EXERCISE 29.6

Q1. i.

Solution:

Given planes are:

$\vec{r}.(2\hat{i} - 3\hat{j} + 4\hat{k}) = 1$ and $\vec{r}.(-\hat{i} + \hat{j}) = 1$

$\vec{r}.(2\hat{i} - 3\hat{j} + 4\hat{k}) = 1$ ...... (1)

$\vec{r}.(-\hat{i} + \hat{j}) = 1$ ...... (2)

We know that the angle between two planes,

$\vec{r}.\vec{n}_1 = d_1$ and $\vec{r}.\vec{n}_2 = d_2$ is given by

$$\cos\theta = \frac{\vec{n}_1.\vec{n}_2}{|\vec{n}_1||\vec{n}_2|}$$

Here, we have

$\vec{n}_1 = 2\hat{i} - 3\hat{j} + 4\hat{k}$ and $\vec{n}_2 = -\hat{i} + \hat{j}$

$$\therefore \cos\theta = \frac{\left((2\hat{i} - 3\hat{j} + 4\hat{k}).(-\hat{i} + \hat{j})\right)}{|2\hat{i} - 3\hat{j} + 4\hat{k}||-\hat{i} + \hat{j}|}$$

$$= \frac{((2)(-1) + (-3)(1) + (4)(0))}{\sqrt{2^2 + 3^2 + 4^2}\sqrt{1^2 + 1^2}}$$

$$= \frac{-2 - 3 + 0}{\sqrt{29}\sqrt{2}}$$

$$= -\frac{5}{\sqrt{58}}$$

Now, as

$$\cos\theta = -\frac{5}{\sqrt{58}}$$

$$\Rightarrow \theta = \cos^{-1}\left(-\frac{5}{\sqrt{58}}\right)$$

Hence the angle between the two planes is:

$\theta = \cos^{-1}\left(-\frac{5}{\sqrt{58}}\right)$

ii.

Solution:

Given planes, $\vec{r}.\left(2\hat{i} - \hat{j} + 2\hat{k}\right) = 6$ and $\vec{r}.\left(3\hat{i} + 6\hat{j} - 2\hat{k}\right) = 9$

$\vec{r}.\left(2\hat{i} - \hat{j} + 2\hat{k}\right) = 6$ …… (1)

$\vec{r}.\left(3\hat{i} + 6\hat{j} - 2\hat{k}\right) = 9$ …… (2)

We know that the angle between two planes,

$\vec{r}.\vec{n}_1 = d_1$ and $\vec{r}.\vec{n}_2 = d_2$ is given by

$$\cos\theta = \frac{\vec{n}_1.\vec{n}_2}{|\vec{n}_1||\vec{n}_2|}$$

Here we have,

$\vec{n}_1 = 2\hat{i} - \hat{j} + 2\hat{k}$ and $\vec{n}_2 = 3\hat{i} + 6\hat{j} - 2\hat{k}$

$$\therefore \cos\theta = \frac{\left(\left(2\hat{i} - \hat{j} + 2\hat{k}\right).\left(3\hat{i} + 6\hat{j} - 2\hat{k}\right)\right)}{|2\hat{i} - \hat{j} + 2\hat{k}||3\hat{i} + 6\hat{j} - 2\hat{k}|}$$

$$= \frac{((2)(3) + (-1)(6) + (2)(-2))}{\sqrt{2^2 + 1^2 + 2^2}\sqrt{3^2 + 6^2 + 2^2}}$$

$$= \frac{6 - 6 - 4}{\sqrt{9}\sqrt{49}}$$

$$= -\frac{4}{3 \times 7} = -\frac{4}{21}$$

Now, as

$$\cos\theta = -\frac{4}{21}$$

$$\Rightarrow \theta = \cos^{-1}\left(-\frac{4}{21}\right)$$

Hence the angle between the two planes is:

$\theta = \cos^{-1}\left(-\frac{4}{21}\right)$

iii.
Solution:

Given planes are:

$\vec{r}.(2\hat{i} + 3\hat{j} - 6\hat{k}) = 6$ and $\vec{r}.(\hat{i} - 2\hat{j} + 2\hat{k}) = 9$

So,

$\vec{r}.(2\hat{i} + 3\hat{j} - 6\hat{k}) = 5$ ...... (1)

$\vec{r}.(\hat{i} - 2\hat{j} + 2\hat{k}) = 9$ ...... (2)

We know that the angle between two planes,

$\vec{r}.\vec{n}_1 = d_1$ and $\vec{r}.\vec{n}_2 = d_2$ is given by

$$\cos\theta = \frac{\vec{n}_1.\vec{n}_2}{|\vec{n}_1||\vec{n}_2|}$$

Here we have

$\vec{n}_1 = 2\hat{i} + 3\hat{j} - 6\hat{k}$ and $\vec{n}_2 = \hat{i} - 2\hat{j} + 2\hat{k}$

$$\therefore \cos\theta = \frac{\left((2\hat{i} + 3\hat{j} - 6\hat{k}).(\hat{i} - 2\hat{j} + 2\hat{k})\right)}{|2\hat{i} + 3\hat{j} - 6\hat{k}||\hat{i} - 2\hat{j} + 2\hat{k}|}$$

$$= \frac{\left((2)(1) + (3)(-2) + (-6)(2)\right)}{\sqrt{2^2 + 3^2 + 6^2}\sqrt{1^2 + 2^2 + 2^2}}$$

$$= \frac{2 - 6 - 12}{\sqrt{49}\sqrt{9}}$$

$$= -\frac{16}{3 \times 7} = -\frac{16}{21}$$

Now, as

$$\cos\theta = -\frac{16}{21}$$

$$\Rightarrow \theta = \cos^{-1}\left(-\frac{16}{21}\right)$$

Hence the angle between the two planes is:

$$\theta = \cos^{-1}\left(-\frac{16}{21}\right)$$

Q2. i

Solution:

Given planes are:

$2x - y + z = 4$ and $x + y + 2z = 3$

We know that angle between two planes,

$a_1x + b_1y + c_1z + d_1 = 0$ and

$a_2x + b_2y + c_2z + d_2 = 0$ is given as

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

So here we have,

$a_1 = 2, b_1 = -1, c_1 = 1$

$a_2 = 1, b_2 = 1, c_2 = 2$

Now let us substitute the values in above expression, we get

$$\therefore \cos\theta = \frac{(2)(1) + (-1)(1) + (1)(2)}{\sqrt{2^2 + 1^2 + 1^2}\sqrt{1^2 + 1^2 + 2^2}}$$

$$= \frac{2 - 1 + 2}{\sqrt{6}\sqrt{6}}$$

$$= \frac{3}{6} = \frac{1}{2}$$

$$\therefore \cos\theta = \frac{1}{2}$$

$$\theta = \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$$

Hence, the angle between planes,

$2x - y + z = 4$ and $x + y + 2z = 3$ is $\frac{\pi}{3}$.

ii.

Solution:

Given planes are:

$x + y - 2z = 3$ and $2x - 2y + z = 5$

We know that angle between two planes,

$a_1x + b_1y + c_1z + d_1 = 0$

$a_2x + b_2y + c_2z + d_2 = 0$ is given as

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

So here we have,

$a_1 = 1, b_1 = 1, c_1 = -2$

$a_2 = 2, b_2 = -2, c_2 = 1$

Now let us substitute the values in above expression, we get

$$\therefore \cos\theta = \frac{(1)(2) + (1)(-2) + (-2)(1)}{\sqrt{1^2 + 1^2 + 2^2}\sqrt{2^2 + 2^2 + 1^2}}$$

$$= \frac{2 - 2 - 2}{\sqrt{6}\sqrt{9}}$$

$$= -\frac{2}{\sqrt{6} \times 3}$$

$$= -\frac{2}{3\sqrt{6}}$$

$$\therefore \cos\theta = -\frac{2}{3\sqrt{6}}$$

$$\theta = \cos^{-1}\left(-\frac{2}{3\sqrt{6}}\right)$$

Hence, the angle between planes $x + y - 2z = 3$ and $2x - 2y + z = 5$ is $\theta = \cos^{-1}\left(-\frac{2}{3\sqrt{6}}\right)$.

iii.

Solution:

Given planes are:

$x - y + z = 5$ and $x + 2y + z = 9$

We know that angle between two planes,

$a_1x + b_1y + c_1z + d_1 = 0$

$a_2x + b_2y + c_2z + d_2 = 0$ is given as

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

So here we have,
$a_1 = 1, b_1 = -1, c_1 = 1$
$a_2 = 1, b_2 = 2, c_2 = 1$
Now let us substitute the values in above expression, we get

$$\therefore \cos\theta = \frac{(1)(1) + (-1)(2) + (1)(1)}{\sqrt{1^2 + 1^2 + 1^2}\sqrt{1^2 + 2^2 + 1^2}}$$

$$= \frac{1-2+1}{\sqrt{3}\sqrt{4}}$$

$$= \frac{0}{\sqrt{3} \times 2} = 0$$

$$\therefore \cos\theta = 0$$

$$\theta = \frac{\pi}{2}$$

Hence, the angle between planes
$x - y + z = 5$ and $x + 2y + z = 9$ is $\theta = \frac{\pi}{2}$.

iv.
Solution:
Given planes are:
$2x - 3y + 4z = 1$ and $-x + y = 4$
We know that angle between two planes,
$a_1x + b_1y + c_1z + d_1 = 0$
$a_2x + b_2y + c_2z + d_2 = 0$ is given as

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

So here we have,
$a_1 = 2, b_1 = -3, c_1 = 4$
$a_2 = -1, b_2 = 1, c_2 = 0$
Now let us substitute the values in above expression, we get

$$\therefore \cos\theta = \frac{(2)(-1) + (-3)(1) + (4)(0)}{\sqrt{2^2 + 3^2 + 4^2}\sqrt{1^2 + 1^2 + 0^2}}$$

$$= \frac{-2-3+0}{\sqrt{29}\sqrt{2}}$$

$$= \frac{-5}{\sqrt{58}}$$

$$\therefore \cos\theta = \frac{-5}{\sqrt{58}}$$

$$\theta = \cos^{-1}\left(\frac{-5}{\sqrt{58}}\right)$$

Hence, the angle between planes
2x – 3y + 4z = 1 and – x + y = 4 is $\theta = \cos^{-1}\left(\frac{-5}{\sqrt{58}}\right)$

v.

Solution:

Given planes are:

2x + y – 2z = 5 and 3x – 6y – 2z = 7

We know that angle between two planes,

$a_1x + b_1y + c_1z + d_1 = 0$

$a_2x + b_2y + c_2z + d_2 = 0$ is given as

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

So here we have,

$a_1 = 2, b_1 = 1, c_1 = -2$

$a_2 = 3, b_2 = -6, c_2 = -2$

Now let us substitute the values in above expression, we get

$$\therefore \cos\theta = \frac{(2)(3) + (1)(-6) + (-2)(-2)}{\sqrt{2^2 + 1^2 + 2^2}\sqrt{3^2 + 6^2 + 2^2}}$$

$$= \frac{6-6+4}{\sqrt{9}\sqrt{49}}$$

$$= \frac{4}{3 \times 7} = \frac{4}{21}$$

$$\therefore \cos\theta = \frac{4}{21}$$

$$\theta = \cos^{-1}\left(\frac{4}{21}\right)$$

Hence, the angle between planes

$2x + y - 2z = 5$ and $3x - 6y - 2z = 7$ is $\theta = \cos^{-1}\left(\frac{4}{21}\right)$.

Q3.i.

Solution:

Given planes,

$\vec{r}.\left(2\hat{i} - \hat{j} + \hat{k}\right) = 5$ and

$\vec{r}.\left(-\hat{i} - \hat{j} + \hat{k}\right) = 3$

We know that planes $\vec{r}.\vec{n}_1 = d_1$ and $\vec{r}.\vec{n}_2 = d_2$ are perpendicular if $\vec{n}_1.\vec{n}_2 = 0$

We have

$\vec{n}_1 = 2\hat{i} - \hat{j} + \hat{k}$ and

$\vec{n}_2 = -\hat{i} - \hat{j} + \hat{k}$

Now substitute the values, we get

$$\vec{n}_1.\vec{n}_2 = \left(2\hat{i} - \hat{j} + \hat{k}\right).\left(-\hat{i} - \hat{j} + \hat{k}\right)$$
$$= (-2 + 1 + 1) = 0$$

Hence, the two given planes are perpendicular.

ii.

Solution:

Given planes,

$x - 2y + 4z = 10$ and

$18x + 17y + 4z = 49$

We know that planes

$a_1x + b_1y + c_1z + d_1 = 0$ and

$a_2x + b_2y + c_2z + d_2 = 0$ are at right angles if,

$a_1a_2 + b_1b_2 + c_1c_2 = 0$ …… (1)

Now we have,
$a_1 = 1$, $b_1 = -2$, $c_1 = 4$ and
$a_2 = 18$, $b_2 = 17$, $c_2 = 4$

Using equation (1) we have,
$a_1a_2 + b_1b_2 + c_1c_2 = (1)(18) + (-2)(17) + (4)(4)$
$= 18 - 34 + 16 = 0$
Hence, the planes are at right angle to each other.

Q4.i.
Solution:
Given planes are,
$\vec{r}.(\hat{i} + 2\hat{j} + 3\hat{k}) = 7$ and
$\vec{r}.(\lambda\hat{i} + 2\hat{j} - 7\hat{k}) = 26$

We know that planes $\vec{r}.\vec{n}_1 = d_1$ and $\vec{r}.\vec{n}_2 = d_2$ are perpendicular if $\vec{n}_1.\vec{n}_2 = 0$
We have
$\vec{n}_1 = \hat{i} + 2\hat{j} + 3\hat{k}$ and
$\vec{n}_2 = \lambda\hat{i} + 2\hat{j} - 7\hat{k}$
Now substitute the values, we get
$\vec{n}_1.\vec{n}_2 = (\hat{i} + 2\hat{j} + 3\hat{k}).(\lambda\hat{i} + 2\hat{j} - 7\hat{k}) = 0$
So, $(\lambda + 4 - 21) = 0$
$\lambda = 21 - 4$
$= 17$
Hence, for $\lambda = 17$ the given planes are perpendicular.

ii.
Solution:
Given planes are,
$2x - 4y + 3z = 5$ and
$x + 2y + \lambda z = 5$
We know that planes
$a_1x + b_1y + c_1z + d_1 = 0$ and
$a_2x + b_2y + c_2z + d_2 = 0$ are at right angles if,

$a_1a_2 + b_1b_2 + c_1c_2 = 0$ …… (1)
We have,
$a_1 = 2, b_1 = -4, c_1 = 3$ and
$a_2 = 1, b_2 = 2, c_2 = \lambda$

Let us substitute the values in equation (1), we get
$a_1a_2 + b_1b_2 + c_1c_2 \Rightarrow (2)(1) + (-4)(2) + (3)(\lambda) = 0$
$\Rightarrow 2 - 8 + 3\lambda = 0$
$6 = 3\lambda$
$2 = \lambda$
Hence, for $\lambda = 2$ the given planes are perpendicular.

iii.
Solution:
Given planes are,
$3x - 6y - 2z = 7$ and
$2x + y - \lambda z = 5$
We know that planes
$a_1x + b_1y + c_1z + d_1 = 0$ and
$a_2x + b_2y + c_2z + d_2 = 0$ are at right angles if,
$a_1a_2 + b_1b_2 + c_1c_2 = 0$ …… (1)

We have,
$a_1 = 3, b_1 = -6, c_1 = -2$ and
$a_2 = 2, b_2 = 1, c_2 = -\lambda$

Let us substitute the values in equation (1), we get
$a_1a_2 + b_1b_2 + c_1c_2 = (3)(2) + (-6)(1) + (-2)(-\lambda) = 0$
$\Rightarrow 6 - 6 + 2\lambda = 0$
$0 = -2\lambda$
$0 = \lambda$
Hence, for $\lambda = 0$ the given planes are perpendicular to each other.

Q5.
Solution:
We know that solution of a plane passing through (x_1, y_1, z_1) is given as:
$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$
The required plane passes through $(-1, -1, 2)$,
So the equation of plane is

$a(x + 1) + b(y + 1) + c(z - 2) = 0$
$\Rightarrow ax + by + cz = 2c - a - b$ …… (1)

Now, the required plane is also perpendicular to the planes,
$3x + 2y - 3z = 1$ and
$5x - 4y + z = 5$

We know that planes
$a_1x + b_1y + c_1z + d_1 = 0$ and
$a_2x + b_2y + c_2z + d_2 = 0$ are at right angles if,
$a_1a_2 + b_1b_2 + c_1c_2 = 0$ …… (2)

Using equation (2) we have,
$3a + 2b - 3c = 0$ …… (3)
$5a - 4b + c = 0$ …… (4)

Now let us solve equation (3) and (4) we get,

$$\frac{a}{(2)(1)-(-3)(-4)} = \frac{b}{(5)(-3)-(3)(1)} = \frac{c}{(3)(-4)-(2)(5)}$$

$$\Rightarrow \frac{a}{2-12} = \frac{b}{-15-3} = \frac{c}{-12-10}$$

$$\Rightarrow \frac{a}{-10} = \frac{b}{-18} = \frac{c}{-22} = \lambda(\text{say})$$

So,
$a = -10\lambda$, $b = -18\lambda$, $c = -22\lambda$
Now substitute the values of a, b, c in equation (1) we get,

$(-10\lambda)x + (-18\lambda)y + (-22\lambda)z = 2(-22)\lambda - (-10\lambda) - (-18\lambda)$
$-10\lambda x - 18\lambda y - 22\lambda z = -44\lambda + 10\lambda + 18\lambda$
$-10\lambda x - 18\lambda y - 22\lambda z = -16\lambda$
Divide both sides by (-2λ), we get
$5x + 9y + 11z = 8$
Hence, the required equation of the plane is $5x + 9y + 11z = 8$.