Question 15: In a Poisson distribution if P (X = 2) = P (X = 3) then, the value of its parameter λ is:

(1) 3 (2) 0 (3) 6 (4) 2

Solution: (1)

Question 16: The surface area of the solid of revolution of the region bounded by $x^2 + y^2 = 4$, x = −2 and x = 2 about x−axis is:

(1) 64π (2) 32π (3) 8π (4) 16π

Solution: (4)

Question 17: If a + b + c = 0, |a| = 3, |b| = 4, |c| = 5 then, the angle between a and b is:

(1) 5π / 3 (2) π / 2 (3) π / 6 (4) 2π / 3

Solution: (2)

a + b + c = 0

|a| = 3

|b| = 4

|c| = 5

$\theta = \cos^{-1} [(a . b) / |a| |b|]$

a + b + c = 0

a + b = - c

$(a + b)^2 = (-c)^2$

$|a|^2 + |b|^2 + 2ab = |c|^2$

$3^2 + 4^2 + 2ab = 5^2$

9 + 16 + 2ab = 25

2ab = 0

ab = 0

$\theta = \cos^{-1} [0 / 3 . 4]$

$\theta = \cos^{-1} [0]$

$\theta = \pi / 2$

Question 18: The tangents at the end of any focal chord to the parabola $y^2 = 12x$ intersect on the line:

(1) $y + 3 = 0$ (2) $y - 3 = 0$ (3) $x - 3 = 0$ (4) $x + 3 = 0$

Solution: (4)

The tangent at the end of any focal chord to the parabola intersects on the directrix.

The equation of the parabola is $y^2 = 12x$

$y^2 = 4 (3) x$

=> $a = 3$

The equation of the directrix is

$x = -3$

$x + 3 = 0$

Question 19: If A is a scalar matrix with scalar $k \neq 0$, of order 3, then A^{-1} is:

(1) $(1 / k) I$ (2) kI (3) $[1 / k^2] I$ (4) $[1 / k^3] I$

Solution: (1)

Question 20: The surface area of a sphere when the volume is increasing at the same rate as its radius, is:

(1) 4π (2) $4\pi / 3$ (3) 1 (4) $1 / 2\pi$

Solution: (3)

Let S be the surface area, V be the volume, r be the radius of a sphere at time t

Given $dv / dt = dr / dt$

$V = (4 / 3) \pi r^3$

Surface area $S = 4\pi r^2$

$V = (4 / 3) \pi r^3$

$dV / dt = (4 / 3)\ 3\pi^2 (dr / dt)$
$dV / dt = 4\pi^2 (dV / dt)$
$1 = S$
$=> S = 1$
Surface area = 1

PART - II

Answer any seven questions. **[7 * 2 = 14]**

Question 21: To find the number of coins, in each category, write the suitable system of equations for the given situation:
"A bag contains 3 types of coins namely Re. 1, Rs. 2 and Rs. 5. There are 30 coins amounting to Rs. 100 in total."

Solution:
Let x, y and z be the number of coins respectively.
$x + y + z = 30$
$x + 2y + 5z = 100$

Question 22: If the two vectors 3i + 2j + 9k and i + mj + 3k are parallel, then prove that m = 2 / 3.

Solution:
$a = 3i + 2j + 9k$
$b = i + mj + 3k$
It is given that the two vectors are parallel.
$a = \lambda b$
$3i + 2j + 9k = \lambda (i + mj + 3k)$
$3i + 2j + 9k = \lambda i + \lambda mj + \lambda 3k$
Equating i, j, k vectors,
$3i = \lambda i$
$\lambda = 3$
$m\lambda = 2$
$m = 2 / 3$

Question 23: Find the least positive integer n such that $[(1 + i) / (1 - i)]^n = 1$.

Solution:

$[(1 + i) / (1 - i)]^n$

$1 = [(1 + i) / (1 - i) * (1 + i) / (1 + i)]^n$

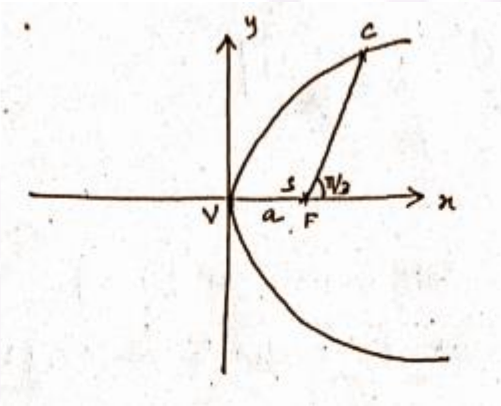
$1 = [(1 - 1 + 2i) / 2]$

$i^n = 1$

$n = 4$

Question 24: Draw the diagram for the given situation: "A comet is moving in a parabolic orbit around the sun which is at the focus of a parabola. When the comet is 80 million kms from the sun, the line segment from the sun to the comet makes an angle of π / 3 radians with the axis of the orbit."

Solution:



Question 25: Find the critical numbers of f (x) = sinx.

Solution:

f (x) = sinx

f ' (x) = cosx

f ' (x) = 0

cosx = 0

x = (2n + 1) **π** / 2, n ∈ Z

x = **π** / 2, 3**π** / 2, …..

Question 26: Write the domain and extent of the function f (x) = $x^3 + 1$.

Solution:

Domain: $(-\infty, \infty)$

Extent:

Vertical : $(-\infty, \infty)$

Horizontal : $(-\infty, \infty)$

Question 27: Prove that $\int_{\pi/6}^{\pi/3} dx / [1 + \sqrt{\cot x}] = \int_{\pi/6}^{\pi/3} dx / [1 + \sqrt{\tan x}]$.

Solution:

LHS = $\int_{\pi/6}^{\pi/3} dx / [1 + \sqrt{\cot x}]$

= $\int_{\pi/6}^{\pi/3} dx / [1 + \sqrt{\cot [(\pi / 6) + (\pi / 3) - x]}]$

= $\int_{\pi/6}^{\pi/3} dx / [1 + \sqrt{\tan x}]$

Question 28: Show that the set of all non-zero rational numbers is not closed under addition.

Solution:

Let G = Q - {0}

$\forall$ a, b $\in$ G $\Rightarrow$ a + b $\notin$ G

So, G is not closed under addition.

Question 29: Prove that F (3) = $1 - e^{-9}$ if the probability density function f (x) is defined as

$$f(x) = \begin{cases} 3e^{-3x}, & x > 0 \\ 0, & x \le 0 \end{cases}.$$

Solution:

F (x) = P (X ≤ x) = $\int_{-\infty}^{x} f(t)\, dt$

F (3) = $\int_0^3 3e^{-3t}\, dt$

$= 3 [e^{-3t} / -3]$
$= [- e^{-3t}]_0^3$
$= - [e^{-9} - 1]$
$= 1 - e^{-9}$

Question 30: Verify Rolle's theorem for the function f (x) = |x − 2| + |x − 5| in [1, 6].

Solution:

$f(x) = |x - 2| + |x - 5|$ in $[1, 6]$

f (x) is continuous on [1, 6]

f (x) is not differentiable on (1, 6)

Rolle's theorem is not satisfied.

PART - III

Answer any seven questions. **[7 * 3 = 21]**

Question 31: Prove that ρ (A) + ρ (B) ≠ ρ (A + B) by giving the suitable matrices A and B of order 3.

Solution:

$$A = \begin{pmatrix} 1 & -2 & 3 \\ -2 & 4 & -6 \\ 5 & 1 & -1 \end{pmatrix} \Rightarrow \rho(A) = 2$$

$$B = \begin{pmatrix} 1 & 1 & -1 \\ 2 & -3 & 4 \\ 3 & -2 & 3 \end{pmatrix} \Rightarrow \rho(B) = 2$$

$$\rho(A) + \rho(B) = 4 \rightarrow (1)$$

$$A + B = \begin{pmatrix} 2 & -1 & 2 \\ 0 & 1 & -2 \\ 8 & -1 & 2 \end{pmatrix}$$

$$\sim \begin{pmatrix} 2 & -1 & 2 \\ 0 & 1 & -2 \\ 0 & 3 & 10 \end{pmatrix} R_3 \to R_3 - 4R_1$$

$$\sim \begin{pmatrix} 2 & -1 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 16 \end{pmatrix} R_3 \to R_3 - 3R_2$$

$$\rho(A+B) = 3 \rightarrow (2)$$

From (1) & (2). $\rho(A+B) \neq \rho(A) + \rho(B)$

Question 32: Find the vectors of magnitude 6 which are perpendicular to both the vectors 4i - j + 3k and - 2i + j - 2k.

Solution:

a = 4i - j + 3k

b = - 2i + j - 2k

Required vector = ± **μ** (a x b) / |a x b|

$a \times b = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 4 & -1 & 3 \\ -2 & 1 & -2 \end{vmatrix}$

$= -i + 2j + 2k$

$|a \times b| = \sqrt{1^2 + 2^2 + 2^2} = \sqrt{9} = 3$

Vector = ± **μ** (a x b) / |a x b|

$= \pm(-2i + 4j + 4k)$

Question 33: If n is a positive integer, prove that $[(1 + \sin\theta - i\cos\theta) / (1 + \sin\theta + i\cos\theta)]^n = \cos n(\pi / 2 - \theta) - i \sin n(\pi / 2 - \theta)$.

Solution:

Let $Z = \sin\theta - i\cos\theta$

$|Z| = 1$

$(1 / Z) = (\bar{Z})$

$(1 / Z) = \sin\theta + i\cos\theta$

LHS $= [(1 + \sin\theta - i\cos\theta) / (1 + \sin\theta + i\cos\theta)]^n$

$= [(1 + Z) / (1 + (1 / Z))]^n$

$= [((1 + Z) / (Z + 1) / Z)]^n$

$= Z^n$

$= (\sin\theta - i\cos\theta)^n$

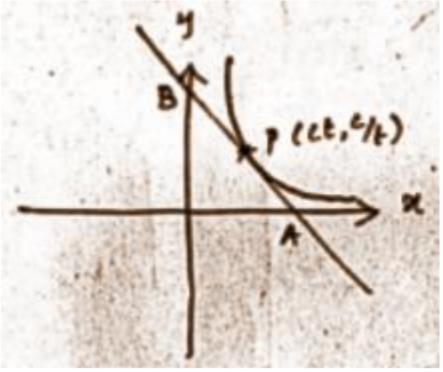
$= [\cos(\pi / 2 - \theta) - i\sin(\pi / 2 - \theta)]^n$

$= \cos n(\pi / 2 - \theta) - i\sin n(\pi / 2 - \theta)$

= RHS

Question 34: Show that the tangent to a rectangular hyperbola terminated by its asymptotes is bisected at the point of contact.

Solution:



Equation of the tangent is $x + yt^2 = 2ct$

Put $x = 0$, $y = 2c / t$, B $(0, 2c / t)$

Put $x = 0$, $x = 2ct$, A $(2ct, 0)$

Midpoint of AB = $(2ct / 2), (2c / t / 2)$

= $(ct, c / t)$

$(ct, c / t)$ is the point of contact.

Question 35: Show that the function $f(x) = \tan^{-1}(\sin x + \cos x)$, $x > 0$ is strictly increasing in the interval $(0, \pi / 4)$.

Solution:

$f(x) = \tan^{-1}(\sin x + \cos x)$

$f'(x) = \{1 / [1 + (\sin x + \cos x)^2]\} * (\cos x - \sin x)$

$= (\cos x - \sin x) / [1 + \sin^2 x + \cos^2 x + 2 \sin x \cos x]$

$= (\cos x - \sin x) / [2 + 2 \sin x \cos x]$

$= (\cos x - \sin x) / [2 + \sin 2x] \ \forall\ x \in (0, \pi / 4)$

$f'(x) > 0$

$f(x)$ is strictly increasing in the interval $(0, \pi / 4)$.

Question 36: If $f(x, y) = 1 / \sqrt{x^2 + y^2}$ then, prove that $x\, \partial f / \partial x + y\, \partial f / \partial y = - f$.

Solution:

$f(x, y) = 1 / \sqrt{x^2 + y^2}$

$f(tx, ty) = 1 / \sqrt{t^2x^2 + t^2y^2}$

$= 1 / t \sqrt{x^2 + y^2}$

$= t^{-1} f(x, y)$

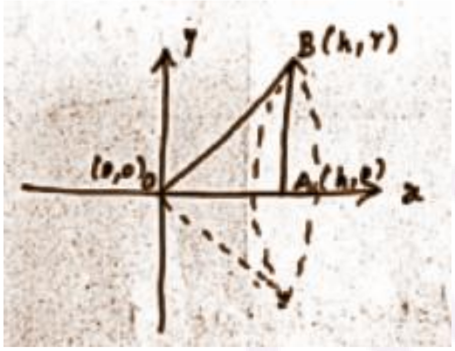
f is a homogeneous function of degree 1.

By Euler's theorem,

$x\, \partial f / \partial x + y\, \partial f / \partial y = - f$

Question 37: Derive the formula for the volume of a cylinder with radius 'r' and height 'h' by using integration.

Solution:



Consider a triangle AOB with vertices O (0, 0), A (h, 0), B (h, r).

Equation of OB, $y = (r / h) x$

$V = \pi \int_0^h y^2\, dx$

$= \pi \int_0^h (r^2 / h^2) x^2$

$= \pi r^2 / h^2 [x^3 / 3]_0^h$

$= \pi r^2 / h^2 [h^3 / 3]$

$= (1 / 3) \pi r^2 h$ cubic units

Question 38: Show that $(p \wedge q) \rightarrow (p \vee q)$ is a tautology.

Solution:

p	q	$p \wedge q$	$p \vee q$	$(p \wedge q) \rightarrow (p \vee q)$
T	T	T	T	T
T	F	F	T	T

F	T	F	T	T
F	F	F	F	T

It is a tautology.

Question 39: A die is thrown 120 times and getting 1 or 5 is considered a success. Find the mean and variance of the number of successes.

Solution:

n = 120
p = 2 / 6 = 1 / 3
q = 1 - p = 1 - (1 / 3) = 2 / 3
Mean = np
= 120 * (1 / 3)
= 40
Variance = npq
= (120) * (1 / 3) * (2 / 3)
= 80 / 3

Question 40: Show that the solution of the differential equation yx^3 dx + e^{-x} dy = 0 is (x^3 - $3x^2$ + 6x - 6) e^x + log y = c.

Solution:

yx^3 dx + e^{-x} dy = 0
e^{-x} dy = - yx^3 dx
dy / y = - x^3 e^x dx
log y = - [x^3 e^x - 3 x^2 e^x + 6x e^x - 6e^x] + c
log y + e^x [x^3 - 3 x^2 + 6x - e^x] = c

PART - IV

Answer all the questions. **[7 * 5 = 35]**

Question 41: (a) For what values of μ the system of homogeneous equations x + y + 3z = 0; 4x + 3y + μz = 0; 2x + y + 2z = 0 have:
(i) only trivial solution
(ii) infinitely many solutions

OR

(b) Prove by vector method that sin (A + B) = sinA cosB + cosA sinB.

Solution:
[a] The system of equations can be written as AX = B.

$$\begin{bmatrix}1 & 1 & 3\\ 4 & 3 & \mu\\ 2 & 1 & 2\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$$[A, B] = \begin{bmatrix}1 & 1 & 3 & 0\\ 4 & 3 & \mu & 0\\ 2 & 1 & 2 & 0\end{bmatrix} \sim \begin{bmatrix}1 & 1 & 3 & 0\\ 0 & -1 & \mu-12 & 0\\ 0 & -1 & -4 & 0\end{bmatrix} \begin{matrix}R_2 \to R_2 - 4R_1\\ R_3 \to R_3 - 2R_1\end{matrix}$$

$$\sim \begin{bmatrix}1 & 1 & 3 & 0\\ 0 & -1 & \mu-12 & 0\\ 0 & 0 & 8-\mu & 0\end{bmatrix} R_3 \to R_3 - R_2$$

Case [i] Let $\mu \neq 8$
ρ [A, B] = 3 and ρ [A] = 3
The given system is consistent and has a trivial solution.
x = 0, y = 0, z = 0
Case [ii]: Let $\mu = 8$
ρ [A, B] = 2 and ρ [A] = 2 < 3 [is equal to number of unknowns]
The corresponding equations are x + y + 3z = 0; y + 4z = 0
Take z = k, y = - 4k and x = k
The solutions set is (x, y, z) = (k, - 4k, k), which are non-trivial.
The given system is consistent and has infinitely many non-trivial solutions.

OR

[b] Let α and β be two unit vectors, and A and B be the angles made by them respectively with the x-axis.

$\alpha = \cos A i + \sin A j$ and $\beta = \cos B i + \sin B j$

Now, $\alpha . \beta = (\cos A i + \sin A j)(\cos B i + \sin B j)$

$\Rightarrow \alpha\beta \cos(A - B) = \cos A \cos B + \sin A \sin B$

$\Rightarrow \cos(A - B) = \cos A \cos B + \sin A \sin B \quad [\because \alpha = 1, \beta = 1]$ ------------ (1)

Putting -B in place of B in (1):-

$\cos(A - (-B)) = \cos A \cos(-B) + \sin A \sin(-B)$

$\Rightarrow \cos(A + B) = \cos A \cos B - \sin A \sin B$

Similarly,

$\alpha \times \beta = (\cos A i + \sin A j) \times (\cos B i + \sin B j)$

$\Rightarrow \alpha \times \beta = \cos A \sin B k - \sin A \cos B k$

$\Rightarrow \alpha\beta \sin(A - B)(-k) = (\sin A \cos B - \cos A \sin B)(-k)$

$\Rightarrow \sin(A - B) = \sin A \cos B - \cos A \sin B$ ------------ (2)

Putting B = - B in (2),

$\sin(A - (-B)) = \sin A \cos(-B) - \cos A \sin(-B)$

$\Rightarrow \sin(A + B) = \sin A \cos B + \cos A \sin B$

Question 42: [a] Find the cartesian equation of the plane containing the line $x - 2 / 2 = y - 2 / 3 = z - 1 / -2$ and passing through the point $(-1, 1, -1)$.

OR

(b) Solve : $x^{11} - x^6 + x^5 - 1 = 0$.

Solution:

[a] $(x_1, y_1, z_1) = (-1, 1, -1)$

$(x_2, y_2, z_2) = (2, 2, 1)$

$(l_1, m_1, n_1) = (2, 3, -2)$

The equation of the plane is

$$\begin{vmatrix} x+1 & y-1 & z+1 \\ 3 & 1 & 2 \\ 2 & 3 & -2 \end{vmatrix} = 0$$

$0 = (x + 1)(-2 - 6) - (y - 1)(-6 - 4) + (z + 1)(9 - 2)$

$0 = (x + 1)(-8) - (y - 1)(-10) + (z + 1)(7)$

$0 = - 8x - 8 + 10y - 10 + 7z + 7$
$8x + 8 - 10y + 10 - 7z - 7 = 0$
$8x - 10y - 7z + 11 = 0$

OR

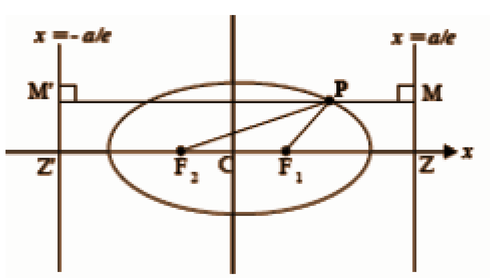
[b] $x^{11} - x^6 + x^5 - 1 = 0$
$x^6 [x^5 - 1] [x^5 - 1] = 0$
$x = (-1)^{1/6} = (\text{cis } \pi)^{1/6}$
$= (\text{cis } (2k\pi + \pi))^{1/6}$ $k = 0, 1, 2, 3, 4, 5$
$x = \text{cis } \pi / 6, \text{cis } 3\pi / 6, \text{cis } 5\pi / 6, \text{cis } 7\pi / 6, \text{cis } 9\pi / 6, \text{cis } 11\pi / 6$
$x = (\text{cis } 0)^{1/5}$
$= (\text{cis } (2k\pi))^{1/5}$
$= \text{cis } 2k\pi / 5$, $k = 0, 1, 2, 3, 4, 5$
$x = 0, \text{cis } 2\pi / 5, \text{cis } 4\pi / 5, \text{cis } 6\pi / 5, \text{cis } 8\pi / 5.$

Question 43: [a] Show that the sum of the focal distances of any point on an ellipse is equal to the length of the major axis and also prove that the locus of a point which moves so that the sum of its distances from (3, 0) and (−3, 0) is 9, is $x^2 / (81 / 4) + y^2 / (45 / 4) = 1$.

OR

(b) Prove that the area of the largest rectangle that can be inscribed in a circle of radius ‘r’ is $2r^2$.

Solution:
[a]



$F_1P + F_2P = 2a$

$x = a / e, x = - a / e$
$F_1P / PM = e$
$F_2P / PM = e$
$F_1P = e\ PM$
$F_2P = e\ PM$
$F_1P + F_2P = e\ (PM + PM)$
$= e\ (MM')$
$= e\ (2a / e)$
$= 2a$
= length of major axis
$2a = 9$
$a = 9 / 2$
$ae = 3$
$b^2 = a^2 - (ae)^2$
$= (81 / 4) - 9$
$= 45 / 4$
The equation is $x^2 / (81 / 4) + y^2 / (45 / 4) = 1$.

OR

[b] $x = 2r \cos \theta$
$y = 2r \sin \theta$
Area of rectangle = 2x.2y
$A (\theta) = 2r \cos\theta \, . \, 2r \sin\theta$
$d (A(\theta)) / d\theta = 4r^2 \cos 2\theta$
$d (A(\theta)) / d\theta = 0$ then $\theta = \pi / 4$
$A'' (\theta) = - 8r^2 \sin 2\theta < 0$ for $\theta = \pi / 4$
So A is maximum for $\theta = \pi / 4$
$X = y = \sqrt{2}r$,
Required area = x.y
$= 2r. 2r$
$= 2r^2$

Question 44: [a] A missile fired from ground level rises x metres vertically upwards in t seconds and $x = 100t - (25 / 2)t^2$.
Find :
(i) the initial velocity of the missile
(ii) the time when the height of the missile is a maximum
(iii) the maximum height reached
(iv) the velocity with which the missile strikes the ground

OR

(b) Find the centre, foci and vertices of the hyperbola $16x^2 - 9y^2 - 32x - 18y + 151 = 0$ and draw the diagram.

Solution:

[a] $x = 100t - (25 / 2)t^2$

(i) To find the initial velocity of the missile.
So, $t = 0$
$v = dx / dt = 100 – (25 / 2) 2t$
$v = 100 - 0$
$=100$

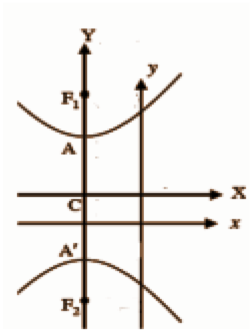
(ii) The time when the height of the missile is a maximum is given by $v = 0$.
$100 - 25t = 0$
$t = 4$ seconds

(iii) The maximum height reached when $t = 4$ seconds
$x = 100 (4) - (25 / 2)16x$
$= 200$ meters

(iv) velocity if $t = 4 + 4 = 8$ seconds
$v = 100 – 25(8)$
$= -100$ m/sec

[b] equation $(y + 1)^2 / 16 - (x - 1)^2 / 9 = 1$
$a^2 = 16$

$b^2 = 9$
e = 5 / 4
Centre: (1, - 1)
Foci: (1, 4), (1, - 6)
Vertices : (1, 3), (1, - 4)



Question 45: [a] The mean score of 1000 students for an examination is 34 and the standard deviation is 16. Determine the limit of the marks of the central 70% of the candidates by assuming the distribution is normal. P [0 < Z < 1.04] = 0.35

OR

(b) Compute the area between the curve y = sinx and y = cosx and the lines x = 0 and x = π.

Solution:
[a] a) P $(Z_1 < Z < Z_2)$ = 70%
P $(Z_1 < Z < Z_2) = 0.70$
$P(Z_1 < Z < 0) = 0.35$
P $(0 < Z < Z_2) = 0.35$
Z_1 and Z_2 lie on the left and right side of the normal curve.
$Z_1 = -1.04$ and $Z_2 = 1.04$
$Z = [x - \mu] / \sigma$
$= [x - 34] / 16$
= - 1.04
X = 17.36
Z = x − 34 / 16 = 1.04
x = 50.64

70% of the students' score lies between 17.36 and 50.64.

[b] y = sinx
y = cosx
Then sinx = cosx
→ $x = \pi / 4 \in (0, \pi)$
Required area = $\int_0^{\pi/4} (\cos x - \sin x)\, dx + \int_{\pi/4}^{\pi} (\sin x - \cos x)\, dx$
= $2\sqrt{2}$ square units

Question 46: [a] If $w = x + 2y + z^2$ and x = cos t; y = sint; z = t find dw / dt by using chain rule. Also find dw / dt by substitution of x, y and z in w and hence verify the result.

OR

(b) A cup of tea at temperature 100°C is placed in a room whose temperature is 15°C and it cools to 60°C in 5 minutes. Find its temperature after a further interval of 5 minutes.

Solution:
[a] $dw / dt = (\partial w / \partial x)(dx / dt) + (\partial w / \partial y)(dy / dt) + (\partial w / \partial z)(dz / dt)$
= - sint + 2 cost + 2t …. (1)
$w = \cos t + 2 \sin t + t^2$
dw / dt = - sint + 2 cost + 2t …. (2)

b) dT / dt = k (T - s) , s = 15
$T = s + ce^{kt}$……. (1)
t = 0 then T = 100
(1) => c = 85
t = 5, T = 60 then $e^{5k} = 45 / 85$
t =10 , then T= ?
$T = 15 + 83\, e^{10k} = 38.82$

Question 47: [a] State all the five properties of groups.

OR

(b) Prove that the solution of the differential equation : $(5D^2 - 8D - 4)y = 5e^{(-\frac{2}{5}x)} + 2e^x + 3$ is $y = Ae^{2x} + Be^{(-\frac{2}{5}x)} - (5 / 12)\, xe^{(-\frac{2}{5}x)} - (2 / 7)\, e^x - (3 / 4)$.

Solution:

[a]

(i) Identity element of group is unique

(ii) Inverse of each element of group is unique

(iii) Reversal law

(iv) Cancellation law

(v) $[a^{-1}]^{-1} = a$

b) C.F = $Ae^{2x} + Be^{(-2/5)x}$

$P.I_1 = (-5 / 12)\, xe^{(-2/5)x}$

$P.I_2 = (-2 / 7)\, e^x$

$P.I_3 = -3 / 4$

General solution $y = C.F + P.I_1 + P.I_2 + P.I_3$

$= Ae^{2x} + Be^{(-2/5)x} + (-5 / 12)\, xe^{(-2/5)x} + (-2 / 7)\, e^x (-3 / 4)$