EXERCISE 29.15

Q1.

Solution:

Given:

Let point P = (0, 0, 0) and M be the image of P in the plane

3x + 4y – 6z + 1 = 0.

Direction ratios of PM are proportional to 3, 4, – 6 as PM is normal to the plane.

The equation of the line passing through (x_1, y_1, z_1) and having direction ratios proportional to l, m, n is given by

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$$

Here,

$(x_1, y_1, z_1) = (0, 0, 0)$ and $(l, m, n) = (3, 4, -6)$

Hence, the equation of PM is

$$\frac{x-0}{3}=\frac{y-0}{4}=\frac{z-0}{-6}$$

$$\Rightarrow \frac{x}{3}=\frac{y}{4}=\frac{z}{-6}=\alpha \text{ (say)}$$

$x = 3\alpha, y = 4\alpha, z = -6\alpha$

Let M = (3α, 4α, –6α).

As M is the image of P in the given plane, the midpoint of PM lies on the plane.

Using the midpoint formula, we have

$$\text{Midpoint of PM} = \left(\frac{0+3\alpha}{2}, \frac{0+4\alpha}{2}, \frac{0-6\alpha}{2}\right)$$

$$= \left(\frac{3\alpha}{2}, 2\alpha, -3\alpha\right)$$

This point lies on the given plane, which means this point satisfies the plane equation.

So,

$$\Rightarrow 3\left(\frac{3\alpha}{2}\right) + 4(2\alpha) - 6(-3\alpha) + 1 = 0$$

$$\frac{9\alpha}{2} + 8\alpha + 18\alpha + 1 = 0$$

$$\frac{9\alpha + 16\alpha + 36\alpha}{2} = -1$$

$61\alpha = -2$

$\alpha = -\frac{2}{61}$

We have M = $(3\alpha, 4\alpha, -6\alpha)$

$$\Rightarrow M = \left(3\left(-\frac{2}{61}\right), 4\left(-\frac{2}{61}\right), -6\left(-\frac{2}{61}\right)\right)$$

$$= \left(-\frac{6}{61}, -\frac{8}{61}, \frac{12}{61}\right)$$

Hence, the image of (0, 0, 0) in the plane $3x + 4y - 6z + 1 = 0$ is $\left(-\frac{6}{61}, -\frac{8}{61}, \frac{12}{61}\right)$

Q2.

Solution:

Given:

Let point P = (1, 2, –1) and M be the image of P in the plane 3x – 5y + 4z = 5.

Direction ratios of PM are proportional to 3, –5, 4 as PM is normal to the plane.

The equation of the line passing through (x_1, y_1, z_1) and having direction ratios proportional to l, m, n is given by

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$$

Here,

$(x_1, y_1, z_1) = (1, 2, -1)$ and $(l, m, n) = (3, -5, 4)$

So, the equation of PM is

$$\frac{x-1}{3} = \frac{y-2}{-5} = \frac{z-(-1)}{4}$$

$$\frac{x-1}{3} = \frac{y-2}{-5} = \frac{z+1}{4} = \alpha \text{ (say)}$$

$x = 3\alpha + 1, y = -5\alpha + 2, z = 4\alpha - 1$

Let M = $(3\alpha + 1, -5\alpha + 2, 4\alpha - 1)$.

As M is the image of P in the given plane, the midpoint of PM lies on the plane.
Using the midpoint formula, we have

$$\text{Midpoint of PM} = \left(\frac{1+(3\alpha+1)}{2}, \frac{2+(-5\alpha+2)}{2}, \frac{-1+(4\alpha-1)}{2}\right)$$

$$\text{Midpoint of PM} = \left(\frac{3\alpha+2}{2}, \frac{-5\alpha+4}{2}, 2\alpha-1\right)$$

This point lies on the given plane, which means this point satisfies the plane equation.

$$3\left(\frac{3\alpha+2}{2}\right) - 5\left(\frac{-5\alpha+4}{2}\right) + 4(2\alpha-1) = 5$$

$$\frac{9\alpha+6}{2} - \frac{-25\alpha+20}{2} + 8\alpha - 4 = 5$$

$$\frac{9\alpha+6+25\alpha-20+16\alpha}{2} = 9$$

$$50\alpha - 14 = 18$$

$$50\alpha = 32$$

$$\alpha = \frac{32}{50} = \frac{16}{25}$$

We have $M = (3\alpha + 1, -5\alpha + 2, 4\alpha - 1)$

$$M = \left(3\left(\frac{16}{25}\right) + 1, -5\left(\frac{16}{25}\right) + 2, 4\left(\frac{16}{25}\right) - 1\right)$$

$$M = \left(\frac{48}{25} + 1, -\frac{16}{5} + 2, \frac{64}{25} - 1\right)$$

$$M = \left(\frac{73}{25}, -\frac{6}{5}, \frac{39}{25}\right)$$

Hence, the image of (1, 2, –1) in the plane $3x - 5y + 4z = 5$ is $\left(\frac{73}{25}, -\frac{6}{5}, \frac{39}{25}\right)$.

Q3.
Solution:

Given:
Let point P = (5, 4, 2) and Q be the foot of the perpendicular drawn from to P the line $\frac{x+1}{2}=\frac{y-3}{3}=\frac{z-1}{-1}$.

Q is a point on the given line. So, for some α, Q is given by
$$\frac{x+1}{2}=\frac{y-3}{3}=\frac{z-1}{-1}=\alpha \text{ (say)}$$
$x = 2\alpha - 1, y = 3\alpha + 3, z = -\alpha + 1$
So, Q = $(2\alpha - 1, 3\alpha + 3, -\alpha + 1)$

Now, let us find the direction ratios of PQ.
The direction ratios of a line joining two points (x_1, y_1, z_1) and (x_2, y_2, z_2) are given by $(x_2 - x_1, y_2 - y_1, z_2 - z_1)$.
Here,
$(x_1, y_1, z_1) = (5, 4, 2)$ and $(x_2, y_2, z_2) = (2\alpha - 1, 3\alpha + 3, -\alpha + 1)$

Direction Ratios of PQ are $((2\alpha - 1) - (5), (3\alpha + 3) - (4), (-\alpha + 1) - (2))$
Direction Ratios of PQ are $(2\alpha - 6, 3\alpha - 1, -\alpha - 1)$
PQ is perpendicular to the given line, whose direction ratios are (2, 3, –1).

We know that if two lines with direction ratios (a_1, b_1, c_1) and (a_2, b_2, c_2) are perpendicular to each other, then $a_1a_2 + b_1b_2 + c_1c_2 = 0$.
$(2)(2\alpha - 6) + (3)(3\alpha - 1) + (-1)(-\alpha - 1) = 0$
$4\alpha - 12 + 9\alpha - 3 + \alpha + 1 = 0$
$14\alpha - 14 = 0$
$14\alpha = 14$
$\alpha = 1$

We have Q = $(2\alpha - 1, 3\alpha + 3, -\alpha + 1)$
Q = $(2\times1 - 1, 3\times1 + 3, -1 + 1)$
Q = (1, 6, 0)

Using the distance formula, we have
$$PQ = \sqrt{(1-5)^2+(6-4)^2+(0-2)^2}$$
$$= \sqrt{(-4)^2+2^2+(-2)^2}$$
$$= \sqrt{16+4+4}$$
$$= \sqrt{24}$$

$= 2\sqrt{6}$

Hence, the required foot of perpendicular is (1, 6, 0) and the length of the perpendicular is $2\sqrt{6}$ units.

Q4.

Solution:

Given:

Let P be the point with position vector $\vec{p} = 3\hat{i} + \hat{j} + 2\hat{k}$ and M be the image of P in the plane $\vec{r}.(2\hat{i} - \hat{j} + \hat{k}) = 4$.

In addition, let Q be the foot of the perpendicular from P on to the given plane. So, Q is the midpoint of PM.

Direction ratios of PM are proportional to 2, –1, 1 as PM is normal to the plane and parallel to $2\hat{i} - \hat{j} + \hat{k}$.

The vector equation of the line passing through the point with position vector $\vec{r}$ and parallel to vector $\vec{b}$ is given by

$\vec{r} = \vec{a} + \lambda\vec{b}$

Here,

$\vec{a} = 3\hat{i} + \hat{j} + 2\hat{k}$ and $\vec{b} = 2\hat{i} - \hat{j} + \hat{k}$

Hence, the equation of PM is

$\vec{r} = (3\hat{i} + \hat{j} + 2\hat{k}) + \lambda(2\hat{i} - \hat{j} + \hat{k})$

$\vec{r} = (3 + 2\lambda)\hat{i} + (1 - \lambda)\hat{j} + (2 + \lambda)\hat{k}$

Let the position vector of M be $\vec{m}$. As M is a point on this line, for some scalar α, we have

$\vec{m} = (3 + 2\alpha)\hat{i} + (1 - \alpha)\hat{j} + (2 + \alpha)\hat{k}$

Now, let us find the position vector of Q, the midpoint of PM.

Let this be $\vec{q}$.

Using the midpoint formula, we have

$$\vec{q} = \frac{\vec{p} + \vec{m}}{2}$$

$$\vec{q} = \frac{[3\hat{i} + \hat{j} + 2\hat{k}] + [(3 + 2\alpha)\hat{i} + (1 - \alpha)\hat{j} + (2 + \alpha)\hat{k}]}{2}$$

$$\vec{q} = \frac{(3 + (3 + 2\alpha))\hat{i} + (1 + (1 - \alpha))\hat{j} + (2 + (2 + \alpha))\hat{k}}{2}$$

$$\vec{q} = \frac{(6 + 2\alpha)\hat{i} + (2 - \alpha)\hat{j} + (4 + \alpha)\hat{k}}{2}$$

$$\vec{q} = (3 + \alpha)\hat{i} + \frac{(2 - \alpha)}{2}\hat{j} + \frac{(4 + \alpha)}{2}\hat{k}$$

This point lies on the given plane, which means this point satisfies the plane equation $\vec{r}.(2\hat{i} - \hat{j} + \hat{k}) = 4$.

$$\left[(3 + \alpha)\hat{i} + \frac{(2 - \alpha)}{2}\hat{j} + \frac{(4 + \alpha)}{2}\hat{k}\right].(2\hat{i} - \hat{j} + \hat{k}) = 4$$

$$2(3 + \alpha) - \left(\frac{2 - \alpha}{2}\right)(1) + \left(\frac{4 + \alpha}{2}\right)(1) = 4$$

$$6 + 2\alpha + \frac{4 + \alpha - (2 - \alpha)}{2} = 4$$

$$2\alpha + (1 + \alpha) = -2$$

$$3\alpha = -3$$

$$\alpha = -1$$

We have the image

$\vec{m} = (3 + 2\alpha)\hat{i} + (1 - \alpha)\hat{j} + (2 + \alpha)\hat{k}$

$\vec{m} = [3 + 2(-1)]\hat{i} + [1 - (-1)]\hat{j} + [2 + (-1)]\hat{k}$

$\vec{m} = \hat{i} + 2\hat{j} + \hat{k}$

Therefore, image is (1, 2, 1)

Foot of the perpendicular:

$\vec{q} = (3 + \alpha)\hat{i} + \frac{(2-\alpha)}{2}\hat{j} + \frac{(4+\alpha)}{2}\hat{k}$

$\vec{q} = [3 + (-1)]\hat{i} + \left[\frac{2 - (-1)}{2}\right]\hat{j} + \left[\frac{4 + (-1)}{2}\right]\hat{k}$

$\vec{q} = 2\hat{i} + \frac{3}{2}\hat{j} + \frac{3}{2}\hat{k}$

Hence, the position vector of the image is $\hat{i} + 2\hat{j} + \hat{k}$ and that of the foot of perpendicular is $2\hat{i} + \frac{3}{2}\hat{j} + \frac{3}{2}\hat{k}$.

Q5.

Solution:

Given:

Let point P = (1, 1, 2) and Q be the foot of the perpendicular drawn from P to the plane 2x – 2y + 4z + 5 = 0.

Direction ratios of PQ are proportional to 2, –2, 4 as PQ is normal to the plane.

The equation of the line passing through (x_1, y_1, z_1) and having direction ratios proportional to l, m, n is given by

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$$

Here,

$(x_1, y_1, z_1) = (1, 1, 2)$ and $(l, m, n) = (2, -2, 4)$

So, the equation of PQ is

$$\frac{x - 1}{2} = \frac{y - 1}{-2} = \frac{z - 2}{4}$$

$$\frac{x - 1}{2} = \frac{y - 1}{-2} = \frac{z - 2}{4} = \alpha \text{ (say)}$$

$x = 2\alpha + 1, y = -2\alpha + 1, z = 4\alpha + 2$

Let $Q = (2\alpha + 1, -2\alpha + 1, 4\alpha + 2)$.

This point lies on the given plane, which means this point satisfies the plane equation.

$2(2\alpha + 1) - 2(-2\alpha + 1) + 4(4\alpha + 2) + 5 = 0$

$4\alpha + 2 + 4\alpha - 2 + 16\alpha + 8 + 5 = 0$

$24\alpha + 13 = 0$

$24\alpha = -13$

$$\alpha = -\frac{13}{24}$$

We have $Q = (2\alpha + 1, -2\alpha + 1, 4\alpha + 2)$

$$Q = \left(2\left(-\frac{13}{24}\right) + 1, -2\left(-\frac{13}{24}\right) + 1, 4\left(-\frac{13}{24}\right) + 2\right)$$

$$= \left(-\frac{13}{12} + 1, \frac{13}{12} + 1, \frac{-13}{6} + 2\right)$$

$$= \left(-\frac{1}{12}, \frac{25}{12}, -\frac{1}{6}\right)$$

The length of the perpendicular drawn from (x_1, y_1, z_1) to the plane $Ax + By + Cz + D = 0$ is given by

$$\left|\frac{Ax_1 + By_1 + Cz_1 + D}{\sqrt{A^2 + B^2 + C^2}}\right|$$

Here,

$(x_1, y_1, z_1) = (1, 1, 2)$ and $(A, B, C, D) = (2, -2, 4, 5)$

$$PQ = \left|\frac{(2)(1)+(-2)(1)+(4)(2)+5}{\sqrt{2^2+(-2)^2+4^2}}\right|$$

$$= \left|\frac{2-2+8+5}{\sqrt{4+4+16}}\right|$$

$$= \left|\frac{13}{\sqrt{24}}\right|$$

$$= \frac{13}{2\sqrt{6}}$$

Hence, the required foot of perpendicular is $\left(-\frac{1}{12}, \frac{25}{12}, -\frac{1}{6}\right)$ and the length of the perpendicular is $\frac{13}{2\sqrt{6}}$ units.