KVPY-2018 (Mathematics)-Stream (SA)

PART-I

1. The number of pairs (a, b) of positive real numbers satisfying $a^4 + b^4 < 1$ and $a^2 + b^2 > 1$ is

   a. 0   b. 1

   c. 2   d. more than 2

2. The number of real roots of the polynomial equation $x^4 - x^2 + 2x - 1 = 0$ is

   a. 0   b. 2

   c. 3   d. 4

3. Suppose the sum of the first m terms of arithmetic progression is n and the sum of its first n terms is m, where $m \neq n$. Then the sum of the first (m + n) terms of the arithmetic progression is

   a. 1–mm   b. mn – 5

   c. –(m + n)   d. m + n

4. Consider the following two statements :

   I. Any pair of consistent linear equations in two variables must have a unique solution

   II. There do not exist two consecutive integers, the sum of whose squares is 365. Then

   a. both I and II are true   b. both I and II are false

   c. I is true and II is   d. I is false and II is true

5. The number of polynomials p(x) with integer coefficients such that the curve y = p(x) passes through (2,2) and (4, 5) is

   a. 0   b. 1

   c. More than 1 but finite   d. infinite

6. The median of all 4-digit number that are divisible by 7 is

   a. 5497
   b. 5498.5
   c. 5499.5
   d. 5490

7. A solid hemisphere is attached to the top of a cylinder, having the same radius as that of the cylinder. If the height of the cylinder were doubled (keeping both radii fixed), the volume of the entire system would have increased by 50%. By what percentage would the volume have increased if the radii of the hemisphere and the cylinder were doubled (keeping the height fixed)?

   a. 300%
   b. 400%
   c. 500%
   d. 600%

8. Consider a triangle PQR in which the relation $QR^2 + PR^2 = 5\ PQ^2$ hold. Let G be the point of intersection of medians PM and QN. Then ∠ QGM is always

   a. Less than 45°
   b. obtuse
   c. A right angle
   d. Acute and larger than 45°

9. Let a,b,c be the side-lengths of a triangle, and l, m, n be the lengths of its medians. Let $K = \frac{l+m+n}{a+b+c}$. Then as a, b, c vary, K can assume every value in the interval

   a. $\left(\frac{1}{4}, \frac{2}{3}\right)$
   b. $\left(\frac{1}{2}, \frac{4}{5}\right)$
   c. $\left(\frac{3}{4}, 1\right)$
   d. $\left(\frac{4}{5}, \frac{5}{4}\right)$

10. Let x_0, y_0 be fixed real numbers such that $x_0^2 + y_0^2 > 1$. If x,y are arbitrary real numbers such that $x^2 + y^2 \le 1$, then the minimum value of $(x - x_0)^2 + (y - y_0)^2$ is

   a. $\left(\sqrt{x_0^2 + y_0^2} - 1\right)^2$
   b. $x_0^2 + y_0^2 - 1$
   c. $\left(|x_0^2| + |y_0^2| - 1\right)^2$
   d. $\left(|x_0| + |y_0|\right)^2 - 1$

11. Let PQR be a triangle in which PQ = 3. From the vertex R, draw the altitude RS to meet PQ at S. Assume that RS = $\sqrt{3}$ and PS = QR. Then PR equals

a. $\sqrt{5}$
b. $\sqrt{6}$
c. $\sqrt{7}$
d. $\sqrt{8}$

12. A 100 mark examination was administered to a class of 50 students. Despite only integer marks being given, the average score of the class was 47.5. Then, the maximum number of students who would get marks more than the class average is

a. 25
b. 35
c. 45
d. 49

13. Let s be the sum of the digits of the number $15^2 \times 5^{18}$ in base 10. Then

a. $s < 6$
b. $6 \le s < 140$
c. $140 \le s < 148$
d. $S \ge 148$

14. Let PQR be an acute-angled triangle in which PQ < QR. From the vertex Q draw the altitude Q Q_1, the angle bisector Q Q_2 and the medium Q Q_3 with Q_1, Q_2, Q_3 lying on PR. Then.

a. $PQ_1 < PQ_2 < PQ_3$
b. $PQ_2 < PQ_1 < PQ_3$
c. $PQ_1 < PQ_3 < PQ_2$
d. $PQ_3 < PQ_1 < PQ_2$

15. All the vertices of rectangle are of the form (a, b) with a, b integers satisfying the equation $(a - 8)^2 - (b - 7)^2 = 5$. Then the perimeter of the rectangle is

a. 20
b. 22
c. 24
d. 26

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PART-II

16. What is the sum of all natural numbers n such that the product of the digits of n (in base 10) is equal to $n^2 - 10n - 36$?

a. 12
b. 13
c. 124
d. 2612

17. Let m (respectively, n) be the number of 5-digit integers obtained by using the digits 1,2,3,4,5 with repetitions (respectively, without repetitions) such that the sum of any two adjacent digits is odd. Then $\frac{m}{n}$ is equal to

a. 9
b. 12
c. 15
d. 18

18. The number of solid cones with integer radius and height each having its volume numerically equal to its total surface area is

a. 0
b. 1
c. 2
d. infinite

19. Let ABCD be a square. An arc of a circle with A as center and AB as radius is drawn inside the square joining the points B and D. Points P on AB, S on AD, Q and R on arc BD are taken such that PQRS is a square. Further suppose that PQ and RS are parallel to AC. Then $\frac{\text{area PQRS}}{\text{area ABCD}}$ is

a. $\frac{1}{8}$
b. $\frac{1}{5}$
c. $\frac{1}{4}$
d. $\frac{2}{5}$

20. Suppose ABCD is a trapezium whose sides and height are integers and AB is parallel to CD. If the area of ABCD is 12 and the sides are distinct, then |AB – CD|

a. is 2
b. is 4
c. is 8
d. cannot be determined from the data

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ANSWER KEYS

1. (d)	2. (b)	3. (c)	4. (b)	5. (a)	6. (b)	7. (c)	8. (c)	9. (c)	10. (a)
11. (c)	12. (d)	13. (b)	14. (a)	15. (a)	16. (b)	17. (c)	18. (b)	19. (d)	20. (b)

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Solution

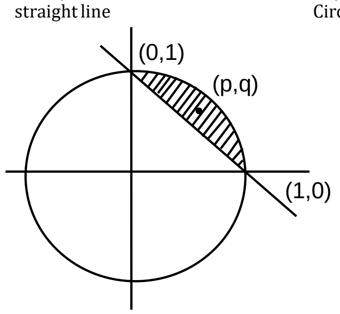
PART-I

1. (d)

Let $a^2 = p$ and $b^2 = q$ here $p > 0$ & $q > 0$

Now, given inequation are,

$\underbrace{p+q>1}_{\text{straight line}}$ and $\underbrace{p^2+q^2<1}_{\text{Circle}}$



Point (p, q) lies above $x + y = 1$ and (p, q) lies inside circle $x^2 + y^2 = 1$ {using concept of position of point}

From common shaded region we can say (p, q) lies in shaded region and number of points in shaded region are infinite. So number of pairs (a, b) are also infinite.

2. (b)

$\Rightarrow x^4 - x^2 + 2x - 1 = 0$ (given)

$\Rightarrow x^4 - [x^2 - 2x + 1] = 0$ $\quad \because A^2 - 2AB + B^2 = (A - B)^2$; $A = x$, $B = 1$

$\Rightarrow x^4 - (x - 1)^2 = 0$

$\Rightarrow (x^2 - x + 1)(x^2 + x - 1) = 0$ $\quad \because a^2 - b^2 = (a - b)(a + b)$; $a = x^2$, $b = x - 1$

$\Rightarrow \underbrace{x^2-x^2+1=0}_{D<0}$ or $\underbrace{x^2+x-1=0}_{D>0}$

no. solution

Here $D = b^2 - 4ac$

$= (-1)^2 - 4 .1 . 1 = -3 < 0$

$\therefore D < 0$

D = Discriminant

b = coefficient of x

a = leading coefficient

c = constant term

two roots

here $D = b^2 - 4ac$

$= (1^2) - 4. (1) (-1) = 5 > 0$

$\therefore D > 0$

3. (c)

$S_m = \frac{m}{2}(2a+(m-1)d) = n$............(1) $\quad \because S_n = \frac{n}{2}(2a+(n-1)d)$

S_n = sum of n terms; n = number of terms

a = first term; d = common difference

$S_n = \frac{n}{2}(2a+(n-1)d) = m$............(2)

From equation(1) – (2)

$= \frac{1}{2}[2a(m-n) + d(m^2 - m - n^2 + n) = n - m$

$= \frac{1}{2}[2a(m-n) + d\,(m-n)(m+n-1)] = n - m$

$= \frac{1}{2}[2a + d\,(m + n - 1)] = -1$

Multiplying both sides by m + n

We get,

$= \frac{m+n}{2}[2a + (m + n - 1)d] = -(m + n)$

$= S_{m+n} = -(m+n)$

4. (b)

(I) clearly statement 1 is false because they can have infinite solution (lines can be coincident)

(II) Let two consecutive integers α, $\alpha + 1$

$\Rightarrow \quad \alpha^2 + (\alpha + 1)^2 = 365 \qquad \because (A + B)^2 = A^2 + 2AB + B^2$; $A = \alpha, B = 1$

$\Rightarrow \quad 2\alpha^2 + 2\alpha - 364 = 0$

Using shreedharacharya formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$\Rightarrow \quad \alpha = \frac{-2 \pm \sqrt{4 - 4 \times 2(-364)}}{2.2}$

$\Rightarrow \quad \alpha = \frac{-2 \pm 54}{4}$

On taking positive sign $\alpha = \frac{-2+54}{4} = 13$

On taking negative sign $\alpha = \frac{-2-54}{4} = -14$

For $\alpha = -14$, consecutive integers are –14, –13

F $\alpha = 13$, consecutive integers are 13, 14

Hence statement (II) is false

5. (a)

$y = a_n x^n + a_{n-1}x^{n-1} + \ldots\ldots\ldots + a_0 \qquad \ldots\ldots\ldots(1)$

Put x = 2 & y = 2 in equation (1)

$\Rightarrow \quad 2 = a_n 2^n + a_{n-1}2^{n-1} + \ldots\ldots\ldots + a_0 \qquad \ldots\ldots\ldots(2)$

$\Rightarrow$ Put x = 4 & y = 5 in equation (1)

$\Rightarrow \quad 5 = a_n 4^n + a_{n-1} 4^{n-1} + \ldots\ldots\ldots + a_n \qquad \ldots\ldots\ldots(3)$

$\Rightarrow$ From equation (3) – (2)

$\Rightarrow \quad 3 = a_n(4^n - 2^n) + a_{n-1}(4^{n-1} - 2^{n-1}) + \ldots\ldots\ldots + 2a_1$

$\Rightarrow$ Odd = even

$\Rightarrow$ Not possible

$\Rightarrow$ No such polynomial exists

6. (b)

Four digit numbers which are divisible by 7 are 1001, 1008 …………..9996

no. of terms

$\Rightarrow 9996 = 1001 + (n - 1)7$ $\quad \because (T_n = a+(n-1)d)$

$\Rightarrow n = 1286$ (even) $\quad T_n = n^{th}$ term; a = first term; d = common difference; n = number of terms

$$\Rightarrow \text{median} = \frac{\left(\frac{n}{2}\right)^{th}\text{term} + \left(\frac{n}{2}+1\right)^{th}\text{term}}{2}$$

$$= \frac{(643)^{th}\text{term} + (644)^{th}\text{term}}{2}$$

$$= \frac{(1001+642\times7)+(1001+643\times7)}{2}$$

$$= \frac{2002+1285\times7}{2}$$

$$= \frac{2002+8995}{2} = \frac{10997}{2} = 5498.5$$

7. (c)

$\because$ Volume of cylinder = $\pi r^2 h$; volume of hemisphere = $\frac{2}{3}\pi r^3$

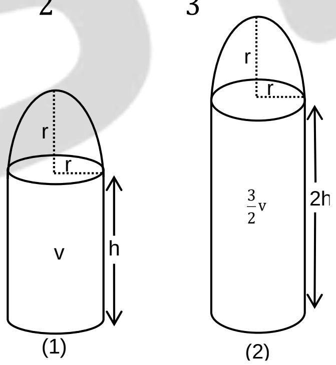
$v_1 = v = \pi r^2 h + \frac{2}{3}\pi r^3$ …….(1)

v_2 = Volume of the entire system increased by 50%.

$v_2 = \frac{3}{2}v = \pi r^2(2h) + \frac{2}{3}\pi r^3$ ……(2)

From eq. (2) – (1)

$\Rightarrow \frac{v}{2} = \pi r^2 h = \frac{2}{3}\pi r^3$ (from eq. (1))……(3)



Let v′ = increased volume when radii of hemisphere and cylinder are doubled,

Now,

$v' = \pi(2r)^2 h + \frac{2}{3}\pi(2r)^3$

$v' = 4\pi r^2 h + \frac{16}{3}\pi r^3$

$v' = 4 \times \frac{v}{2} + 8 \times \frac{v}{2}$ (from eq. (3))

$v' = 2v + 4v$

$v' = 6v$

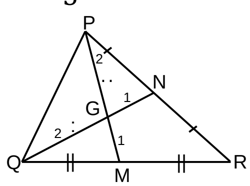
$\% \text{ change} = \frac{\Delta v}{v} \times 100 = \frac{v_{final} - v_{initial}}{v_{initial}} \times 100 = \frac{6v - v}{v} \times 100 = 500\%$

8. (c)

Let $PQ = r, QR = p, \ RP = q$

$GM = \frac{1}{3} PM$

$QG = \frac{2}{3} QN$



Now,

$QG^2 + GM^2 = \frac{4}{9}(QN)^2 + \frac{1}{9}(PM)^2$

$= \frac{4(QN)^2 + (PM)^2}{9}$ (using formula of length of median $\ell = \frac{1}{2}\sqrt{2b^2 + 2c^2 - a^2}$)

$\{QN = \frac{1}{2}\sqrt{2r^2 + 2p^2 - q^2}\ ; PM = \frac{1}{2}\sqrt{2r^2 + 2q^2 - p^2}\ \}$

$= \frac{1}{9}\left[4 \cdot \frac{1}{4}(2r^2 + 2p^2 - q^2) + \frac{1}{4}(2r^2 + 2q^2 - p^2)\right]$

$= \frac{1}{9}\left[2r^2 + 2p^2 - q^2 + \frac{1}{4}(2r^2 + 2q^2 - p^2)\right]$

$= \frac{1}{36}\left[8r^2 + 8p^2 - 4q^2 + 2r^2 + 2q^2 - p^2\right]$

$= \frac{1}{36}\left[10r^2 + 7p^2 - 2q^2\right]$

$= \frac{1}{36}\left[2p^2 + 2q^2 + 7p^2 - 2q^2\right]$ $\because \begin{Bmatrix} QR^2 + PR^2 = 5PQ^2 \text{ (given)} \\ P^2 + q^2 = 5r^2 \end{Bmatrix}$

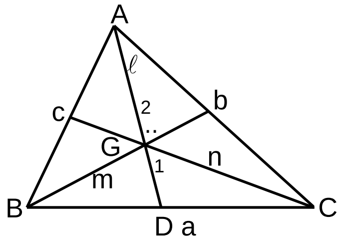
$= \frac{p^2}{4} = (QM)^2$

$\Rightarrow$ Hence angle QGM = 90° (using property of right angle Δ)

9. (c)

AD is median = $AD < \frac{AB + AC}{2}$

$\Rightarrow \ell < \frac{b+c}{2}$



Similarly $m < \frac{c+a}{2}$ and $n < \frac{a+b}{2}$

$\Rightarrow \ell + m + n < a + b + c$

$\Rightarrow \frac{\ell + m + n}{a+b+c} < 1$ ....(1)

Also in the ΔBGC

BG + GC> BC

$\Rightarrow \frac{2}{3}(m+n) > a$

Similarly $\frac{2}{3}(n+\ell) > b$ and $\frac{2}{3}(\ell + m) > c$

Hence, $\frac{4}{3}(\ell + m + n) > a + b + c$

$\Rightarrow \frac{\ell + m + n}{a+b+c} > \frac{3}{4}$ ....(2)

From eq. (1) & (2)

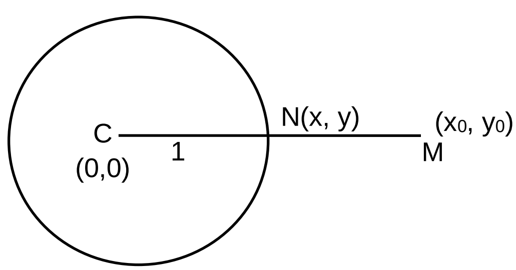
$\frac{\ell + m + n}{a+b+c} \in \left(\frac{3}{4}, 1\right)$

10. (a)

Let circle $x^2 + y^2 = 1$;

C : (0, 0) & r = 1

$\Rightarrow x_0^2 + y_0^2 > 1$ (given)



$\therefore$ point (x_0y_0) lies outside the circle $x^2 + y^2 = 1$

{using concept of position of point with respect to circle}

And $x^2 + y^2 \le 1$ (given)

$\therefore$ point (x, y) lies inside or on the circle

{Using concept of position of point with respect to circle}

Now,

$(x-x_0)^2 + (y-y_0)^2$ will be minimum when the variable point (x, y) will be on circle

$$\min\left\{(x-x_0)^2 + (y-y_0)^2\right\} = (CM-CN)^2 = \left(\sqrt{x_0^2+y_0^2}-1\right)^2$$

{use $d = \sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$ }

11. (c)

In Δ SRQ

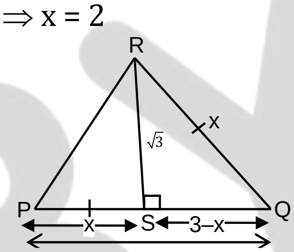
Using Pythagoras theorem $(RQ)^2 = (RS)^2 + (SQ)^2$

$\Rightarrow x^2 = \left(\sqrt{3}\right)^2 + (3-x)^2$

$\Rightarrow x^2 = 3 + 9 + x^2 - 6x$

$\Rightarrow 6x = 12$

$\Rightarrow x = 2$



Now,

In ΔPSR

Using Pythagoras theorem $(PR)^2 = (PS)^2 + (RS)^2$

$\Rightarrow x^2 + \left(\sqrt{3}\right)^2 = (PR)^2$ $\quad\because x = 2$

$\Rightarrow 4 + 3 = (PR)^2$

$\Rightarrow PR = \sqrt{7}$

12. (d)

Total students in class = 50

Average score of class = 47.5

Total score of class = 50 × 47.5 = 2375

Student can obtain only integer marks. If 48 marks are obtained by student then

maximum number of students can be = $\left[\frac{2375}{48}\right] = 49$

Here [·] is greatest integer function

13. (b)

$N = 15^2 \times 5^{18}$

$N = 3^2 \times 5^{20} \quad = 3^2 \times \left(\frac{10}{2}\right)^{20}$

On taking log base 10 both sides

$\log_{10} N = \log_{10} 3^2 \times \left(\frac{10}{2}\right)^{20}$

$= 2\log_{10} 3 + 20\log_{10} 10 - 20\log_{10} 2$ {using log properties}

$= 2 \times .4771 + 20\,(1 - .3010) = 14.64$ {$\log_{10} 3 = 0.4771$; $\log_{10} 2 = 0.3010$}

Hence number of digits are 14 + 1 = 15

If each digits are 9 then maximum sum = 15 × 9 = 135 and

If number has only 5 & 3 as factor

So number would end with 5 hence sum of digits would be atleast 5, also as 3 is a factor so sum of digits should be divisible by 3

Hence, 6 would be minimum sum of digits

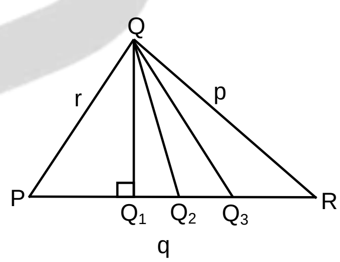
$\Rightarrow \quad 6 \le S \le 135$

From option $\quad 6 \le S < 140$

14. (a)

$PQ_3 = Q_3R$ ($\therefore$ QQ_3 is median)

$PQ_3 = \frac{1}{2}PR$



$PQ_2 : Q_2R = r : p$ {By property of angle bisector}

$\frac{Q_2R}{PQ_2} = \frac{p}{r}$

Apply componendo

$$\frac{PQ_2 + Q_2R}{PQ_2} = \frac{p+r}{r}$$

$PQ_2 = \left(\frac{r}{r+p}\right)PR$ $\because$ $\{PQ_2 + Q_2R = PR\}$

But $r < p$ (given)

$\therefore PQ_2 < \frac{1}{2}PR$

Comparison between altitude and angle bisector

$\Rightarrow$ $\angle QPQ_2 + \angle PQ_2Q + \angle PQQ_2 = \angle RQQ_2 + \angle QQ_2R + \angle QRQ_2$

$\therefore$ $\angle PQQ_2 = \angle RQQ_2$ {Since angle bisector}

$\angle QPQ_2 + \angle PQ_2Q = \angle QQ_2R + \angle QRQ_2$

$\therefore$ $PQ < QR$ then $\angle QPQ_2 > \angle QRQ_2$

Hence $\angle QQ_2P < \angle QQ_2R$

But $\angle QQ_2P + \angle QQ_2R = 180°$

Hence $\angle QQ_2P < 90°$ & $\angle QQ_2R > 90°$

$\Rightarrow$ foot from Q to side PR lies inside ΔPQQ_2

$\Rightarrow$ $PQ_1 < PQ_2 < PQ_3$

15. (a)

$(a-8)^2 - (b-7)^2 = 5$ (given)

Let $a - 8 = x$ & $b - 7 = y$

$\Rightarrow$ $x^2 - y^2 = 5$ {using $A^2 - B^2 = (A - B)(A + B)$; where $A = x$, $B = y$

$\Rightarrow$ $(x - y)(x + y) = 5$

Cases -

$x-y$	$x+y$
5	1
1	5
–5	–1
–1	–5

Case I

$x - y = 5$ & $x + y = 1$

on solving

$\Rightarrow$ $x = 3$ & $y = -2$ $\because$ $a - 8 = x$; $b - 7 = y$

$\Rightarrow$ $a - 8 = 3$ & $b - 7 = -2$

$\Rightarrow$ $a = 11$ & $b = 5$

Case II

$x - y = 1$ & $x + y = 5$

$\Rightarrow$ $x = 3$ & $y = 2$ $\because a - 8 = x; b - 7 = y$

$\Rightarrow$ $a = 11$ & $b = 9$

Case III

$x - y = -5$ & $x + y = -1$

$\Rightarrow$ $x = -3$ & $y = 2$ $\because a - 8 = x; b - 7 = y$

$\Rightarrow$ $a = 5$ & $b = 9$

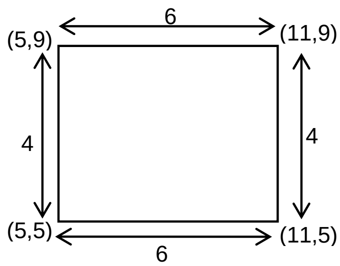
Case IV

$x - y = -1$ & $x + y = -5$

$\Rightarrow$ $x = -3$ & $y = -2$ $\because a - 8 = x; b - 7 = y$

$\Rightarrow$ $a = 5$ & $b = 5$

Points of rectangle are (11, 5), (11, 9), (5, 5), (5, 9)



Perimeter = 4 + 4 + 6 + 6 = 20

PART-II

16. (b)

Let product of digits of n be $P_{(n)}$

Let $n = a_k 10^k + a_{k-1} 10^{k-1} + \ldots\ldots\ldots + a_0$

$P_{(n)} = a_k \cdot a_{k-1} \cdot \ldots\ldots\ldots a_1 \cdot a_0$

Also

$a_0, a_1 \ldots\ldots\ldots a_{k-1}$ can take values from 0 to 9

$\therefore$ $(a_0 \cdot a_1 \ldots\ldots\ldots a_{k-1})_{max} = 9^k$ $\{a_0 = a_1 = a_2 = \ldots a_{k-1} = 9\}$

$\therefore$ $P_{(n)} \le a_k 9^k$

$P_{(n)} \le a_k (10)^k$ ....(1)

Also

$A_k 10^k \le a_k 10^k + a_{k-1} 10^{k-1} + \ldots\ldots\ldots + a_0 \le n$ ....(2)

From (1) & (2)

$P_{(n)} \le n$

$\Rightarrow$ $n^2 - 10n - 36 \le n$ $\Rightarrow$ $-2.64 \le n \le 13.64$ ....(3)

Also $P_{(n)} \ge 0$

$\Rightarrow$ $n^2 - 10n - 36 \ge 0$

$\Rightarrow$ $n \ge 12.81$ ....(4)

From (3) & (4)

$12.81 \le n \le 13.64$

Hence n = 13 ($\because n \in I$)

17. (c)

Case I

For (n); Repetition not allowed

$\Rightarrow$ OEOEO is the only possibility of arrangement of digits, where 0 = odd digit, E = even digits

n =

O	E	O	E	O
3	2	2	1	1

$n = 3 \times 2 \times 2 \times 1 \times 1 = 12$

Case II

For (m) ; Repetition is allowed

$\Rightarrow$ two possibilities

(a) OEOEO

Arrangements (m_1) =

O	E	O	E	O
3	2	3	2	3

$m_1 = 3 \times 2 \times 3 \times 2 \times 3 = 108$

(b) EOEOE

Arrangement (m_2) =

E	O	E	O	E
2	3	2	3	2

$m_2 = 2 \times 3 \times 2 \times 3 \times 2 = 72$

$m = m_1 + m_2$

$m = 108 + 72 = 180$

$\Rightarrow \frac{m}{n} = \frac{180}{12} = 15$

18. (b)

Let height = h, radius = r & slant height = ℓ

Given, Volume = surface area

$\Rightarrow \quad \frac{1}{3}\pi r^2 h = \pi r\ell + \pi r^2 \qquad [\because \ \ell = \sqrt{r^2 + h^2}\,]$

$\Rightarrow \quad rh = 3\sqrt{r^2+h^2} + 3r$

$\Rightarrow \quad rh - 3r = 3\sqrt{r^2+h^2}$

On squaring both sides

$\Rightarrow \quad (rh - 3r)^2 = \left(3\sqrt{r^2+h^2}\right)^2$ {using $(A - B)^2 = A^2 + B^2 - 2AB$}

$\Rightarrow \quad r^2h^2 + 9r^2 - 6r^2h = 9r^2 + 9h^2$

$\Rightarrow \quad r^2(h - 6) = 9h$

$\Rightarrow \quad r^2 = \frac{9h}{h-6}$

$\Rightarrow \quad r^2 = \frac{9(h-6)+54}{h-6} = 9 + \frac{54}{h-6}$

There exists only one solution which satisfy above condition is h = 8, and r = 6

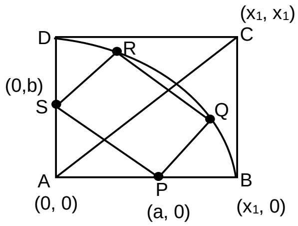
19. (d)

Let, A(0, 0), B (x_1, 0), C(x_1, x_1) & D(0, x_1)

Area of ABCD = $(x_1)^2$ $\because$ area of square = (side)^2

Let P(a, 0) & S(0, b)

Equation of circle is $x^2 + y^2 = x_1^2$



Equation of circle is $x^2 + y^2 = x_1^2$

Slope of AC = $\frac{x_1 - 0}{x_1 - 0} = 1$ $\{\because$ slope = $\frac{y_2 - y_1}{x_2 - x_1}\}$

$\therefore$ Slope of PQ & SR = 1 $\Rightarrow \theta = \frac{\pi}{4}$

Let PQ = SR = d

Using parametric coordinate ($x_1 \pm r\cos\theta$, $y_1 \pm r\sin\theta$)

Q = (a + dcos45°, 0 + dsin45°) = $\left(a + \frac{d}{\sqrt{2}}, \frac{d}{\sqrt{2}}\right)$ { Here (x_1, y_1) = (a, 0); r = d; $\theta = \frac{\pi}{4}$}

R = (0 + dcos45°; b + dsin45°) = $\left(\frac{d}{\sqrt{2}}, b + \frac{d}{\sqrt{2}}\right)$ { Here (x_1, y_1) = (0, b); r = d; $\theta = \frac{\pi}{4}$}

Q and R lies on Arc BD

$\Rightarrow \left(a + \frac{d}{\sqrt{2}}\right)^2 + \left(\frac{d}{\sqrt{2}}\right)^2 = x_1^2$

$\Rightarrow a^2 + d^2 + \sqrt{2}ad = x_1^2$ ....(1)

Similarly $b^2 + d^2 + \sqrt{2}bd = x_1^2$ ....(2)

And

$(PS)^2 = (d)^2 \Rightarrow a^2 + b^2 = d^2$ ....(3)

$\Rightarrow$ Slope of PS = –1 (PS $\perp$ AC) $\{\because$ slope of AC = 1}

$\Rightarrow \frac{-b}{a} = -1$ $\Rightarrow$ a = b ....(4)

From (3) & (4)

$\Rightarrow 2a^2 = d^2$

$\Rightarrow a = \frac{d}{\sqrt{2}}$ = b put in eq (1)

Now $\frac{d^2}{2} + d^2 + d^2 = x_1^2 \Rightarrow d^2 = \frac{2}{5}x_1^2$ = Area (PQRS)

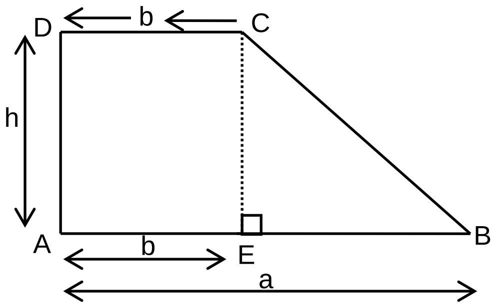
$\Rightarrow \quad \frac{\text{area PQRS}}{\text{area ABCD}} = \frac{\frac{2}{5}x_1^2}{x_1^2} = \frac{2}{5}$

20. (b)

Area = 12

$\Rightarrow \quad \frac{1}{2}(a + b) \times h = 12$

$\Rightarrow \quad (a + b) \times h = 24$



Possible →	24	×	1
Cases	12	×	2
	6	×	4
	8	×	3

In ΔEBC, possible height for integer sides, h = 4, 3

Case I When h = 4

Then possible triplet (3, 4, 5)

i.e. EC = 4, EB = 3, BC = 5

it EB = 3, 2b = 3

$b = \frac{3}{2}$ (not possible because $b \in I$)

Case II When h = 3

Then EB = 4, 2b = 4 $\Rightarrow$ b = 2

∴ CD = 2, AB = 6

∴ |AB – CD| = 4