Exercise 2a

1.

Solution:

Consider A and B as two non empty sets. We know that f which associates to each element $x \in A$ is a unique element which is denoted by f(x) of B is a function from A to B and we write it as

$f: A \to B$

So the domain, co domain and range of function $f: A \to B$ where A is the domain of f, B is the codomain of f and $f(A) = \{f(x): x \in A\}$ is the range of f.

Example:

Consider A = {1, 2, 3, 4} and B = {1, 4, 9, 16, 25}

We know that $f: A \to B: f(x) = x^2 \ \forall\ x \in A$

All the elements of A has unique image in B where f is a function from A to B

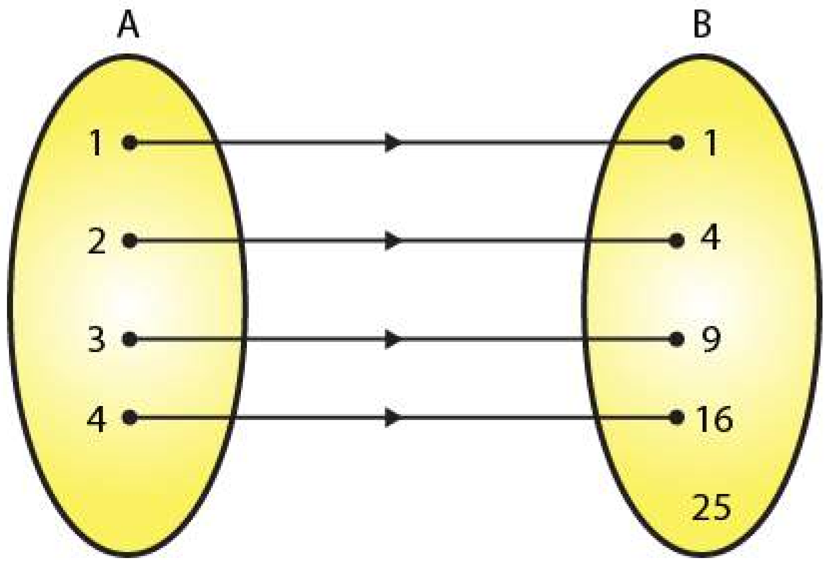
By substituting the value

$f(1) = 1^2 = 1$

$f(2) = 2^2 = 4$

$f(3) = 3^2 = 9$

$f(4) = 4^2 = 16$



dom (f) = {1, 2, 3, 4} = A

codom (f) = {1, 4, 9, 16, 25} = B

range (f) = {1, 4, 9, 16}

Here, 25 ∈ B does not have pre image in A.

2.

Solution:

(i) Injective function

A function f: A → B is said to be one-one if distinct elements in A have distinct images in B.

Example:

Let N be the set of all natural numbers.

Let f: N → N: f(x) = 2x ∀ x ∈ N

Then, $f(x_1) = f(x_2)$

$2x_1 = 2x_2$

$x_1 = x_2$

Hence, f is one-one.

(ii) Surjective Function

A function f: A → B is said to be onto if every element in B has at least one pre-image in A. Thus, if f is onto then for each y ∈ B ∃ at least one element x ∈ A such that y = f(x)

Also, f is onto range f = B

Example:

Let N be the set of all natural numbers and let E be the set of all even natural numbers

Let f: N → E: f(x) = 2x ∀ x ∈ N

Then, y = 2x => x = ½ y

Thus, for each y ∈ E there exists ½ y ∈ N such that

f(1/2 y) = (2 × ½ y) = y

Hence, f is onto.

(iii) Bijective Function

A one-one onto function is said to be bijective or a one-to-one correspondence.

Example:

If $f(x) = x^2$ from set of positive real numbers to positive real numbers which is both surjective and injective.

Hence, f is bijective function.

(iv) Many-one Function

A function f: A→ B is said to be many-one if two or more than two elements in A have the same image in B.

Example:

Let A = {-1, 1, 2, 3} and B = {1, 4, 9}

Let f: A → B: $f(x) = x^2$ ∀ x ∈ A

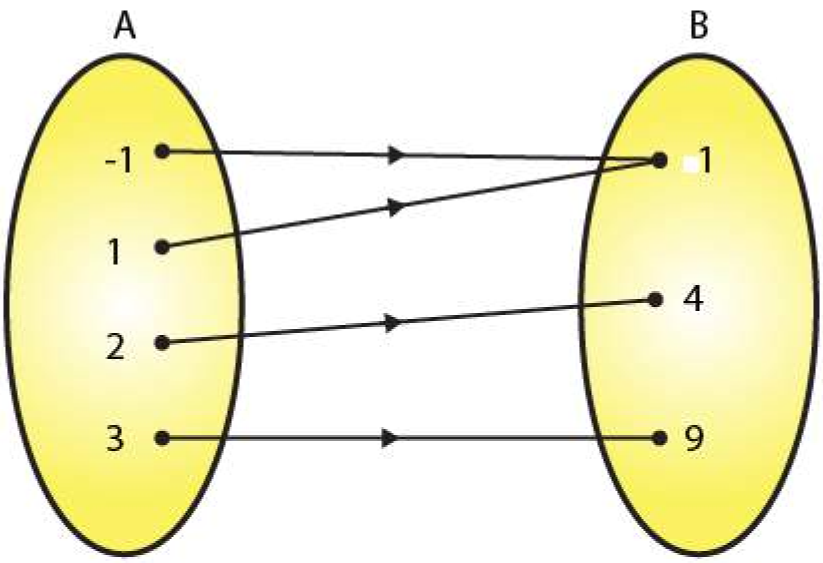
Then, each element in A has a unique image under f in B f is a function from A to B such that

$f(-1) = (-1)^2 = 1$

$f(1) = 1^2 = 1$

$f(2) = 2^2 = 4$

$f(3) = 3^2 = 9$



Here, two elements namely -1 and 1 have same image 1 ∈ B.

Hence, f is many-one.

(v) Into Function

A function f: A → B is said to be into if there exists even a single element in B having no pre-image in A.

So, f is into range (f) ⊂ B.

Example:

Let A = {2, 3, 5, 7} and B = {0, 1, 3, 5, 7}

Let f: A → B: f(x) = (x – 2) then

f(2) = (2 – 2) = 0

f(3) = (3 – 2) = 1

f(5) = (5 – 2) = 3

f(7) = (7 – 2) = 5

Thus, every element in A has a unique image in B

Now ∃ 7 ∈ B having no pre-image in A

Hence, f is into.

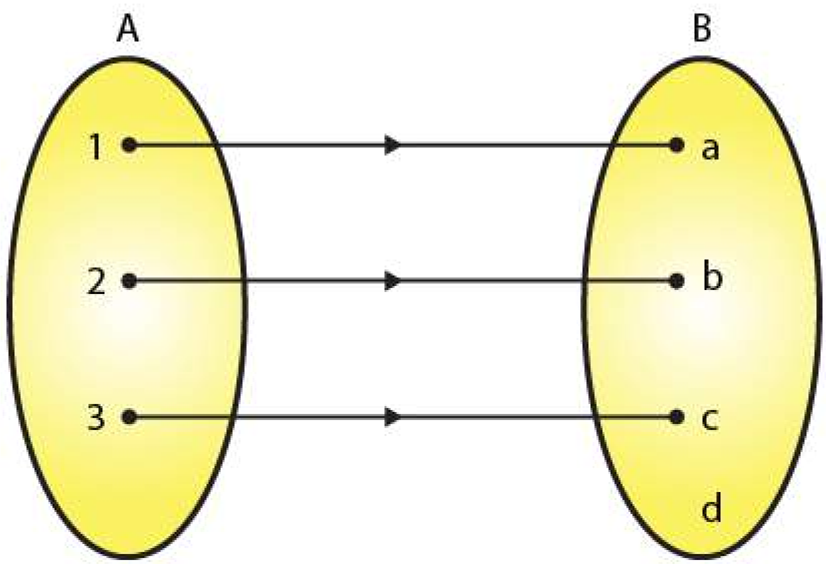
3.

Solution:

(i) One-one but not onto

Consider A = {1, 2, 3} and B = {a, b, c, d}

So we get f = {(1, a), (2, b), (3, c)}



(ii) One-one and onto

We know that f(x) = 2x

Infectivity:

Consider x_1, $x_2 \in R$ where f(x1) = f(x2)

So we get

$2x_1 = 2x_2$

$x_1 = x_2$

Hence, f: R → R is one-one

Subjectivity:

Consider y be any real number in R which is the co-domain

f(x) = y

We get

2x = y

It can be written as

x = y/2

We know that y/2 ∈ R for y ∈ R where

f(y/2) = 2(y/2) = y

For y ∈ R(co-domain) there exists x = y/2 ∈ R (domain) where f(x) = y

Here, each element in co-domain has pre-image in domain

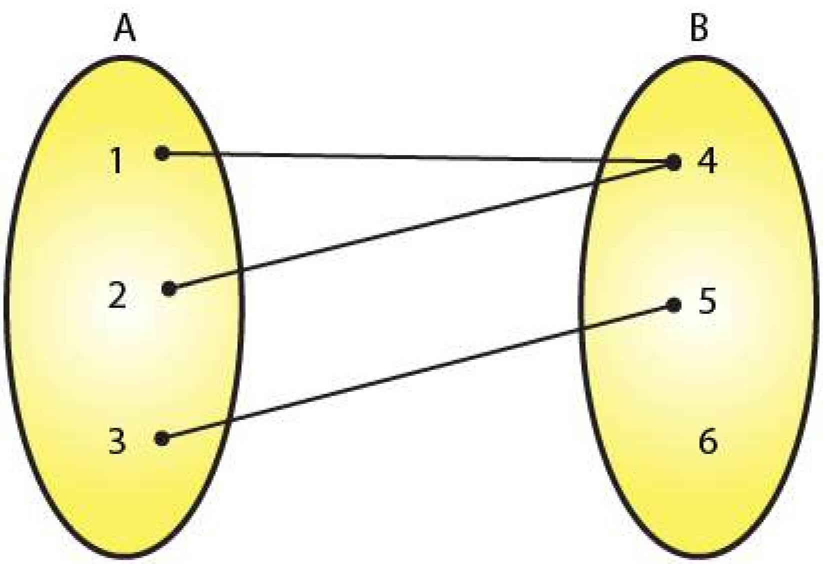
Thus, f: R → R is bijective.

(iii) Neither one-one nor onto

Consider A = {1, 2, 3} and B = {4, 5, 6}

We get

f = {(1, 4), (2, 4), (3, 5)}

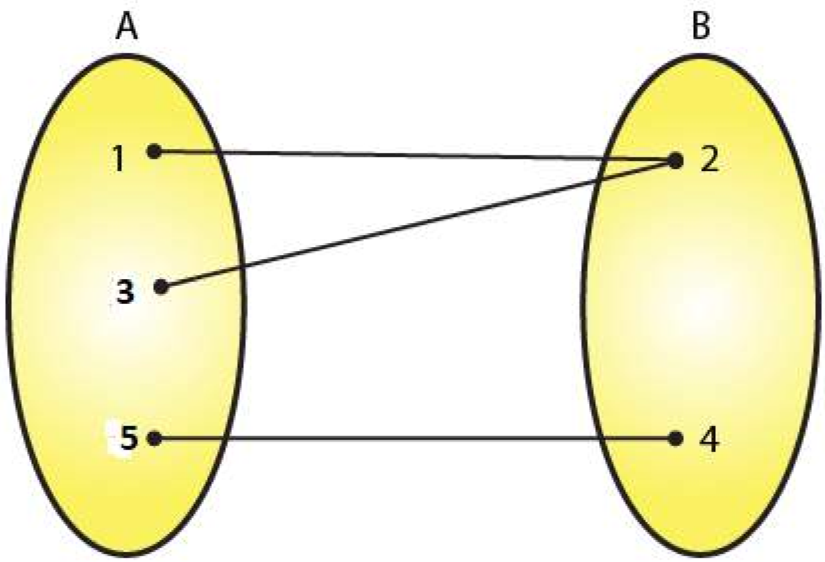


(iv) Onto but not one-one

Consider A = {1, 2, 3} and B = {4, 5, 6}

We get

f = {(1, 2), (3, 2), (5, 4)}



4.

Find (i) f(2) (ii) f(4) (iii) f(-1) (iv) f(-3).

Solution:

(i) We know that

$f(x) = x^2 - 2$

By substituting the value we get

$f(2) = 2^2 – 2 = 4 – 2 = 2$

(ii) We know that

$f(x) = 3x – 1$

By substituting the value we get

$f(x) = 3 (4) – 1 = 12 – 1 = 11$

(iii) We know that

$f(x) = x^2 - 2$

By substituting the value we get

$f(-1) = (-1)^2 – 2 = 1 – 2 = -1$

(iv) We know that

$f(x) = 2x + 3$

By substituting the value we get

$f(-3) = 2 (-3) + 3 = - 6 + 3 = - 3$

5.

Solution:

We know that for every negative and positive value of x we get the same value of f(x) which is not one-one that is many one

$1 + x^2 \geq 1$ for all values of x

Thus, range $[1, \infty) \subset R$

Here, the function is into function.

6.

Solution:

We know that for every negative and positive value of x we get the same value of f(x) which is not one-one that is many one.

$x^4 \geq 0$ for all values of f(x)

Thus, range $[0, \infty) \subset R$

Here, the function is into function.

7.

Solution:

Injectivity:

Consider $x, y \in R$ where $f(x) = f(y)$

We get

$x^5 = y^5$

It can be written as

$x^5 - y^5 = 0$

By multiplying and dividing by 2 on both sides

$(x^{5/2})^2 - (y^{5/2})^2 = 0$

We know that

$(x^{5/2} + y^{5/2})(x^{5/2} - y^{5/2}) = 0$

So we get

$x^{5/2} - y^{5/2} = 0$

where $x = y$

Hence, $f(x) = f(y)$ is $x = y$ for all $x, y \in R$

f is injective.

Surjectivity:

Consider y as an arbitrary element of R

We know that

$f(x) = y$

It can be written as

$x^5 = y$

So we get

$x^5 - y = 0$

Odd degree equation has one real root

Thus, for every real value of y

$x^5 - y = 0$ has real root α where

$\alpha^5 - y = 0$

So $\alpha^5 = y$

Thus, $f(\alpha) = y$

For every $y \in R$ there exists $\alpha \in R$ where $f(\alpha) = y$

f is surjective

Thus, $f: R \to R$ is bijective.

8.

Solution:

For any two distinct elements x_1 and x_2 in $[0, \pi/2]$

We know that

$\sin x_1 \neq \sin x_2$ and $\cos x_1 \neq \cos x_2$

So we get

$f(x_1) \neq f(x_2)$ and $g(x_1) \neq g(x_2)$

Here, both f and g are one-one

We know that

$(f + g)(x) = f(x) + g(x) = \sin x + \cos x$

By substituting the values

$(f + g)(0) = \sin 0 + \cos 0 = 1$

$(f + g)(\pi/2) = \sin \pi/2 + \cos \pi/2 = 1$

We know that $0 \neq \pi/2$

Here, $(f + g)(0) = (f + g)(\pi/2)$

So, f + g is not one-one.

9.

Solution:

(i) Consider N as the set of all natural numbers

It is given that $f: N \rightarrow N: f(x) = x^2 \ \forall \ x \in N$

We know that

$f(x_1) = f(x_2)$

It can be written as

$x_1^2 = x_2^2$

So we get

$x_1^2 - x_2^2 = 0$

We get

$(x_1 - x_2)(x_1 + x_2) = 0$

Here, $(x_1 - x_2) = 0$ where $x_1 = x_2$

Hence, f is one-one.

Consider A = {1, 2, 3, 4} and B = {1, 4, 9, 16, 25}

Here, f: A → B: $f(x) = x^2$

By substituting the values

$f(1) = 1^2 = 1$

$f(2) = 2^2 = 4$

$f(3) = 3^2 = 9$

$f(4) = 4^2 = 16$

Hence, every moment in A has unique image in B. Here, ∃ 25 ∈ B which has no pre-image in A

f is into.

(ii) Consider A = {-1, 1, 2, 3} and B = {1, 4, 9}

It is given that f: A → B: $f(x) = x^2 \ \forall \ x \in A$

Each element in A has unique image in B where f is a function from A to B

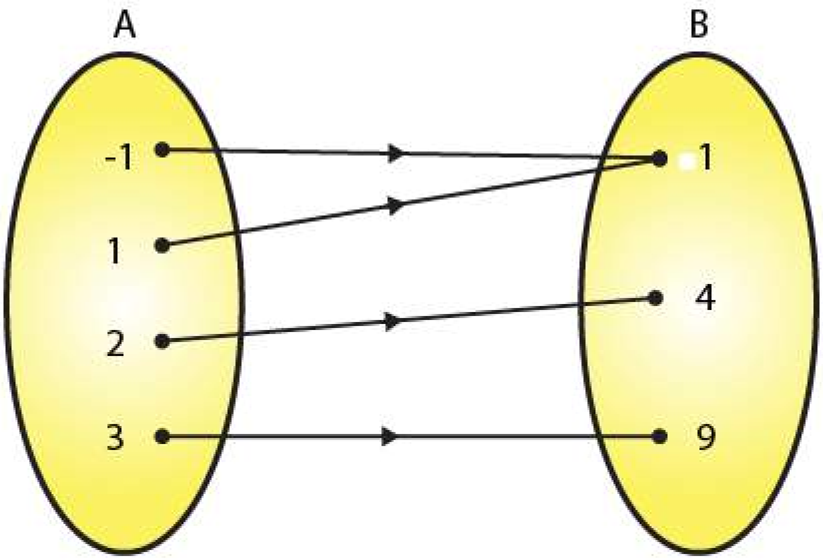
By substituting the values

$f(-1) = (-1)^2 = 1$

$f(1) = 1^2 = 1$

$f(2) = 2^2 = 4$

$f(3) = 3^2 = 9$



Here, two elements -1 and 1 have same image $1 \in B$

Hence, f is many-one.