\sqrt{2}} \right| \\
 &= \frac{5}{\sqrt{2}}
 \end{aligned}$$

Hence, the shortest distance is $5/\sqrt{2}$ units.