EXERCISE 29.5

Q1.

Solution:

Given:

The plane passes through points,

(1, 1, 1), (1, -1, 1) and (-7, -3, -5)

So by using these points we know that equation of plane passes through these points,

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix} = 0$$

By substituting the values, we get

$$\begin{vmatrix} x-1 & y-1 & z-1 \\ 1-1 & -1-1 & 1-1 \\ -7-1 & -3-1 & -5-1 \end{vmatrix} = 0$$

$$\begin{vmatrix} x-1 & y-1 & z-1 \\ 0 & -2 & 0 \\ -8 & -4 & -6 \end{vmatrix} = 0$$

Let us simplify, we get

$$(x-1)(12-0)-(y-1)(0-0)+(z-1)(0-16)=0$$
$$(x-1)(12)-(y-1)(0)+(z-1)(-16)=0$$
$$12x-12-0-16z+16=0$$
$$12x-16z+4=0$$

Divide by 4, we get

$$3x-4z+1=0$$
$$\left(x\hat{i}+y\hat{j}+z\hat{k}\right)\left(3\hat{i}+0\hat{j}-4\hat{k}\right)+1=0$$
$$\vec{r}.\left(3\hat{i}-4\hat{k}\right)+1=0$$

Hence, the required equation of plane is

$$\vec{r}.\left(3\hat{i}-4\hat{k}\right)+1=0$$

Q2.

Solution:

Let us consider three points be P, Q and R.

So P(2, 5, -3), Q(-2, -3, 5) and R(5, 3, -3) are the points on the plane having position vectors $\vec{p}$, $\vec{q}$ and $\vec{s}$.

Then the vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ are in the same plane.

Therefore, $\overrightarrow{PQ} \times \overrightarrow{PR}$ is a vector perpendicular to the plane.
Let $\vec{n} = \overrightarrow{PQ} \times \overrightarrow{PR}$

$\overrightarrow{PQ} = (-2-2)\hat{i} + (-3-5)\hat{j} + (5-(-3))\hat{k}$
$\Rightarrow \overrightarrow{PQ} = -4\hat{i} - 8\hat{j} + 8\hat{k}$
Similarly,
$\overrightarrow{PR} = (5-2)\hat{i} + (3-5)\hat{j} + (-3-(-3))\hat{k}$
$\Rightarrow \overrightarrow{PR} = 3\hat{i} - 2\hat{j} + 0\hat{k}$
Thus
$\vec{n} = \overrightarrow{PQ} \times \overrightarrow{PR}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -4 & -8 & 8 \\ 3 & -2 & 0 \end{vmatrix}$$

$= 16\hat{i} + 24\hat{j} + 32\hat{k}$

The plane passes through the point P with position vector $\vec{p} = 2\hat{i} + 5\hat{j} - 3\hat{k}$
Hence the vector equation is

$\{\vec{r} - (2\hat{i} + 5\hat{j} - 3\hat{k})\} \cdot (16\hat{i} + 24\hat{j} + 32\hat{k}) = 0$
$\Rightarrow \vec{r} \cdot (16\hat{i} + 24\hat{j} + 32\hat{k}) - (32 + 120 - 96) = 0$
$\Rightarrow \vec{r} \cdot (16\hat{i} + 24\hat{j} + 32\hat{k}) - 56 = 0$
$\Rightarrow \vec{r} \cdot (16\hat{i} + 24\hat{j} + 32\hat{k}) = 56$
$\Rightarrow \vec{r} \cdot (2\hat{i} + 3\hat{j} + 4\hat{k}) = 7$

Q3.
Solution:
Let us consider three points be A, B and C.
So A(a, 0, 0), B(0, b, 0) and C(0, 0, c) are the points on the plane having position vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$.

Then vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are in the same plane.

Therefore, $\overrightarrow{AB} \times \overrightarrow{AC}$ is a vector perpendicular to the plane.
Let $\vec{n} = \overrightarrow{AB} \times \overrightarrow{AC}$

$\overrightarrow{AB} = (0-a)\hat{i} + (b-0)\hat{j} + (0-0)\hat{k}$
$\Rightarrow \overrightarrow{AB} = -a\hat{i} + b\hat{j} + 0\hat{k}$
Similarly,
$\overrightarrow{AC} = (0-a)\hat{i} + (0-0)\hat{j} + (c-0)\hat{k}$
$\Rightarrow \overrightarrow{AC} = -a\hat{i} + 0\hat{j} + c\hat{k}$
Thus
$\vec{n} = \overrightarrow{AB} \times \overrightarrow{AC}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -a & b & 0 \\ -a & 0 & c \end{vmatrix}$$

$$\vec{n} = bc\hat{i} + ac\hat{j} + ab\hat{k}$$
$$\Rightarrow \hat{n} = \frac{bc\hat{i} + ac\hat{j} + ab\hat{k}}{\sqrt{b^2c^2 + a^2c^2 + a^2b^2}}$$

The plane passes through the point P with position vector $\vec{a} = a\hat{i} + 0\hat{j} + 0\hat{k}$.

Thus, the vector equation in the normal form is given as,

$$\{\vec{r} - ((a\hat{i} + 0\hat{j} + 0\hat{k})\} \cdot \left(\frac{bc\hat{i} + ac\hat{j} + ab\hat{k}}{\sqrt{b^2c^2 + a^2c^2 + a^2b^2}}\right) = 0$$
$$\Rightarrow \vec{r} \cdot \frac{(bc\hat{i} + ac\hat{j} + ab\hat{k})}{\sqrt{b^2c^2 + a^2c^2 + a^2b^2}} = \frac{abc}{\sqrt{b^2c^2 + a^2c^2 + a^2b^2}}$$
$$\Rightarrow \vec{r} \cdot \frac{(bc\hat{i} + ac\hat{j} + ab\hat{k})}{\sqrt{b^2c^2 + a^2c^2 + a^2b^2}} = \frac{1}{\sqrt{\frac{b^2c^2 + a^2c^2 + a^2b^2}{a^2b^2c^2}}}$$
$$\Rightarrow \vec{r} \cdot \frac{(bc\hat{i} + ac\hat{j} + ab\hat{k})}{\sqrt{b^2c^2 + a^2c^2 + a^2b^2}} = \frac{1}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}}} \ldots(1)$$

We know that the vector equation of the plane with distance d from origin and normal to unit vector $\vec{n}$ is given as:

$$\vec{r} \cdot \hat{n} = d \quad \ldots\ldots.. (2)$$

It is given that, plane is at a distance 'p' from the origin.
Let us compare equations (1) and (2), we get

$$d = p = \frac{1}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}}}$$
$$\Rightarrow \frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}$$

Q4.

Solution:

Let us consider three points be P, Q and R.
So P(1, 1, -1), Q(6, 4, -5) and R(-4, -2, 3) are the points on the plane having position vectors $\vec{p}$, $\vec{q}$ and $\vec{s}$.

Then the vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ are in the same plane.

Therefore, $\overrightarrow{PQ} \times \overrightarrow{PR}$ is a vector perpendicular to the plane.
Let $\vec{n} = \overrightarrow{PQ} \times \overrightarrow{PR}$

$$\overrightarrow{PQ} = (6-1)\hat{i} + (4-1)\hat{j} + (-5-(-1))\hat{k}$$
$$\Rightarrow \overrightarrow{PQ} = 5\hat{i} + 3\hat{j} - 4\hat{k}$$

Similarly,

$$\overrightarrow{PR} = (-4-1)\hat{i} + (-2-1)\hat{j} + (3-(-1))\hat{k}$$
$$\Rightarrow \overrightarrow{PR} = -5\hat{i} - 3\hat{j} + 4\hat{k}$$

Thus

Here, $\overrightarrow{PQ} = -\overrightarrow{PR}$

Hence, we can say that the given points are collinear.

Thus, $\vec{n} = a\hat{i} + b\hat{j} + c\hat{k}$ where, $5a + 3b - 4c = 0$

The plane passes through the point P with position vector $\vec{p} = \hat{i} + \hat{j} - \hat{k}$.

Thus, the vector equation in the normal form is given as,

$\{\vec{r} - (\hat{i} + \hat{j} - \hat{k})\} \cdot (a\hat{i} + b\hat{j} + c\hat{k}) = 0$, where, $5a + 3b - 4c = 0$

Q5.

Solution:

Let us consider three points be A, B and C.

So now let A, B, C be the points with position vector

$(3\hat{i} + 4\hat{j} + 2\hat{k})$, $(2\hat{i} - 2\hat{j} - \hat{k})$ and $(7\hat{i} + 6\hat{k})$.

Then,

$\overrightarrow{AB}$ = Position vector of B – Poosition vector of A

$= (2\hat{i} - 2\hat{j} - \hat{k}) - (3\hat{i} + 4\hat{j} + 2\hat{k})$

$= 2\hat{i} - 2\hat{j} - \hat{k} - 3\hat{i} - 4\hat{j} - 2\hat{k}$

$= -\hat{i} - 6\hat{j} - 3\hat{k}$

$\overrightarrow{BC}$ = Position vector of C – Poosition vector of B

$= (7\hat{i} + 6\hat{k}) - (2\hat{i} - 2\hat{j} - \hat{k})$

$= 7\hat{i} + 6\hat{k} - 2\hat{i} + 2\hat{j} + \hat{k}$

$= 5\hat{i} + 2\hat{j} + 7\hat{k}$

We know that, a vector normal to A, B, C is a vector perpendicular to $\overrightarrow{AB} \times \overrightarrow{BC}$.

So,

$\vec{n} = \overrightarrow{AB} \times \overrightarrow{BC}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & -6 & -3 \\ 5 & 2 & 7 \end{vmatrix}$$

Let us simplify, we get

$\vec{n} = \hat{i}(-42 + 6) - \hat{j}(-7 + 15) + \hat{k}(-2 + 30)$

$= -36\hat{i} - 8\hat{j} + 28\hat{k}$

We know that the vector equation of the plane passing through vector $\vec{a}$ and perpendicular to vector $\vec{n}$ is given as:

$\vec{r}.\vec{n} = \vec{a}.\vec{n}$ ...... (1)

Substitute the values of $\vec{a}$ and $\vec{n}$ in equation (1), we get

$$\vec{r}.\left(-36\hat{i} - 8\hat{j} + 28\hat{k}\right) = \left(3\hat{i} + 4\hat{j} + 2\hat{k}\right)\left(-36\hat{i} - 8\hat{j} + 28\hat{k}\right)$$
$$= (3)(-36) + (4)(-8) + (2)(28)$$
$$= -108 - 32 + 56$$
$$= -140 + 56$$
$$= -84$$

Divide by -4, we get

$$\vec{r}.\left(9\hat{i} + 2\hat{j} - 7\hat{k}\right) = 21$$

Hence, the required equation of plane is,

$$\vec{r}.\left(9\hat{i} + 2\hat{j} - 7\hat{k}\right) = 21$$