EXERCISE 7.1 PAGE NO: 7.22

1. Find the adjoint of each of the following matrices:

$(i)\begin{bmatrix}-3 & 5\\ 2 & 4\end{bmatrix}$

$(ii)\begin{bmatrix}a & b\\ c & d\end{bmatrix}$

$(iii)\begin{bmatrix}cos\alpha & sin\alpha\\ sin\alpha & cos\alpha\end{bmatrix}$

$(iv)\begin{bmatrix}1 & tan\frac{\alpha}{2}\\ -tan\frac{\alpha}{2} & 1\end{bmatrix}$

Verify that (adj A) A = |A| I = A (adj A) for the above matrices.

Solution:

(i) Let

A = $\begin{bmatrix}-3 & 5\\ 2 & 4\end{bmatrix}$

Cofactors of A are

$C_{11} = 4$

$C_{12} = -2$

$C_{21} = -5$

$C_{22} = -3$

Since, adj A = $\begin{bmatrix}C_{11} & C_{12}\\ C_{21} & C_{22}\end{bmatrix}^T$

(adj A) = $\begin{bmatrix}4 & -2\\ -5 & -3\end{bmatrix}^T$

= $\begin{bmatrix}4 & -5\\ -2 & -3\end{bmatrix}$

Now, (adj A) A = $\begin{bmatrix}4 & -5\\ -2 & -3\end{bmatrix}\begin{bmatrix}-3 & 5\\ 2 & 4\end{bmatrix} = \begin{bmatrix}-12-10 & 20-20\\ 6-6 & -10-12\end{bmatrix}$

(adj A)A = $\begin{bmatrix}-22 & 0\\ 0 & -22\end{bmatrix}$

And, |A| I = $\begin{vmatrix}-3 & 5\\ 2 & 4\end{vmatrix}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} = (-22)\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} = \begin{bmatrix}-22 & 0\\ 0 & -22\end{bmatrix}$

Also, A (adj A) = $\begin{bmatrix} -3 & 5 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 4 & -5 \\ -2 & -3 \end{bmatrix} = \begin{bmatrix} -12-10 & 20-20 \\ 6-6 & -10-12 \end{bmatrix}$

A (adj A) = $\begin{bmatrix} -22 & 0 \\ 0 & -22 \end{bmatrix}$

Hence, (adj A) A = |A|I = A (adj A)

(ii) Let

A = $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Therefore cofactors of A are

$C_{11} = d$

$C_{12} = -c$

$C_{21} = -b$

$C_{22} = a$

We know that, adj A = $\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}^T$

Therefore by substituting these values we get,

(adj A) = $\begin{bmatrix} d & -c \\ -b & a \end{bmatrix}^T$

= $\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Now, (adj A) A = $\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ad-bc & bd-bd \\ -ac+ac & -bc+ad \end{bmatrix}$

(adj A)A = $\begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix}$

And, |A|.I = $\begin{vmatrix} a & b \\ c & d \end{vmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = (ad-bc)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix}$

Also,

A (adj A) = $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix}$

Hence, (adj A) A = |A| I = A (adj A)

(iii) Let

A = $\begin{bmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$

Therefore cofactors of A are

C_{11} = cos α

C_{12} = - sin α

C_{21} = - sin α

C_{22} = cos α

We know that, adj A = $\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}^T$

(adj A) = $\begin{bmatrix} \cos\alpha & -\sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}^T$

= $\begin{bmatrix} \cos\alpha & -\sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$

Now, (adj A) A = $\begin{bmatrix} \cos\alpha & -\sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}\begin{bmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$

$= \begin{bmatrix} -\sin^2\alpha + \cos^2\alpha & \cos\alpha.\sin\alpha - \sin\alpha.\cos\alpha \\ -\cos\alpha\sin\alpha + \sin\alpha\cos\alpha & -\sin^2\alpha + \cos^2\alpha \end{bmatrix}$

(adj A) A = $\begin{bmatrix} \cos2\alpha & 0 \\ 0 & \cos2\alpha \end{bmatrix}$

And, |A|I = $\begin{vmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & \cos\alpha \end{vmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

= $(\cos^2\alpha - \sin^2\alpha)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

= $\begin{bmatrix} \cos^2\alpha - \sin^2\alpha & 0 \\ 0 & \cos^2\alpha - \sin^2\alpha \end{bmatrix}$

$= \begin{bmatrix} \cos 2\alpha & 0 \\ 0 & \cos 2\alpha \end{bmatrix}$

Also, A (adj A)

$= \begin{bmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \cos\alpha & -\sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} = \begin{bmatrix} \cos^2\alpha - \sin^2\alpha & 0 \\ 0 & \cos^2\alpha - \sin^2\alpha \end{bmatrix}$

$= \begin{bmatrix} \cos 2\alpha & 0 \\ 0 & \cos 2\alpha \end{bmatrix}$

Hence, (adj A) A = |A|I = A (adj A)

(iv) Let

A = $\begin{bmatrix} 1 & \tan\frac{\alpha}{2} \\ -\tan\frac{\alpha}{2} & 1 \end{bmatrix}$

Therefore cofactors of A are

$C_{11} = 1$

$C_{12} = \tan \alpha/2$

$C_{21} = - \tan \alpha/2$

$C_{22} = 1$

We know that, adj A = $\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}^T$

(adj A) = $\begin{bmatrix} 1 & \tan\frac{\alpha}{2} \\ -\tan\frac{\alpha}{2} & 1 \end{bmatrix}^T$

$= \begin{bmatrix} 1 & -\tan\frac{\alpha}{2} \\ \tan\frac{\alpha}{2} & 1 \end{bmatrix}$

Now, (adj A) A = $\begin{bmatrix} 1 & -\tan\frac{\alpha}{2} \\ \tan\frac{\alpha}{2} & 1 \end{bmatrix} \begin{bmatrix} 1 & \tan\frac{\alpha}{2} \\ -\tan\frac{\alpha}{2} & 1 \end{bmatrix}$

$= \begin{bmatrix} 1 + \tan^2\frac{\alpha}{2} & \tan\frac{\alpha}{2} - \tan\frac{\alpha}{2} \\ \tan\frac{\alpha}{2} - \tan\frac{\alpha}{2} & 1 + \tan^2\frac{\alpha}{2} \end{bmatrix}$

(adj A)A = $\begin{bmatrix} 1 + \tan^2\frac{\alpha}{2} & 0 \\ 0 & 1 + \tan^2\frac{\alpha}{2} \end{bmatrix}$

And, |A|.I = $\begin{vmatrix} 1 & \tan\frac{\alpha}{2} \\ -\tan\frac{\alpha}{2} & 1 \end{vmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \left(1 + \tan^2\frac{\alpha}{2}\right) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1 + \tan^2\frac{\alpha}{2} & 0 \\ 0 & 1 + \tan^2\frac{\alpha}{2} \end{bmatrix}$

Also, A (adj A) = $\begin{bmatrix} 1 & \tan\frac{\alpha}{2} \\ -\tan\frac{\alpha}{2} & 1 \end{bmatrix} \begin{bmatrix} 1 & -\tan\frac{\alpha}{2} \\ \tan\frac{\alpha}{2} & 1 \end{bmatrix}$

$= \begin{bmatrix} 1 + \tan^2\frac{\alpha}{2} & \tan\frac{\alpha}{2} - \tan\frac{\alpha}{2} \\ \tan\frac{\alpha}{2} - \tan\frac{\alpha}{2} & 1 + \tan^2\frac{\alpha}{2} \end{bmatrix}$

$= \begin{bmatrix} 1 + \tan^2\frac{\alpha}{2} & 0 \\ 0 & 1 + \tan^2\frac{\alpha}{2} \end{bmatrix}$

Hence, (adj A) A = |A|I = A (adj A)

2. Compute the adjoint of each of the following matrices.

$(i) \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$

$(ii) \begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 1 \\ -1 & 1 & 1 \end{bmatrix}$

$(iii)\ \begin{bmatrix} 2 & -1 & 3 \\ 4 & 2 & 5 \\ 0 & 4 & -1 \end{bmatrix}$

$(iv)\ \begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 1 & 1 & 3 \end{bmatrix}$

Solution:

(i) Let

$A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$

Therefore cofactors of A are

$C_{11} = -3$

$C_{21} = 2$

$C_{31} = 2$

$C_{12} = 2$

$C_{22} = -3$

$C_{23} = 2$

$C_{13} = 2$

$C_{23} = 2$

$C_{33} = -3$

$\text{adj } A = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T$

$= \begin{bmatrix} -3 & 2 & 2 \\ 2 & -3 & 2 \\ 2 & 2 & -3 \end{bmatrix}$

Now, $(\text{adj } A)\, A = \begin{bmatrix} -3 & 2 & 2 \\ 2 & -3 & 2 \\ 2 & 2 & -3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$

$= \begin{bmatrix} -3+4+4 & -6+2+4 & -6+4+2 \\ 2-3+4 & 4-3+4 & 4-6+2 \\ 2+4-6 & 4+2-6 & 4+4-3 \end{bmatrix}$

$= \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

Also, $|A|I = \begin{vmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{vmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = (-3+4+4) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

Then, $A\,(\text{adj } A) = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} -3 & 2 & 2 \\ 2 & -3 & 2 \\ 2 & 2 & -3 \end{bmatrix}$

$= \begin{bmatrix} -3+4+4 & -6+2+4 & -6+4+2 \\ 2-3+4 & 4-3+4 & 4-6+2 \\ 2+4-6 & 4+2-6 & 4+4-3 \end{bmatrix}$

$= \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

Since, (adj A) A = |A|I = A (adj A)

(ii) Let

$A = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 1 \\ -1 & 1 & 1 \end{bmatrix}$

Cofactors of A

$C_{11} = 2$

$C_{21} = 3$

$C_{31} = -13$

$C_{12} = -3$

$C_{22} = 6$

$C_{32} = 9$

$C_{13} = 5$

$C_{23} = -3$

$C_{33} = -1$

adj A = $\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T$

= $\begin{bmatrix} 2 & -3 & 5 \\ 3 & 6 & -3 \\ -13 & 9 & -1 \end{bmatrix}^T$

adj A = $\begin{bmatrix} 2 & 3 & -13 \\ -3 & 6 & 9 \\ 5 & -3 & -1 \end{bmatrix}$

Now, (adj A) A = $\begin{bmatrix} 2 & 3 & -13 \\ -3 & 6 & 9 \\ 5 & -3 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 1 \\ -1 & 1 & 1 \end{bmatrix}$

= $\begin{bmatrix} 2 + 6 + 13 & 4 + 9 - 13 & 10 + 3 - 13 \\ -3 + 12 - 9 & -6 + 18 + 9 & -15 + 6 + 9 \\ 5 - 6 + 1 & 10 - 9 - 1 & 25 - 3 - 1 \end{bmatrix}$

= $\begin{bmatrix} 21 & 0 & 0 \\ 0 & 21 & 0 \\ 0 & 0 & 21 \end{bmatrix}$

Also, |A|I = $\begin{vmatrix} 1 & 2 & 5 \\ 2 & 3 & 1 \\ -1 & 1 & 1 \end{vmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

= [1(3 – 1) – 2(2 + 1) + 5(2 + 3)] $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

= 21 $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

= $\begin{bmatrix} 21 & 0 & 0 \\ 0 & 21 & 0 \\ 0 & 0 & 21 \end{bmatrix}$

Then, A (adj A) = $\begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 1 \\ -1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 3 & -13 \\ -3 & 6 & 9 \\ 5 & -3 & -1 \end{bmatrix}$

= $\begin{bmatrix} 2-6+25 & 3+12-15 & -13+18-5 \\ 4-9+5 & 6+18-3 & -26+27-1 \\ -2-3+5 & -3+6-3 & 13+9-1 \end{bmatrix}$

= $\begin{bmatrix} 21 & 0 & 0 \\ 0 & 21 & 0 \\ 0 & 0 & 21 \end{bmatrix}$

Hence, (adj A) A = |A|I = A (adj A)

(iii) Let

A = $\begin{bmatrix} 2 & -1 & 3 \\ 4 & 2 & 5 \\ 0 & 4 & -1 \end{bmatrix}$

Therefore cofactors of A

$C_{11} = -22$

$C_{21} = 11$

$C_{31} = -11$

$C_{12} = 4$

$C_{22} = -2$

$C_{32} = 2$

$C_{13} = 16$

$C_{23} = -8$

$C_{33} = 8$

We know that adj A = $\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T$

Now by substituting the values in above matrix we get,

= $\begin{bmatrix} -22 & 4 & 16 \\ 11 & -2 & -8 \\ -11 & 2 & 8 \end{bmatrix}^T$

adj A = $\begin{bmatrix} -22 & 11 & -11 \\ 4 & -2 & 2 \\ 16 & -8 & 8 \end{bmatrix}$

Now, (adj A) A = $\begin{bmatrix} -22 & 11 & -11 \\ 4 & -2 & 2 \\ 16 & -8 & 8 \end{bmatrix}\begin{bmatrix} 2 & -1 & 3 \\ 4 & 2 & 5 \\ 0 & 4 & -1 \end{bmatrix}$

= $\begin{bmatrix} -44 + 44 + 0 & 22 + 22 - 44 & -66 + 55 + 11 \\ 8 - 8 + 0 & -4 - 4 + 8 & 12 - 10 - 2 \\ 32 - 32 + 0 & -16 - 16 + 32 & 48 - 40 - -8 \end{bmatrix}$

= $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Now, |A|I = $\begin{vmatrix} 2 & -1 & 3 \\ 4 & 2 & 5 \\ 0 & 4 & -1 \end{vmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

= [2(– 2 – 20) + 1(– 4 – 0) + 3(16 – 0)] $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

= (– 44 – 4 + 48) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

= $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Then, A (adj A) = $\begin{bmatrix} 2 & -1 & 3 \\ 4 & 2 & 5 \\ 0 & 4 & -1 \end{bmatrix}\begin{bmatrix} -22 & 11 & -11 \\ 4 & -2 & 2 \\ 16 & -8 & 8 \end{bmatrix}$

= $\begin{bmatrix} -44 - 4 + 48 & 22 + 2 - 24 & -22 - 2 + 24 \\ -88 + 8 + 80 & 44 - 4 - 40 & -44 + 4 + 40 \\ 0 + 16 - 16 & 0 - 8 + 8 & 0 + 8 - 8 \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Hence, (adj A) A = |A|I = A (adj A)

(iv) Let

$A = \begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 1 & 1 & 3 \end{bmatrix}$

Therefore cofactors of A

$C_{11} = 3$

$C_{21} = -1$

$C_{31} = 1$

$C_{12} = -15$

$C_{22} = 7$

$C_{32} = -5$

$C_{13} = 4$

$C_{23} = -2$

$C_{33} = 2$

$\text{adj } A = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T$

$= \begin{bmatrix} 3 & -15 & 4 \\ -3 & 7 & -2 \\ 1 & -5 & 2 \end{bmatrix}^T$

$\text{adj } A = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 7 & -5 \\ 4 & -2 & 2 \end{bmatrix}$

Now, $(\text{adj } A) A = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 7 & -5 \\ 4 & -2 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 1 & 1 & 3 \end{bmatrix}$

$= \begin{bmatrix} 6-5+1 & 0-1+1 & -3+0+3 \\ -30+35-5 & 0+7-5 & 15-0-15 \\ 8-10+2 & 0-2+2 & -4-0+6 \end{bmatrix}$

$= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

Also, $|A|I = \begin{vmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 1 & 1 & 3 \end{vmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$= [2(3-0) + 0(15-0) - 1(5-1)] \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$= (6-4) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

Then, $A\,(\text{adj } A) = \begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 1 & 1 & 3 \end{bmatrix} \begin{bmatrix} 3 & -1 & 1 \\ -15 & 7 & -5 \\ 4 & -2 & 2 \end{bmatrix}$

$= \begin{bmatrix} 6+0-4 & -2+0+2 & 2-0-2 \\ 15-15+0 & -5+7+0 & 5-5+0 \\ 3-15+12 & -1+7-6 & 1-5+6 \end{bmatrix}$

$= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

Hence, (adj A) A = |A|I = A (adj A)

3. $\boldsymbol{For\ the\ matrix\ A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 3 & 0 \\ 18 & 2 & 10 \end{bmatrix},\ show\ that\ A(adj\,A) = 0}$

Solution:

Given

A = $\begin{bmatrix} 1 & -1 & 1 \\ 2 & 3 & 0 \\ 18 & 2 & 10 \end{bmatrix}$

Therefore cofactors of A

$C_{11} = 30$

$C_{21} = 12$

$C_{31} = -3$

$C_{12} = -20$

$C_{22} = -8$

$C_{32} = 2$

$C_{13} = -50$

$C_{23} = -20$

$C_{33} = 5$

We know that adj A = $\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T$

By substituting these values in above matrix we get,

= $\begin{bmatrix} 30 & -20 & -50 \\ 12 & -8 & -20 \\ -3 & 2 & 5 \end{bmatrix}^T$

So, adj (A) = $\begin{bmatrix} 30 & 12 & -3 \\ -20 & -8 & 2 \\ -50 & -20 & 5 \end{bmatrix}$

Now, A (adj A) = $\begin{bmatrix} 1 & -1 & 1 \\ 2 & 3 & 0 \\ 18 & 2 & 10 \end{bmatrix}\begin{bmatrix} 30 & 12 & -3 \\ -20 & -8 & 2 \\ -50 & -20 & 5 \end{bmatrix}$

= $\begin{bmatrix} 30 + 20 - 50 & 12 + 8 - 20 & -3 - 2 + 5 \\ 60 - 60 + 0 & 24 - 24 + 0 & -6 + 6 + 0 \\ 540 - 40 - 500 & 216 - 16 - 200 & -54 + 4 + 50 \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Hence, A (adj A) = 0

$$4.\ If\ A = \begin{bmatrix} -4 & -3 & -3 \\ 1 & 0 & 1 \\ 4 & 4 & 3 \end{bmatrix},\ show\ that\ adj\,A = A$$

Solution:

Given

$A = \begin{bmatrix} -4 & -3 & -3 \\ 1 & 0 & 1 \\ 4 & 4 & 3 \end{bmatrix}$

Cofactors of A

$C_{11} = -4$

$C_{21} = -3$

$C_{31} = -3$

$C_{12} = 1$

$C_{22} = 0$

$C_{32} = 1$

$C_{13} = 4$

$C_{23} = 4$

$C_{33} = 3$

We know that adj A = $\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T$

$= \begin{bmatrix} -4 & 1 & 4 \\ -3 & 0 & 4 \\ -3 & 1 & 3 \end{bmatrix}^T$

So, adj A = $\begin{bmatrix} -4 & -3 & -3 \\ 1 & 0 & 1 \\ 4 & 4 & 3 \end{bmatrix}$

Hence, adj A = A

5. $If\ A = \begin{bmatrix} -1 & -2 & -2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix},\ show\ that\ adj\,A = 3A^T.$

Solution:

Given

A = $\begin{bmatrix} -1 & -2 & -2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$

Cofactors of A are

$C_{11} = -3$

$C_{21} = 6$

$C_{31} = 6$

$C_{12} = -6$

$C_{22} = 3$

$C_{32} = -6$

$C_{13} = -6$

$C_{23} = -6$

$C_{33} = 3$

adj A = $\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T$

= $\begin{bmatrix} -3 & -6 & -6 \\ 6 & 3 & -6 \\ 6 & -6 & 3 \end{bmatrix}^T$

So, adj A = $\begin{bmatrix} -3 & 6 & 6 \\ -6 & 3 & -6 \\ -6 & -6 & 3 \end{bmatrix}$

Now, $3A^T = 3\begin{bmatrix} -1 & -2 & -2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix} = \begin{bmatrix} -3 & 6 & 6 \\ -6 & 3 & -6 \\ -6 & -6 & 3 \end{bmatrix}$

Hence, adj A = $3.A^T$

$$6.\ Find\ A(adjA)\ for\ the\ matrix\ A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & 2 & -1 \\ -4 & 5 & 2 \end{bmatrix}$$

Solution:

Given

A = $\begin{bmatrix} 1 & -2 & 3 \\ 0 & 2 & -1 \\ -4 & 5 & 2 \end{bmatrix}$

Cofactors of A are

C_{11} = 9

C_{21} = 19

C_{31} = – 4

C_{12} = 4

C_{22} = 14

C_{32} = 1

C_{13} = 8

C_{23} = 3

C_{33} = 2

We know that adj A = $\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T$

= $\begin{bmatrix} 9 & 4 & 8 \\ 19 & 14 & 3 \\ -4 & 1 & 2 \end{bmatrix}^T$

So, adj A = $\begin{bmatrix} 9 & 19 & -4 \\ 4 & 14 & 1 \\ 8 & 3 & 2 \end{bmatrix}$

Now, A adj A = $\begin{bmatrix} 1 & -2 & 3 \\ 0 & 2 & -1 \\ -4 & 5 & 2 \end{bmatrix}\begin{bmatrix} 9 & 19 & -4 \\ 4 & 14 & 1 \\ 8 & 3 & 2 \end{bmatrix}$

= $\begin{bmatrix} 9-8+24 & 19-28+9 & -4-2+6 \\ 0+8-8 & 0+28-3 & 0+2-2 \\ -36+20+16 & -76+70+6 & 16+5+4 \end{bmatrix}$

$= \begin{bmatrix} 25 & 0 & 0 \\ 0 & 25 & 0 \\ 0 & 0 & 25 \end{bmatrix}$

Hence, A adj A = 25 I_3

7. Find the inverse of each of the following matrices:

$(i) \begin{bmatrix} cos\theta & sin\theta \\ -sin\theta & cos\theta \end{bmatrix}$

$(ii) \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$(iii) \begin{bmatrix} a & b \\ c & \frac{1+bc}{a} \end{bmatrix}$

$(iv) \begin{bmatrix} 2 & 5 \\ -3 & 1 \end{bmatrix}$

Solution:

(i) The criteria of existence of inverse matrix is the determinant of a given matrix should not equal to zero.

Now, |A| = cos θ (cos θ) + sin θ (sin θ)

= 1

Hence, A^{-1} exists.

Cofactors of A are

C_{11} = cos θ

C_{12} = sin θ

C_{21} = - sin θ

C_{22} = cos θ

Since, adj A = $\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}^T$

(adj A) = $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}^T$

$= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

Now, $A^{-1} = \frac{1}{|A|}$.adj A

$A^{-1} = \frac{1}{1}\begin{bmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{bmatrix}$

$A^{-1} = \begin{bmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{bmatrix}$

(ii) The criteria of existence of inverse matrix is the determinant of a given matrix should not equal to zero.
Now, $|A| = -1 \neq 0$
Hence, A^{-1} exists.
Cofactors of A are
$C_{11} = 0$
$C_{12} = -1$
$C_{21} = -1$
$C_{22} = 0$

Since, adj A = $\begin{bmatrix}C_{11} & C_{12} \\ C_{21} & C_{22}\end{bmatrix}^T$

(adj A) = $\begin{bmatrix}0 & -1 \\ -1 & 0\end{bmatrix}^T$

$= \begin{bmatrix}0 & -1 \\ -1 & 0\end{bmatrix}$

Now, $A^{-1} = \frac{1}{|A|}$ adj A

$A^{-1} = \frac{1}{-1}\begin{bmatrix}0 & -1 \\ -1 & 0\end{bmatrix}$

$A^{-1} = \begin{bmatrix}0 & 1 \\ 1 & 0\end{bmatrix}$

(iii) The criteria of existence of inverse matrix is the determinant of a given matrix should not equal to zero.

Now, $|A| = \frac{a + abc}{a} - bc = \frac{a + abc - abc}{a} = 1 \neq 0$

Hence, A^{-1} exists.

Cofactors of A are

$C_{11} = \frac{1 + bc}{a}$

$C_{12} = -c$

$C_{21} = -b$

$C_{22} = a$

Since, adj A = $\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}^T$

(adj A) = $\begin{bmatrix} \frac{1 + bc}{a} & -c \\ -b & a \end{bmatrix}^T$

$= \begin{bmatrix} \frac{1 + bc}{a} & -b \\ -c & a \end{bmatrix}$

Now, $A^{-1} = \frac{1}{|A|}$ adj A

$A^{-1} = \frac{1}{1}\begin{bmatrix} \frac{1 + bc}{a} & -b \\ -c & a \end{bmatrix}$

$A^{-1} = \begin{bmatrix} \frac{1 + bc}{a} & -b \\ -c & a \end{bmatrix}$

(iv) The criteria of existence of inverse matrix is the determinant of a given matrix should not equal to zero.
Now, $|A| = 2 + 15 = 17 \neq 0$
Hence, A^{-1} exists.
Cofactors of A are
$C_{11} = 1$

$C_{12} = 3$
$C_{21} = -5$
$C_{22} = 2$

Since, adj A = $\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}^T$

(adj A) = $\begin{bmatrix} 1 & 3 \\ -5 & 2 \end{bmatrix}^T$

= $\begin{bmatrix} 1 & -5 \\ 3 & 2 \end{bmatrix}$

Now, $A^{-1} = \frac{1}{|A|}.\text{adj } A$

$A^{-1} = \frac{1}{17}\begin{bmatrix} 1 & -5 \\ 3 & 2 \end{bmatrix}$

8. Find the inverse of each of the following matrices.

$(i) \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix}$

$(ii) \begin{bmatrix} 1 & 2 & 5 \\ 1 & -1 & -1 \\ 2 & 3 & -1 \end{bmatrix}$

$(iii) \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

$(iv) \begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$

$(v) \begin{bmatrix} 0 & 1 & -1 \\ 4 & -3 & 4 \\ 3 & -3 & 4 \end{bmatrix}$

$(vi) \begin{bmatrix} 0 & 0 & -1 \\ 3 & 4 & 5 \\ -2 & -4 & -7 \end{bmatrix}$

$(vii)\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos\alpha & sin\alpha \\ 0 & sin\alpha & -cos\alpha \end{bmatrix}$

Solution:

(i) The criteria of existence of inverse matrix is the determinant of a given matrix should not equal to zero.

$|A| = 1\begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} - 2\begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} + 3\begin{vmatrix} 2 & 3 \\ 3 & 1 \end{vmatrix}$

= 1(6 – 1) – 2(4 – 3) + 3(2 – 9)

= 5 – 2 – 21

= – 18≠ 0

Hence, A^{-1} exists

Cofactors of A are

$C_{11} = 5$

$C_{21} = -1$

$C_{31} = -7$

$C_{12} = -1$

$C_{22} = -7$

$C_{32} = 5$

$C_{13} = -7$

$C_{23} = 5$

$C_{33} = -1$

We know that adj A = $\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T$

$= \begin{bmatrix} 5 & -1 & -7 \\ -1 & -7 & 5 \\ -7 & 5 & -1 \end{bmatrix}^T$

So, adj A = $\begin{bmatrix} 5 & -1 & -7 \\ -1 & -7 & 5 \\ -7 & 5 & -1 \end{bmatrix}$

Now, $A^{-1} = \frac{1}{|A|}$ adj A

So, $A^{-1} = \frac{1}{(-18)}\begin{bmatrix} 5 & -1 & -7 \\ -1 & -7 & 5 \\ -7 & 5 & -1 \end{bmatrix}$

Hence, $A^{-1} = \begin{bmatrix} \frac{-5}{18} & \frac{1}{18} & \frac{7}{18} \\ \frac{1}{18} & \frac{7}{18} & \frac{-5}{18} \\ \frac{7}{18} & \frac{-5}{18} & \frac{1}{18} \end{bmatrix}$

(ii) The criteria of existence of inverse matrix is the determinant of a given matrix should not equal to zero.

$|A| = 1\begin{vmatrix} -1 & -1 \\ 3 & -1 \end{vmatrix} - 2\begin{vmatrix} 1 & -1 \\ 2 & -1 \end{vmatrix} + 5\begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix}$

= 1 (1 + 3) – 2 (– 1 + 2) + 5 (3 + 2)

= 4 – 2 + 25

= 27≠ 0

Hence, A^{-1} exists

Cofactors of A are

$C_{11} = 4$

$C_{21} = 17$

$C_{31} = 3$

$C_{12} = -1$

$C_{22} = -11$

$C_{32} = 6$

$C_{13} = 5$

$C_{23} = 1$

$C_{33} = -3$

adj A = $\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T$

$= \begin{bmatrix} 4 & -1 & 5 \\ 17 & -11 & 1 \\ 3 & 6 & -3 \end{bmatrix}^T$

So, adj A = $\begin{bmatrix} 4 & 17 & 3 \\ -1 & -11 & 6 \\ 5 & 1 & -3 \end{bmatrix}$

Now, $A^{-1} = \frac{1}{|A|}$ adj A

So, $A^{-1} = \frac{1}{(27)} \begin{bmatrix} 4 & 17 & 3 \\ -1 & -11 & 6 \\ 5 & 1 & -3 \end{bmatrix}$

Hence, $A^{-1} = \begin{bmatrix} \frac{4}{27} & \frac{17}{27} & \frac{3}{27} \\ \frac{-1}{27} & \frac{-11}{27} & \frac{6}{27} \\ \frac{5}{27} & \frac{1}{27} & \frac{-3}{27} \end{bmatrix} = \begin{bmatrix} \frac{4}{27} & \frac{17}{27} & \frac{1}{9} \\ \frac{-1}{27} & \frac{-11}{27} & \frac{2}{9} \\ \frac{5}{27} & \frac{1}{27} & \frac{-1}{9} \end{bmatrix}$

(iii) The criteria of existence of inverse matrix is the determinant of a given matrix should not equal to zero.

$|A| = 2\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} + 1\begin{vmatrix} -1 & -1 \\ 1 & 2 \end{vmatrix} + 1\begin{vmatrix} -1 & 2 \\ 1 & -1 \end{vmatrix}$

= 2(4 – 1) + 1(– 2 + 1) + 1(1 – 2)

= 6 – 2

= – 4≠ 0

Hence, A^{-1} exists

Cofactors of A are

$C_{11} = 3$

$C_{21} = 1$

$C_{31} = -1$

$C_{12} = +1$

$C_{22} = 3$

$C_{32} = 1$

$C_{13} = -1$

$C_{23} = 1$

$C_{33} = 3$

We know that adj A = $\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T$

$= \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}^T$

So, adj A = $\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$

Now, $A^{-1} = \frac{1}{|A|}$ adj A

So, $A^{-1} = \frac{1}{4}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$

Hence, $A^{-1} = \begin{bmatrix} \frac{3}{4} & \frac{1}{4} & \frac{-1}{4} \\ \frac{1}{4} & \frac{3}{4} & \frac{1}{4} \\ \frac{-1}{4} & \frac{1}{4} & \frac{1}{4} \end{bmatrix}$

(iv) The criteria of existence of inverse matrix is the determinant of a given matrix should not equal to zero.

$|A| = 2\begin{vmatrix} 1 & 0 \\ 1 & 3 \end{vmatrix} - 0\begin{vmatrix} 5 & 0 \\ 0 & 3 \end{vmatrix} - 1\begin{vmatrix} 5 & 1 \\ 0 & 1 \end{vmatrix}$

= 2(3 – 0) – 0 – 1(5)

= 6 – 5

= 1≠ 0

Hence, A^{-1} exists

Cofactors of A are

$C_{11} = 3$

$C_{21} = -1$

$C_{31} = 1$

$C_{12} = -15$

$C_{22} = 6$
$C_{32} = -5$
$C_{13} = 5$
$C_{23} = -2$
$C_{33} = 2$

We know that adj A = $\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T$

= $\begin{bmatrix} 3 & -15 & 5 \\ -1 & 6 & -2 \\ 1 & -5 & 2 \end{bmatrix}^T$

So, adj A = $\begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$

Now, $A^{-1} = \frac{1}{|A|}$ adj A

So, $A^{-1} = \frac{1}{1}\begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$

Hence, $A^{-1} = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$

(v) The criteria of existence of inverse matrix is the determinant of a given matrix should not equal to zero.

$|A| = 0\begin{vmatrix} -3 & 0 \\ -3 & 4 \end{vmatrix} - 1\begin{vmatrix} 4 & 4 \\ 3 & 4 \end{vmatrix} - 1\begin{vmatrix} 4 & -3 \\ 3 & -3 \end{vmatrix}$

= 0 – 1 (16 – 12) – 1 (– 12 + 9)
= – 4 + 3
= – 1≠ 0
Hence, A^{-1} exists
Cofactors of A are
$C_{11} = 0$
$C_{21} = -1$

$C_{31} = 1$
$C_{12} = -4$
$C_{22} = 3$
$C_{32} = -4$
$C_{13} = -3$
$C_{23} = 3$
$C_{33} = -4$

We know that adj A = $\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T$

= $\begin{bmatrix} 0 & -4 & -3 \\ -1 & 3 & 3 \\ 1 & -4 & -4 \end{bmatrix}^T$

So, adj A = $\begin{bmatrix} 0 & -1 & 1 \\ -4 & 3 & -4 \\ -3 & 3 & -4 \end{bmatrix}$

Now, $A^{-1} = \frac{1}{|A|}$ adj A

So, $A^{-1} = \frac{1}{-1}\begin{bmatrix} 0 & -1 & 1 \\ -4 & 3 & -4 \\ -3 & 3 & -4 \end{bmatrix}$

Hence, $A^{-1} = \begin{bmatrix} 0 & 1 & -1 \\ 4 & -3 & 4 \\ 3 & -3 & 4 \end{bmatrix}$

(vi) The criteria of existence of inverse matrix is the determinant of a given matrix should not equal to zero.

$|A| = 0\begin{vmatrix} 4 & 5 \\ -4 & -7 \end{vmatrix} - 0\begin{vmatrix} 3 & 5 \\ -2 & -7 \end{vmatrix} - 1\begin{vmatrix} 3 & 4 \\ -2 & -4 \end{vmatrix}$

$= 0 - 0 - 1(-12 + 8)$

$= 4 \neq 0$

Hence, A^{-1} exists

Cofactors of A are

$C_{11} = -8$

$C_{21} = 4$
$C_{31} = 4$
$C_{12} = 11$
$C_{22} = -2$
$C_{32} = -3$
$C_{13} = -4$
$C_{23} = 0$
$C_{33} = 0$

We know that adj A = $\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T$

$= \begin{bmatrix} 8 & 11 & -4 \\ 4 & -2 & 0 \\ 4 & -3 & 0 \end{bmatrix}^T$

So, adj A = $\begin{bmatrix} 8 & 4 & 4 \\ 11 & -2 & -3 \\ -4 & 0 & 0 \end{bmatrix}$

Now, $A^{-1} = \frac{1}{|A|}$ adj A

So, $A^{-1} = \frac{1}{4}\begin{bmatrix} 8 & 4 & 4 \\ 11 & -2 & -3 \\ -4 & 0 & 0 \end{bmatrix}$

Hence, $A^{-1} = \begin{bmatrix} 2 & 1 & 1 \\ \frac{11}{4} & \frac{-1}{2} & \frac{-3}{4} \\ -1 & 0 & 0 \end{bmatrix}$

(vii) The criteria of existence of inverse matrix is the determinant of a given matrix should not equal to zero.

$|A| = 1\begin{vmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{vmatrix} - 0 + 0$
$= -(\cos^2\alpha - \sin^2\alpha)$
$= -1 \neq 0$
Hence, A^{-1} exists
Cofactors of A are

$C_{11} = -1$
$C_{21} = 0$
$C_{31} = 0$
$C_{12} = 0$
$C_{22} = -\cos\alpha$
$C_{32} = -\sin\alpha$
$C_{13} = 0$
$C_{23} = -\sin\alpha$
$C_{33} = \cos\alpha$

We know that adj A = $\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T$

$= \begin{bmatrix} -1 & 0 & 0 \\ 0 & -\cos\alpha & -\sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{bmatrix}^T$

So, adj A = $\begin{bmatrix} -1 & 0 & 0 \\ 0 & -\cos\alpha & -\sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{bmatrix}$

Now, $A^{-1} = \frac{1}{|A|}$ adj A

So, $A^{-1} = \frac{1}{-1}\begin{bmatrix} -1 & 0 & 0 \\ 0 & -\cos\alpha & -\sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{bmatrix}$

Hence, $A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & \sin\alpha & -\cos\alpha \end{bmatrix}$

9. Find the inverse of each of the following matrices and verify that $A^{-1}A = I_3$.

$(i)\ \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$

$(ii)\ \begin{bmatrix} 2 & 3 & 1 \\ 3 & 4 & 1 \\ 3 & 7 & 2 \end{bmatrix}$

Solution:

(i) We have

$|A| = 1\begin{vmatrix}4 & 3\\3 & 4\end{vmatrix} - 3\begin{vmatrix}1 & 3\\1 & 4\end{vmatrix} + 3\begin{vmatrix}1 & 4\\1 & 3\end{vmatrix}$

= 1(16 – 9) – 3(4 – 3) + 3(3 – 4)

= 7 – 3 – 3

= 1≠ 0

Hence, A^{-1} exists

Cofactors of A are

$C_{11} = 7$

$C_{21} = -3$

$C_{31} = -3$

$C_{12} = -1$

$C_{22} = 1$

$C_{32} = 0$

$C_{13} = -1$

$C_{23} = 0$

$C_{33} = 1$

We know that adj A = $\begin{bmatrix}C_{11} & C_{12} & C_{13}\\C_{21} & C_{22} & C_{23}\\C_{31} & C_{32} & C_{33}\end{bmatrix}^T$

$= \begin{bmatrix}7 & -1 & -1\\-3 & 1 & 0\\-3 & 0 & 1\end{bmatrix}^T$

So, adj A = $\begin{bmatrix}7 & -3 & -3\\-1 & 1 & 0\\-1 & 0 & 1\end{bmatrix}$

Now, $A^{-1} = \frac{1}{|A|}$ adj A = $\frac{1}{1}\begin{bmatrix}7 & -3 & -3\\-1 & 1 & 0\\-1 & 0 & 1\end{bmatrix}$

Also, $A^{-1}A = \begin{bmatrix}7 & -3 & -3\\-1 & 1 & 0\\-1 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 3 & 3\\1 & 4 & 3\\1 & 3 & 4\end{bmatrix}$

$= \begin{bmatrix} 7-3-3 & 21-12-9 & 21-9-12 \\ -1+1+0 & -3+4+0 & -3+3+0 \\ -1+0+1 & -3+0+3 & -3+0+4 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Hence, $A^{-1}A = I_3$

(ii) We have

$|A| = 2\begin{vmatrix} 4 & 1 \\ 7 & 2 \end{vmatrix} - 3\begin{vmatrix} 3 & 1 \\ 3 & 2 \end{vmatrix} + 1\begin{vmatrix} 3 & 4 \\ 3 & 7 \end{vmatrix}$

= 2(8 – 7) – 3(6 – 3) + 1(21 – 12)

= 2 – 9 + 9

= 2≠ 0

Hence, A^{-1} exists

Cofactors of A are

$C_{11} = 1$

$C_{21} = 1$

$C_{31} = -1$

$C_{12} = -3$

$C_{22} = 1$

$C_{32} = 1$

$C_{13} = 9$

$C_{23} = -5$

$C_{33} = -1$

We know that adj A = $\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T$

= $\begin{bmatrix} 1 & -3 & 9 \\ 1 & 1 & -5 \\ -1 & 1 & -1 \end{bmatrix}^T$

So, adj A = $\begin{bmatrix} 1 & 1 & -1 \\ -3 & 1 & 1 \\ 9 & -5 & -1 \end{bmatrix}$

Now, $A^{-1} = \frac{1}{|A|}$ adj A = $\frac{1}{2}\begin{bmatrix} 1 & 1 & -1 \\ -3 & 1 & 1 \\ 9 & -5 & -1 \end{bmatrix}$

Also, $A^{-1}.A = \frac{1}{2}\begin{bmatrix} 1 & 1 & -1 \\ -3 & 1 & 1 \\ 9 & -5 & -1 \end{bmatrix}\begin{bmatrix} 2 & 3 & 1 \\ 3 & 4 & 1 \\ 3 & 7 & 2 \end{bmatrix}$

= $\frac{1}{2}\begin{bmatrix} 2+3-3 & 3+4-7 & 1+1-2 \\ -6+3+3 & -9+4+7 & -3+1+2 \\ 18-15-3 & 27-20-7 & 9-5-2 \end{bmatrix}$

= $\frac{1}{2}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Hence, $A^{-1}.A = I_3$

10. For the following pair of matrices verify that $(AB)^{-1} = B^{-1}A^{-1}$.

$$(i) A = \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix} \ and\ B = \begin{bmatrix} 4 & 6 \\ 3 & 2 \end{bmatrix}$$

$$(ii) A = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix} \ and\ B = \begin{bmatrix} 4 & 5 \\ 3 & 4 \end{bmatrix}$$

Solution:

(i) Given

$A = \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}$,

$|A| = 1 \neq 0$

Then, adj A = $\begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix}$

$A^{-1} = \frac{\text{adj } A}{|A|} = \frac{1}{1}\begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix}$

$B = \begin{bmatrix} 4 & 6 \\ 3 & 2 \end{bmatrix}$,

$|B| = -10 \neq 0$

Then, adj B = $\begin{bmatrix} 2 & -6 \\ -3 & 4 \end{bmatrix}$

$B^{-1} = \frac{\text{adj } B}{|B|} = -\frac{1}{10}\begin{bmatrix} 2 & -6 \\ -3 & 4 \end{bmatrix}$

Also, A.B = $\begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}\begin{bmatrix} 4 & 6 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 12 + 6 & 18 + 4 \\ 28 + 15 & 42 + 10 \end{bmatrix}$

$AB = \begin{bmatrix} 18 & 22 \\ 43 & 52 \end{bmatrix}$

$|AB| = 936 - 946 = -10 \neq 0$

Adj (AB) = $\begin{bmatrix} 52 & -22 \\ -43 & 18 \end{bmatrix}$

$(AB)^{-1} = \frac{\text{adj } AB}{|AB|} = \frac{1}{-10}\begin{bmatrix} 52 & -22 \\ -43 & 18 \end{bmatrix} = \begin{bmatrix} -52 & 22 \\ 43 & -18 \end{bmatrix}$

Now $B^{-1}A^{-1} = \frac{1}{-10}\begin{bmatrix} 2 & -6 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix}$

$= \frac{1}{-10}\begin{bmatrix} 10 + 42 & -4 - 18 \\ -15 - 28 & 6 + 12 \end{bmatrix}$

$= \frac{1}{10}\begin{bmatrix} -52 & 22 \\ 43 & -18 \end{bmatrix}$

Hence, $(AB)^{-1} = B^{-1}A^{-1}$

(ii) Given

$|A| = 1 \neq 0$

Adj A = $\begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$

$A^{-1} = \frac{\text{adj A}}{|A|} = \frac{1}{1}\begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$

B = $\begin{bmatrix} 4 & 5 \\ 3 & 4 \end{bmatrix}$

$|B| = 1 \neq 0$

adj B = $\begin{bmatrix} 4 & -5 \\ -3 & 4 \end{bmatrix}$

$B^{-1} = \frac{\text{adj B}}{|B|} = \frac{1}{1}\begin{bmatrix} 4 & -5 \\ -3 & 4 \end{bmatrix}$

Also, AB = $\begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}\begin{bmatrix} 4 & 5 \\ 3 & 4 \end{bmatrix}$

= $\begin{bmatrix} 11 & 14 \\ 29 & 37 \end{bmatrix}$

$|AB| = 407 - 406 = 1 \neq 0$

And, adj (AB) = $\begin{bmatrix} 37 & -14 \\ -29 & 11 \end{bmatrix}$

$(AB)^{-1} = \frac{\text{adj AB}}{|AB|} = \frac{1}{1}\begin{bmatrix} 37 & -14 \\ -29 & 11 \end{bmatrix}$

= $\begin{bmatrix} 37 & -14 \\ -29 & 11 \end{bmatrix}$

Now, $B^{-1}A^{-1} = \begin{bmatrix} 4 & -5 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$

= $\begin{bmatrix} 37 & -14 \\ -29 & 11 \end{bmatrix}$

Hence, $(AB)^{-1} = B^{-1}A^{-1}$

$$11.\ Let\ A = \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}\ and\ B = \begin{bmatrix} 6 & 7 \\ 8 & 9 \end{bmatrix}.\ Find (AB)^{-1}$$

Solution:

Given

A = $\begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}$

|A| = 15 – 14 = 1 ≠ 0

Therefore adj A = $\begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix}$

$A^{-1} = \frac{adj}{|A|} = \frac{1}{1}\begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix}$

B = $\begin{bmatrix} 6 & 7 \\ 8 & 9 \end{bmatrix}$

|B| = 54 – 56 = – 2 ≠ 0

adj B = $\begin{bmatrix} 9 & -7 \\ -8 & 6 \end{bmatrix}$

$B^{-1} = \frac{adjB}{|B|} = \frac{1}{-2}\begin{bmatrix} 9 & -7 \\ -8 & 6 \end{bmatrix}$

Now, $(AB)^{-1} = B^{-1}A^{-1}$

$= \frac{1}{-2}\begin{bmatrix} 9 & -7 \\ -8 & 6 \end{bmatrix}\begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix}$

$= \frac{1}{-2}\begin{bmatrix} 45 + 49 & -18 - 21 \\ -40 - 42 & 16 + 18 \end{bmatrix}$

$= \frac{1}{-2}\begin{bmatrix} 94 & -39 \\ -82 & 34 \end{bmatrix}$

$(AB)^{-1} = \begin{bmatrix} -47 & \frac{39}{2} \\ 41 & -17 \end{bmatrix}$

$$12.\ Given\ A = \begin{bmatrix} 2 & -3 \\ -4 & 7 \end{bmatrix},\ compute\ A^{-1}\ and\ show\ that\ 2A^{-1} = 9I - A.$$

Solution:

Given

A = $\begin{bmatrix} 2 & -3 \\ -4 & 7 \end{bmatrix}$

|A| = 14 – 12 = 2 ≠ 0

adj A = $\begin{bmatrix} 7 & 3 \\ 4 & 2 \end{bmatrix}$

$A^{-1} = \frac{1}{2}\begin{bmatrix} 7 & 3 \\ 4 & 2 \end{bmatrix}$

To Show: $2A^{-1} = 9I - A$

We have

L.H.S = $2A^{-1} = 2.\frac{1}{2}\begin{bmatrix} 7 & 3 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 3 \\ 4 & 2 \end{bmatrix}$

R.H.S = $9I - A = \begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix} - \begin{bmatrix} 2 & -3 \\ -4 & 7 \end{bmatrix}$

$= \begin{bmatrix} 7 & 3 \\ 4 & 2 \end{bmatrix}$

Hence, $2A^{-1} = 9I - A$

13. $If\ A = \begin{bmatrix} 4 & 5 \\ 2 & 1 \end{bmatrix}$, *then show that* $A - 3I = 2(I + 3A^{-1})$.

Solution:

Given

$A = \begin{bmatrix} 4 & 5 \\ 2 & 1 \end{bmatrix}$

$|A| = 4 - 10 = -6 \neq 0$

adj $A = \begin{bmatrix} 1 & -5 \\ -2 & 4 \end{bmatrix}$

$A^{-1} = \frac{1}{-6}\begin{bmatrix} 1 & -5 \\ -2 & 4 \end{bmatrix}$

To Show: $A - 3I = 2(I + 3A^{-1})$

We have

LHS = $A - 3I$

$= \begin{bmatrix} 4 & 5 \\ 2 & 1 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1 & 5 \\ 2 & -2 \end{bmatrix}$

R.H.S = $2(I + 3A^{-1}) = 2I + 6A^{-1}$

$= 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + 6\frac{1}{6}\begin{bmatrix} -1 & 5 \\ 2 & -4 \end{bmatrix}$

$= \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} -1 & 5 \\ 2 & -4 \end{bmatrix}$

$= \begin{bmatrix} 1 & 5 \\ 2 & -2 \end{bmatrix}$

Hence, $A - 3I = 2(I + 3A^{-1})$

14. *Find the inverse of the matrix* $A = \begin{bmatrix} a & b \\ c & \frac{1+bc}{a} \end{bmatrix}$, *and show that* $aA^{-1} = (a^2 + bc + 1)I - aA$.

Solution:

$A = \begin{bmatrix} a & b \\ c & \frac{1+bc}{a} \end{bmatrix}$

Now, $|A| = \frac{a+abc}{a} - bc = \frac{a+abc-abc}{a} = 1 \neq 0$

Hence, A^{-1} exists.

Cofactors of A are

$C_{11} = \frac{1+bc}{a}$

$C_{12} = -c$

$C_{21} = -b$

$C_{22} = a$

Since, adj A = $\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}^T$

Adj A = $\begin{bmatrix} \frac{1+bc}{a} & -c \\ -b & a \end{bmatrix}^T$

$= \begin{bmatrix} \frac{1+bc}{a} & -b \\ -c & a \end{bmatrix}$

Now, $A^{-1} = \frac{1}{|A|}.\text{adj } A$

$A^{-1} = \frac{1}{1}\begin{bmatrix} \frac{1+bc}{a} & -b \\ -c & a \end{bmatrix}$

$A^{-1} = \begin{bmatrix} \frac{1+bc}{a} & -b \\ -c & a \end{bmatrix}$

To show a $A^{-1} = (a^2 + bc + 1)\, I - aA$.

LHS = a A^{-1}

$= a\begin{bmatrix} \frac{1+bc}{a} & -b \\ -c & a \end{bmatrix}$

$= \begin{bmatrix} 1+bc & -ab \\ -ac & a^2 \end{bmatrix}$

RHS = $(a^2 + bc + 1)\, I - a\, A$

$= \begin{bmatrix} a2+bc+1 & 0 \\ 0 & a2+bc+1 \end{bmatrix} - \begin{bmatrix} a^2 & ab \\ ac & 1+bc \end{bmatrix} = \begin{bmatrix} 1+bc & -ab \\ -ac & a^2 \end{bmatrix}$

Hence, LHS = RHS

$$15.\ Given\ A = \begin{bmatrix} 5 & 0 & 4 \\ 2 & 3 & 2 \\ 1 & 2 & 1 \end{bmatrix},\ B^{-1} = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}.Compute(AB)^{-1}$$

Solution:

Given

$A = \begin{bmatrix} 5 & 0 & 4 \\ 2 & 3 & 2 \\ 1 & 2 & 1 \end{bmatrix}$ and $B^{-1} = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$

Here, $(AB)^{-1} = B^{-1} A^{-1}$

$|A| = -5 + 4 = -1$

Cofactors of A are

$C_{11} = -1$
$C_{21} = 8$
$C_{31} = -12$
$C_{12} = 0$
$C_{22} = 1$
$C_{32} = -2$
$C_{13} = 1$
$C_{23} = -10$
$C_{33} = 15$

Adj A = $\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T$

$= \begin{bmatrix} -1 & 0 & 1 \\ 8 & 1 & -10 \\ -12 & -2 & 15 \end{bmatrix}^T$

So, adj A = $\begin{bmatrix} -1 & 8 & -12 \\ 0 & 1 & -2 \\ 1 & -10 & 15 \end{bmatrix}$

Now, $A^{-1} = \frac{1}{|A|}.\text{adj } A = \frac{1}{-1}\begin{bmatrix} -1 & 8 & -12 \\ 0 & 1 & -2 \\ 1 & -10 & 15 \end{bmatrix}$

$(AB)^{-1} = B^{-1} A^{-1}$

$= \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}\begin{bmatrix} 1 & -8 & 12 \\ 0 & -1 & 2 \\ -1 & 10 & -15 \end{bmatrix}$

$= \begin{bmatrix} 1 + 0 - 3 & -8 - 3 + 30 & 12 + 6 - 45 \\ 1 + 0 - 3 & -8 - 4 + 30 & 12 + 8 - 45 \\ 1 + 0 - 4 & -8 - 3 + 40 & 12 + 6 - 60 \end{bmatrix}$

Hence, $(AB)^{-1} = \begin{bmatrix} -2 & 19 & -27 \\ -2 & 18 & -25 \\ -3 & 29 & -42 \end{bmatrix}$

16. Let $F(\alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $G(\beta) = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix}$. Show that

(i) $[F(\alpha)]^{-1} = F(-\alpha)$
(ii) $[G(\beta)]^{-1} = G(-\beta)$
(iii) $[F(\alpha) G(\beta)]^{-1} = G(-\beta) F(-\alpha)$

Solution:
(i) Given

$$F(\alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$|F(\alpha)| = \cos^2\alpha + \sin^2\alpha = 1 \neq 0$

Cofactors of A are

$C_{11} = \cos\alpha$
$C_{21} = \sin\alpha$
$C_{31} = 0$
$C_{12} = -\sin\alpha$
$C_{22} = \cos\alpha$
$C_{32} = 0$
$C_{13} = 0$
$C_{23} = 0$
$C_{33} = 1$

$$\text{Adj } F(\alpha) = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T$$

$$= \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}^T$$

$$\text{So, adj } F(\alpha) = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\text{Now, } [F(\alpha)]^{-1} = \frac{1}{|F(\alpha)|} \text{ adj } F(\alpha) = \frac{1}{1}\begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \ \ldots\ldots \text{(i)}$$

And, $F(-\alpha) = \begin{bmatrix} \cos(-\alpha) & \sin(-\alpha) & 0 \\ \sin(-\alpha) & \cos(-\alpha) & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$ ...... (ii)

Hence, $[F(\alpha)]^{-1} = F(-\alpha)$

(ii) We have

$|G(\beta)| = \cos^2\beta + \sin^2\beta = 1$

Cofactors of A are

$C_{11} = \cos\beta$

$C_{21} = 0$

$C_{31} = -\sin\beta$

$C_{12} = 0$

$C_{22} = 1$

$C_{32} = 0$

$C_{13} = \sin\beta$

$C_{23} = 0$

$C_{33} = \cos\beta$

Adj $G(\beta) = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T$

$= \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix}^T$

So, adj $G(\beta) = \begin{bmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{bmatrix}$

Now, $[G(\beta)]^{-1} = \frac{1}{1}\begin{bmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{bmatrix}$ ...... (i)

And, G (– β) = $\begin{bmatrix} \cos(-\beta) & 0 & \sin(-\beta) \\ 0 & 1 & 0 \\ \sin(-\beta) & 0 & \cos(-\beta) \end{bmatrix}$

= $\begin{bmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{bmatrix}$ …… (ii)

Hence, $[G(\beta)]^{-1} = G - \beta)$

(iii) Now we have to show that

$[F(\alpha) G(\beta)]^{-1} = G(-\beta) F(-\alpha)$

We have already know that

$[G(\beta)]^{-1} = G(-\beta)$

$[F(\alpha)]^{-1} = F(-\alpha)$

And LHS = $[F(\alpha) G(\beta)]^{-1}$

= $[G(\beta)]^{-1} [F(\alpha)]^{-1}$

= $G(-\beta) F(-\alpha)$

Hence = RHS

17. $If\ A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}\ verify\ that\ A^2 - 4A + I = 0,\ where\ I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\ and\ O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. Hence\ find\ A^{-1}.$

Solution:

Consider,

$A^2 = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 4+3 & 6+6 \\ 2+2 & 3+4 \end{bmatrix}$

$= \begin{bmatrix} 7 & 12 \\ 4 & 7 \end{bmatrix}$

4A = $4\begin{bmatrix}2 & 3\\1 & 2\end{bmatrix} = \begin{bmatrix}8 & 12\\4 & 8\end{bmatrix}$

I = $\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$

Now, $A^2 - 4A + I = \begin{bmatrix}7 & 12\\4 & 7\end{bmatrix} - \begin{bmatrix}8 & 12\\4 & 8\end{bmatrix} + \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$

= $\begin{bmatrix}7-8+1 & 12-2+0\\4-4+0 & 7-8+1\end{bmatrix}$

Hence, = $\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$

Now, $A^2 - 4A + I = 0$

$A.A - 4A = -I$

Multiply by A^{-1} both sides we get

$A.A(A^{-1}) - 4A A^{-1} = -IA^{-1}$

$AI - 4I = -A^{-1}$

$A^{-1} = 4I - AI = \begin{bmatrix}4 & 0\\0 & 4\end{bmatrix} - \begin{bmatrix}2 & 3\\1 & 2\end{bmatrix}$

$A^{-1} = \begin{bmatrix}2 & -3\\-1 & 2\end{bmatrix}$

18. *Show that* $A = \begin{bmatrix}-8 & 5\\2 & 4\end{bmatrix}$ *satisfies the equation* $A^2 + 4A - 42I = 0$. *Hence find* A^{-1}.

Solution:

Given

A = $\begin{bmatrix}-8 & 5\\2 & 4\end{bmatrix}$

$A^2 = \begin{bmatrix} -8 & 5 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} -8 & 5 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 64+10 & -40+20 \\ -16+8 & 10+16 \end{bmatrix}$

$= \begin{bmatrix} 74 & -20 \\ -8 & 26 \end{bmatrix}$

$4A = 4\begin{bmatrix} -8 & 5 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} -32 & 20 \\ 8 & 16 \end{bmatrix}$

$42I = 42\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 42 & 0 \\ 0 & 42 \end{bmatrix}$

Now,

$A^2 + 4A - 42I = \begin{bmatrix} 74 & -20 \\ -8 & 26 \end{bmatrix} + \begin{bmatrix} -32 & 20 \\ 8 & 16 \end{bmatrix} - \begin{bmatrix} 42 & 0 \\ 0 & 42 \end{bmatrix}$

$= \begin{bmatrix} 74-74 & -20+20 \\ -8+8 & 42-42 \end{bmatrix}$

Hence, $= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Now, $A^2 + 4A - 42I = 0$

$= A^{-1} A.A + 4A^{-1}.A - 42A^{-1}I = 0$

$= IA + 4I - 42A^{-1} = 0$

$= 42A^{-1} = A + 4I$

$= A^{-1} = \frac{1}{42}[A + 4I]$

$= \frac{1}{42}\left[\begin{bmatrix} -8 & 5 \\ 2 & 4 \end{bmatrix} + \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}\right]$

$A^{-1} = \frac{1}{42}\left[\begin{bmatrix} -4 & 5 \\ 2 & 8 \end{bmatrix}\right]$

19. $If\, A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$ $show\ that\ A^2 - 5A + 7I = 0. Hence\ find\ A^{-1}.$

Solution:

Given

A = $\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$

$A^2 = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 9-1 & 3+2 \\ -3-2 & -1+4 \end{bmatrix}$

$= \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix}$

Now, $A^2 - 5A + 7I = \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix} - 5\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} + 7\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 8-15+7 & 5-5+0 \\ -5+5+0 & 3-10+7 \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

So, $A^2 - 5A + 7I = 0$

Multiply by A^{-1} both sides

$A.A\,A^{-1} - 5A.\,A^{-1} + 7I.\,A^{-1} = 0$

$A - 5I + 7A^{-1} = 0$

$A^{-1} = \frac{1}{7}[5I - A]$

$A^{-1} = \frac{1}{7}.\begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} - \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$

$A^{-1} = \frac{1}{7}.\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$

EXERCISE 7.2 PAGE NO: 7.34

Find the inverse of the following matrices by using elementary row transformations:

1. $\begin{bmatrix} 7 & 1 \\ 4 & -3 \end{bmatrix}$

Solution:

For row transformation we have

A = IA

$\Rightarrow \begin{bmatrix} 7 & 1 \\ 4 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A$

Applying $r_1 \to \frac{1}{7} r_1$

$\Rightarrow \begin{bmatrix} 1 & \frac{1}{7} \\ 4 & -3 \end{bmatrix} = \begin{bmatrix} \frac{1}{7} & 0 \\ 0 & 1 \end{bmatrix} A$

Applying $r_2 \to r_2 - 4r_1$

$\Rightarrow \begin{bmatrix} 1 & \frac{1}{7} \\ 0 & \frac{-25}{7} \end{bmatrix} = \begin{bmatrix} \frac{1}{7} & 0 \\ -\frac{4}{7} & 1 \end{bmatrix} A$

Applying $r_2 \to -\frac{7}{25} r_2$

$\Rightarrow \begin{bmatrix} 1 & \frac{1}{7} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{7} & 0 \\ \frac{4}{25} & -\frac{7}{25} \end{bmatrix} A$

Applying $r_1 \to r_1 - \frac{1}{7} r_2$

$\Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{21}{175} & \frac{1}{25} \\ \frac{4}{25} & -\frac{7}{25} \end{bmatrix} A$

So, as we know that

$I = A^{-1}A$

Therefore

$$\Rightarrow A^{-1} = \begin{bmatrix} \frac{21}{175} & \frac{1}{25} \\ \frac{4}{25} & -\frac{7}{25} \end{bmatrix}$$

2. $\begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}$

Solution:

For row transformation we have,

A = IA

$$\Rightarrow \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A$$

Applying $r_1 \to \frac{1}{5} r_1$

$$\Rightarrow \begin{bmatrix} 1 & \frac{2}{5} \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{5} & 0 \\ 0 & 1 \end{bmatrix} A$$

Applying $r_2 \to r_2 - 2r_1$

$$\Rightarrow \begin{bmatrix} 1 & \frac{2}{5} \\ 0 & \frac{1}{5} \end{bmatrix} = \begin{bmatrix} \frac{1}{5} & 0 \\ -\frac{2}{5} & 1 \end{bmatrix} A$$

Applying $r_2 \to 5r_2$

$$\Rightarrow \begin{bmatrix} 1 & \frac{2}{5} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{5} & 0 \\ -2 & 5 \end{bmatrix} A$$

Applying $r_1 \to r_1 - \frac{2}{5}r_2$

$\Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix} A$

So, as we know that

$I = A^{-1}A$

Therefore

$\Rightarrow A^{-1} = \begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix}$

3. $\begin{bmatrix} 1 & 6 \\ -3 & 5 \end{bmatrix}$

Solution:

For row transformation we have

A = IA

$\Rightarrow \begin{bmatrix} 1 & 6 \\ -3 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A$

Applying $r_2 \to r_2 + 3r_1$

$\Rightarrow \begin{bmatrix} 1 & 6 \\ 0 & 23 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} A$

Applying $r_2 \to \frac{1}{23} r_2$

$\Rightarrow \begin{bmatrix} 1 & 6 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{3}{23} & \frac{1}{23} \end{bmatrix} A$

Applying $r_1 \to r_1 - 6r_2$

$\Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{5}{23} & \frac{-6}{23} \\ \frac{3}{23} & \frac{1}{23} \end{bmatrix} A$

So, as we know that

$I = A^{-1}A$

Therefore

$\Rightarrow A^{-1} = \frac{1}{23}\begin{bmatrix} 5 & -6 \\ 3 & 1 \end{bmatrix}$

4. $\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$

Solution:

For elementary row operation we have

A = IA

$\Rightarrow \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A$

Applying $r_1 \to \frac{1}{2} r_1$

$\Rightarrow \begin{bmatrix} 1 & \frac{5}{2} \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} A$

Applying $r_2 \to r_2 - r_1$

$\Rightarrow \begin{bmatrix} 1 & \frac{5}{2} \\ 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ -\frac{1}{2} & 1 \end{bmatrix} A$

Applying $r_2 \to 2r_2$

$\Rightarrow \begin{bmatrix} 1 & \frac{5}{2} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ -1 & 2 \end{bmatrix} A$

Applying $r_1 \to r_1 - \frac{5}{2} r_2$

$\Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix} A$

So, as we know that

$I = A^{-1}A$

Therefore

$\Rightarrow A^{-1} = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$

5. $\begin{bmatrix} 3 & 10 \\ 2 & 7 \end{bmatrix}$

Solution:

For elementary row operation we have

$A = IA$

$\Rightarrow \begin{bmatrix} 3 & 10 \\ 2 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A$

Applying $r_1 \to \frac{1}{3} r_1$

$\Rightarrow \begin{bmatrix} 1 & \frac{10}{3} \\ 2 & 7 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{bmatrix} A$

Applying $r_2 \to r_2 - 2r_1$

$\Rightarrow \begin{bmatrix} 1 & \frac{10}{3} \\ 0 & \frac{1}{3} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & 0 \\ -\frac{2}{3} & 1 \end{bmatrix} A$

Applying $r_2 \to 3r_2$

$$\Rightarrow \begin{bmatrix} 1 & \frac{10}{3} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & 0 \\ -2 & 3 \end{bmatrix} A$$

Applying $r_1 \to r_1 - \frac{10}{3} r_2$

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & -10 \\ -2 & 3 \end{bmatrix} A$$

So, as we know that

$I = A^{-1}A$

Therefore

$$\Rightarrow A^{-1} = \begin{bmatrix} 7 & -10 \\ -2 & 3 \end{bmatrix}$$

6. $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$

Solution:

For elementary row operation we have,

A = IA

$$\Rightarrow \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Applying $r_1 \leftrightarrow r_2$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Applying $r_3 \to r_3 - 3r_1$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & -5 & -8 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & -3 & 1 \end{bmatrix} A$$

Applying $r_1 \to r_1 - 2r_2$ and $r_3 \to r_3 + 5r_2$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 0 \\ 5 & -3 & 1 \end{bmatrix} A$$

Applying $r_3 \to \frac{1}{2} r_3$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 0 \\ \frac{5}{2} & \frac{-3}{2} & \frac{1}{2} \end{bmatrix} A$$

Applying $r_1 \to r_1 + r_3$ and $r_2 \to r_2 - 2r_3$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -4 & 3 & 1 \\ \frac{5}{2} & -\frac{3}{2} & \frac{1}{2} \end{bmatrix} A$$

So, as we know that

$I = A^{-1}A$

Therefore

$$\Rightarrow A^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -4 & 3 & 1 \\ \frac{5}{2} & -\frac{3}{2} & \frac{1}{2} \end{bmatrix}$$

7. $\begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$

Solution:

For elementary row operation we have,

A = IA

$$\Rightarrow \begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Applying $r_1 \to \frac{1}{2} r_1$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Applying $r_2 \to r_2 - 5r_1$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & \frac{5}{2} \\ 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ -\frac{5}{2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Applying $r_3 \to r_3 - r_2$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & \frac{5}{2} \\ 0 & 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ -\frac{5}{2} & 1 & 0 \\ \frac{1}{2} & -1 & 1 \end{bmatrix} A$$

Applying $r_3 \to 2r_3$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & \frac{5}{2} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ -\frac{5}{2} & 1 & 0 \\ 5 & -2 & 2 \end{bmatrix} A$$

Applying $r_1 \to r_1 + \frac{1}{2} r_3$ and $r_2 \to r_2 - \frac{5}{2} r_3$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix} A$$

So, as we know that

$I = A^{-1}A$

Therefore

$$\Rightarrow A^{-1} = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$$

8. $\begin{bmatrix} 2 & 3 & 1 \\ 2 & 4 & 1 \\ 3 & 7 & 2 \end{bmatrix}$

Solution:

For row transformation we have

A = IA

$$\Rightarrow \begin{bmatrix} 2 & 3 & 1 \\ 2 & 4 & 1 \\ 3 & 7 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Applying $r_1 \to \frac{1}{2} r_1$

$$\Rightarrow \begin{bmatrix} 1 & \frac{3}{2} & \frac{1}{2} \\ 2 & 4 & 1 \\ 3 & 7 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Applying $r_2 \to r_2 - 2r_1 \; and \; r_3 \to r_3 - 3r_1$

$$\Rightarrow \begin{bmatrix} 1 & \frac{3}{2} & \frac{1}{2} \\ 0 & 1 & 0 \\ 0 & \frac{5}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ -1 & 1 & 0 \\ -\frac{3}{2} & 0 & 1 \end{bmatrix} A$$

Applying $r_1 \to r_1 - \frac{3}{2}r_2$ and $r_3 \to r_3 - \frac{5}{2}r_2$

$$\Rightarrow \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 2 & -\frac{3}{2} & 0 \\ -1 & 1 & 0 \\ 1 & -\frac{5}{2} & 1 \end{bmatrix} A$$

Applying $r_3 \to 2r_3$

$$\Rightarrow \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -\frac{3}{2} & 0 \\ -1 & 1 & 0 \\ 2 & -5 & 2 \end{bmatrix} A$$

Applying $r_1 \to r_1 - \frac{1}{2}r_3$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 \\ -1 & 1 & 0 \\ 2 & -5 & 2 \end{bmatrix} A$$

So, as we know that

$I = A^{-1}A$

Therefore

$$\Rightarrow A^{-1} = \begin{bmatrix} 1 & 1 & -1 \\ -1 & 1 & 0 \\ 2 & -5 & 2 \end{bmatrix}$$