EXERCISE 29.14

Q1.
Solution:

Given the two lines,
Let the two lines be l_1 and l_2.

So, $l_1: \frac{x-2}{-1} = \frac{y-5}{2} = \frac{z-0}{3}$ and $l_2: \frac{x-0}{2} = \frac{y+1}{-1} = \frac{z-1}{2}$

We need to find the shortest distance between l_1 and l_2.

So, recall the shortest distance between the lines: $\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$ and $\frac{x-x_2}{a_2} = \frac{y-y_2}{b_2} = \frac{z-z_2}{c_2}$ is given by

$$\left| \frac{\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{(b_1c_2 - b_2c_1)^2 + (c_1a_2 - c_2a_1)^2 + (a_1b_2 - a_2b_1)^2}} \right|$$

Here,
$(x_1, y_1, z_1) = (2, 5, 0)$ and $(x_2, y_2, z_2) = (0, -1, 1)$
Also
$(a_1, b_1, c_1) = (-1, 2, 3)$ and $(a_2, b_2, c_2) = (2, -1, 2)$

We shall evaluate the numerator first.
Let

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = N$$

Substitute the values, we get

$$N = \begin{vmatrix} 0 - 2 & -1 - 5 & 1 - 0 \\ -1 & 2 & 3 \\ 2 & -1 & 2 \end{vmatrix}$$

$$= \begin{vmatrix} -2 & -6 & 1 \\ -1 & 2 & 3 \\ 2 & -1 & 2 \end{vmatrix}$$

Let us simplify,

N = (–2)[(2)(2) – (–1)(3)] – (–6)[(–1)(2) – (2)(3)] + (1)[(–1)(–1) – (2)(2)]

= –2(4 + 3) + 6(–2 – 6) + (1 – 4)

= –14 – 48 – 3

= –65

Now, we shall evaluate the denominator.

Let

$\sqrt{(b_1c_2 - b_2c_1)^2 + (c_1a_2 - c_2a_1)^2 + (a_1b_2 - a_2b_1)^2} = D$

$b_1c_2 - b_2c_1 = (2)(2) - (-1)(3) = 4 - (-3) = 7$

$c_1a_2 - c_2a_1 = (3)(2) - (2)(-1) = 6 - (-2) = 8$

$a_1b_2 - a_2b_1 = (-1)(-1) - (2)(2) = 1 - 4 = -3$

$D = \sqrt{7^2 + 8^2 + (-3)^2}$

$= \sqrt{49 + 64 + 9}$

$= \sqrt{122}$

So, shortest distance = $\left|\frac{-65}{\sqrt{122}}\right| = \frac{65}{\sqrt{122}}$

Hence, the required shortest distance is $\frac{65}{\sqrt{122}}$ units.

Q2.

Solution:

Given the two lines,

Let the two lines be l_1 and l_2.

So, $l_1: \frac{x+1}{7} = \frac{y+1}{-6} = \frac{z+1}{1}$ and $l_2: \frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-7}{1}$

We need to find the shortest distance between l_1 and l_2.

So, recall the shortest distance between the lines: $\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$ and $\frac{x-x_2}{a_2} = \frac{y-y_2}{b_2} = \frac{z-z_2}{c_2}$ is given by

$$\left|\frac{\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{(b_1c_2 - b_2c_1)^2 + (c_1a_2 - c_2a_1)^2 + (a_1b_2 - a_2b_1)^2}}\right|$$

Here,
$(x_1, y_1, z_1) = (-1, -1, -1)$ and $(x_2, y_2, z_2) = (3, 5, 7)$
Also
$(a_1, b_1, c_1) = (7, -6, 1)$ and $(a_2, b_2, c_2) = (1, -2, 1)$

We shall evaluate the numerator first.
Let

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = N$$

$$N = \begin{vmatrix} 3 - (-1) & 5 - (-1) & 7 - (-1) \\ 7 & -6 & 1 \\ 1 & -2 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 4 & 6 & 8 \\ 7 & -6 & 1 \\ 1 & -2 & 1 \end{vmatrix}$$

$N = (4)[(-6)(1) - (-2)(1)] - (6)[(7)(1) - (1)(1)] + (8)[(7)(-2) - (1)(-6)]$
$= 4(-6 + 2) - 6(7 - 1) + 8(-14 + 6)$
$= -16 - 36 - 64$
$= -116$

Now, we shall evaluate the denominator.
Let

$$\sqrt{(b_1c_2 - b_2c_1)^2 + (c_1a_2 - c_2a_1)^2 + (a_1b_2 - a_2b_1)^2} = D$$

$b_1c_2 - b_2c_1 = (-6)(1) - (-2)(1) = -6 + 2 = -4$

$c_1a_2 - c_2a_1 = (1)(1) - (1)(7) = 1 - 7 = -6$

$a_1b_2 - a_2b_1 = (7)(-2) - (1)(-6) = -14 + 6 = -8$

$D = \sqrt{(-4)^2 + (-6)^2 + (-8)^2}$

$= \sqrt{16 + 36 + 64}$

$= \sqrt{116}$

So, shortest distance = $\left|\frac{-116}{\sqrt{116}}\right| = \sqrt{116} = 2\sqrt{29}$

Hence, the required shortest distance is $2\sqrt{29}$ units.

Q3.
Solution:
Given the two lines,
Let the two lines be l_1 and l_2.
So,
$l_1: \frac{x-1}{2} = \frac{y-3}{4} = \frac{z+2}{1}$ and l_2: $3x - y - 2z + 4 = 0 = 2x + y + z + 1$

We need to find the shortest distance between l_1 and l_2.
The equation of a plane containing the line l_2 is given by
$(3x - y - 2z + 4) + \lambda(2x + y + z + 1) = 0$
$(3 + 2\lambda)x + (\lambda - 1)y + (\lambda - 2)z + (4 + \lambda) = 0$

Direction ratios of l_1 are 2, 4, 1 and those of the line containing the shortest distance are proportional to $3 + 2\lambda$, $\lambda - 1$ and $\lambda - 2$.

We know that if two lines with direction ratios (a_1, b_1, c_1) and (a_2, b_2, c_2) are perpendicular to each other, then $a_1a_2 + b_1b_2 + c_1c_2 = 0$.
$(3 + 2\lambda)(2) + (\lambda - 1)(4) + (\lambda - 2)(1) = 0$
$6 + 4\lambda + 4\lambda - 4 + \lambda - 2 = 0$
$9\lambda = 0$
$\lambda = 0$

Thus, the plane containing line l_2 is $3x - y - 2z + 4 = 0$.
We have
$l_1: \frac{x-1}{2} = \frac{y-3}{4} = \frac{z+2}{1} = \alpha$ (say)
When $\alpha = 0$, $(x, y, z) = (1, 3, -2)$
So, the point $(1, 3, -2)$ lies on the line l_1.

Hence, the shortest distance between the two lines is same as the distance of the perpendicular from (1, 3, –2) on to the plane 3x – y – 2z + 4 = 0.

The length of the perpendicular drawn from (x_1, y_1, z_1) to the plane Ax + By + Cz + D = 0 is given by

$$\left|\frac{Ax_1 + By_1 + Cz_1 + D}{\sqrt{A^2 + B^2 + C^2}}\right|$$

Here,

$(x_1, y_1, z_1) = (1, 3, -2)$ and $(A, B, C, D) = (3, -1, -2, 4)$

Now, substitute the values, we get

$$d = \left|\frac{(3)(1) + (-1)(3) + (-2)(-2) + 4}{\sqrt{3^2 + (-1)^2 + (-2)^2}}\right|$$

$$= \left|\frac{3 - 3 + 4 + 4}{\sqrt{9 + 1 + 4}}\right|$$

$$= \left|\frac{8}{\sqrt{14}}\right|$$

$$= \frac{8}{\sqrt{14}}$$

Hence, the required shortest distance is $\frac{8}{\sqrt{14}}$ units.