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1. Discuss the applicability of Rolle's Theorem for the following functions on the indicated intervals:

$(i)\ f(x) = 3 + (x-2)^{\frac{2}{3}}\ on\ [1, 3]$

Solution:

Given function is

$\Rightarrow f(x) = 3 + (x-2)^{\frac{2}{3}}$ on [1, 3]

Let us check the differentiability of the function f(x).

Now we have to find the derivative of f(x),

$\Rightarrow f'(x) = \frac{d}{dx}\left(3 + (x-2)^{\frac{2}{3}}\right)$

$\Rightarrow f'(x) = \frac{d(3)}{dx} + \frac{d\left((x-2)^{\frac{2}{3}}\right)}{dx}$

$\Rightarrow f'(x) = 0 + \frac{2}{3}(x-2)^{\frac{2}{3}-1}$

$\Rightarrow f'(x) = \frac{2}{3}(x-2)^{-\frac{1}{3}}$

$\Rightarrow f'(x) = \frac{2}{3(x-2)^{\frac{1}{3}}}$

Now we have to check differentiability at the value of x = 2

$\Rightarrow \lim_{x\to 2} f'(x) = \lim_{x\to 2} \frac{2}{3(x-2)^{\frac{1}{3}}}$

$\Rightarrow \lim_{x\to 2} f'(x) = \frac{2}{3(2-2)^{\frac{1}{3}}}$

$\Rightarrow \lim_{x\to 2} f'(x) = \frac{2}{3(0)}$

$\Rightarrow \lim_{x \to 2} f'(x) = \text{undefined}$

∴ f is not differentiable at x = 2, so it is not differentiable in the closed interval (1, 3).

So, Rolle's theorem is not applicable for the function f on the interval [1, 3].

(ii) f (x) = [x] for -1 ≤ x ≤ 1, where [x] denotes the greatest integer not exceeding x

Solution:

Given function is f (x) = [x], – 1≤x≤1 where [x] denotes the greatest integer not exceeding x.

Let us check the continuity of the function f.

Here in the interval x ϵ [– 1, 1], the function has to be Right continuous at x = 1 and left continuous at x = 1.

$\Rightarrow \lim_{x \to 1+} f(x) = \lim_{x \to 1+} [x]$

$\Rightarrow \lim_{x \to 1+} f(x) = \lim_{x \to 1+h} [x]$ Where h>0.

$\Rightarrow \lim_{x \to 1+} f(x) = \lim_{h \to 0} 1$

$\Rightarrow \lim_{x \to 1+} f(x) = 1$ ...... (1)

$\Rightarrow \lim_{x \to 1-} f(x) = \lim_{x \to 1-} [x]$

$\Rightarrow \lim_{x \to 1-} f(x) = \lim_{x \to 1-h} [x]$, where h>0

$\Rightarrow \lim_{x \to 1-} f(x) = \lim_{h \to 0} 0$

$\Rightarrow \lim_{x \to 1-} f(x) = 0$ ...... (2)

From (1) and (2), we can see that the limits are not the same so, the function is not continuous in the interval [– 1, 1].

∴ Rolle's Theorem is not applicable for the function f in the interval [– 1, 1].

(iii) $f(x) = \sin \frac{1}{x}$ **for** $-1 \le x \le 1$

Solution:

Given function is $f(x) = \sin\left(\frac{1}{x}\right)$ for $-1 \le x \le 1$

Let us check the continuity of the function 'f' at the value of x = 0.We cannot directly find the value of limit at x = 0, as the function is not valid at x = 0. So, we take the limit on either sides or x = 0, and we check whether they are equal or not.

So consider RHL:

$$\Rightarrow \lim_{x \to 0+} f(x) = \lim_{x \to 0+} \sin\left(\frac{1}{x}\right)$$

We assume that the limit $\lim_{h \to 0} \sin\left(\frac{1}{h}\right) = k$, k ϵ [– 1, 1].

$$\Rightarrow \lim_{x \to 0+} f(x) = \lim_{x \to 0+h} \sin\left(\frac{1}{x}\right)$$, where h>0

$$\Rightarrow \lim_{x \to 0+} f(x) = \lim_{h \to 0} \sin\left(\frac{1}{h+0}\right)$$

$$\Rightarrow \lim_{x \to 0+} f(x) = \lim_{h \to 0} \sin\left(\frac{1}{h}\right)$$

$$\Rightarrow \lim_{x \to 0+} f(x) = k \quad \text{...... (1)}$$

Now consider LHL:

$$\Rightarrow \lim_{x \to 0-} f(x) = \lim_{x \to 0-} \sin\left(\frac{1}{x}\right)$$

$$\Rightarrow \lim_{x \to 0-} f(x) = \lim_{x \to 0-h} \sin\left(\frac{1}{x}\right)$$, where h>0

$$\Rightarrow \lim_{x \to 0-} f(x) = \lim_{h \to 0} \sin\left(\frac{1}{0-h}\right)$$

$$\Rightarrow \lim_{x \to 0-} f(x) = \lim_{h \to 0} \sin\left(\frac{1}{-h}\right)$$

$\Rightarrow \lim_{x\to 0-} f(x) = \lim_{h\to 0} -\sin\left(\frac{1}{h}\right)$

$\Rightarrow \lim_{x\to 0-} f(x) = -\lim_{h\to 0} \sin\left(\frac{1}{h}\right)$

$\Rightarrow \lim_{x\to 0-} f(x) = -k$ ...... (2)

From (1) and (2), we can see that the Right hand and left – hand limits are not equal, so the function 'f' is not continuous at x = 0.

∴ Rolle's Theorem is not applicable to the function 'f' in the interval [– 1, 1].

(iv) f (x) = $2x^2 - 5x + 3$ on [1, 3]

Solution:

Given function is f (x) = $2x^2 - 5x + 3$ on [1, 3]

Since given function f is a polynomial. So, it is continuous and differentiable everywhere.

Now, we find the values of function at the extreme values.

⇒ f (1) = $2(1)^2 – 5(1) + 3$

⇒ f (1) = 2 – 5 + 3

⇒ f (1) = 0...... (1)

⇒ f (3) = $2(3)^2 – 5(3) + 3$

⇒ f (3) = 2(9)–15 + 3

⇒ f (3) = 18 – 12

⇒ f (3) = 6...... (2)

From (1) and (2), we can say that, f (1) ≠ f (3)

∴ Rolle's Theorem is not applicable for the function f in interval [1, 3].

(v) f (x) = $x^{2/3}$ on [-1, 1]

Solution:

Given function is $f(x) = x^{\frac{2}{3}}$ on [– 1, 1]

Now we have to find the derivative of the given function:

$\Rightarrow f'(x) = \frac{d\left(x^{\frac{2}{3}}\right)}{dx}$

$\Rightarrow f'(x) = \frac{2}{3}x^{\frac{2}{3}-1}$

$\Rightarrow f'(x) = \frac{2}{3}x^{-\frac{1}{3}}$

$\Rightarrow f'(x) = \frac{2}{3x^{\frac{1}{3}}}$

Now we have to check the differentiability of the function at x = 0.

$\Rightarrow \lim_{x\to 0} f'(x) = \lim_{x\to 0}\frac{2}{3x^{\frac{1}{3}}}$

$\Rightarrow \lim_{x\to 0} f'(x) = \frac{2}{3(0)^{\frac{1}{3}}}$

$\Rightarrow \lim_{x\to 0} f'(x) = \text{undefined}$

Since the limit for the derivative is undefined at x = 0, we can say that f is not differentiable at x = 0.

∴ Rolle's Theorem is not applicable to the function 'f' on [– 1, 1].

(vi) $f(x) = \begin{cases} -4x+5, & 0 \le x \le 1 \\ 2x-3, & 1 < x \le 2 \end{cases}$

Solution:

Given function is $f(x) = \begin{cases} -4x + 5, 0 \le x \le 1 \\ 2x - 3, 1 < x \le 2 \end{cases}$

Now we have to check the continuity at x = 1 as the equation of function changes.

Consider LHL:

$\Rightarrow \lim_{x\to 1-} f(x) = \lim_{x\to 1-} -4x + 5$

$\Rightarrow \lim_{x\to 1-} f(x) = -4(1) + 5$

$\Rightarrow \lim_{x\to1-} f(x) = 1$ ...... (1)

Now consider RHL:

$\Rightarrow \lim_{x\to1+} f(x) = \lim_{x\to1+} 2x - 3$

$\Rightarrow \lim_{x\to1+} f(x) = 2(0) - 3$

$\Rightarrow \lim_{x\to1+} f(x) = -1$ ...... (2)

From (1) and (2), we can see that the values of both side limits are not equal.
So, the function 'f' is not continuous at x = 1.

∴ Rolle's Theorem is not applicable to the function 'f' in the interval [0, 2].

2. Verify the Rolle's Theorem for each of the following functions on the indicated intervals:
(i) f (x) = x^2 – 8x + 12 on [2, 6]

Solution:
Given function is f (x) = x^2 – 8x + 12 on [2, 6]
Since, given function f is a polynomial it is continuous and differentiable everywhere i.e., on R.
Let us find the values at extremes:
⇒ f (2) = 2^2 – 8(2) + 12
⇒ f (2) = 4 – 16 + 12
⇒ f (2) = 0
⇒ f (6) = 6^2 – 8(6) + 12
⇒ f (6) = 36 – 48 + 12
⇒ f (6) = 0
∴ f (2) = f(6), Rolle's theorem applicable for function f on [2,6].
Now we have to find the derivative of f(x)

$\Rightarrow f'(x) = \frac{d(x^2 - 8x + 12)}{dx}$

$\Rightarrow f'(x) = \frac{d(x^2)}{dx} - \frac{d(8x)}{dx} + \frac{d(12)}{dx}$

$\Rightarrow f'(x) = 2x - 8 + 0$

$\Rightarrow f'(x) = 2x - 8$

We have f'(c) = 0 ϵ [2, 6], from the above definition

$\Rightarrow f'(c) = 0$

$\Rightarrow 2c - 8 = 0$

$\Rightarrow 2c = 8$

$\Rightarrow c = \frac{8}{2}$

$\Rightarrow C = 4 \in [2, 6]$

∴ Rolle's Theorem is verified.

(ii) f(x) = $x^2 - 4x + 3$ on [1, 3]

Solution:

Given function is f (x) = $x^2 - 4x + 3$ on [1, 3]

Since, given function f is a polynomial it is continuous and differentiable everywhere i.e., on R. Let us find the values at extremes:

$\Rightarrow f(1) = 1^2 - 4(1) + 3$

$\Rightarrow f(1) = 1 - 4 + 3$

$\Rightarrow f(1) = 0$

$\Rightarrow f(3) = 3^2 - 4(3) + 3$

$\Rightarrow f(3) = 9 - 12 + 3$

$\Rightarrow f(3) = 0$

∴ f (1) = f(3), Rolle's theorem applicable for function 'f' on [1,3].

Let's find the derivative of f(x)

$\Rightarrow f'(x) = \frac{d(x^2 - 4x + 3)}{dx}$

$\Rightarrow f'(x) = \frac{d(x^2)}{dx} - \frac{d(4x)}{dx} + \frac{d(3)}{dx}$

$\Rightarrow f'(x) = 2x - 4 + 0$

$\Rightarrow f'(x) = 2x - 4$

We have f'(c) = 0, c ϵ (1, 3), from the definition of Rolle's Theorem.

⇒ f'(c) = 0

⇒ 2c – 4 = 0

⇒ 2c = 4

⇒ c = 4/2

⇒ C = 2 ϵ (1, 3)

∴ Rolle's Theorem is verified.

(iii) f (x) = (x – 1) (x – 2)^2 on [1, 2]

Solution:

Given function is f (x) = (x – 1) (x – 2)^2 on [1, 2]

Since, given function f is a polynomial it is continuous and differentiable everywhere that is on R.

Let us find the values at extremes:

⇒ f (1) = (1 – 1) (1 – 2)^2

⇒ f (1) = 0(1)^2

⇒ f (1) = 0

⇒ f (2) = (2 – 1)(2 – 2)^2

⇒ f (2) = 0^2

⇒ f (2) = 0

∴ f (1) = f (2), Rolle's Theorem applicable for function 'f' on [1, 2].

Let's find the derivative of f(x)

$$\Rightarrow f'(x) = \frac{d((x-1)(x-2)^2)}{dx}$$

Differentiating by using product rule, we get

$$\Rightarrow f'(x) = (x-2)^2 \times \frac{d(x-1)}{dx} + (x-1) \times \frac{d((x-2)^2)}{dx}$$

⇒ f'(x) = ((x – 2)^2×1) + ((x – 1) × 2 × (x – 2))

⇒ f'(x) = x^2 – 4x + 4 + 2(x^2 – 3x + 2)

⇒ f'(x) = 3x^2 – 10x + 8

We have f'(c) = 0 c ϵ (1, 2), from the definition of Rolle's Theorem.

⇒ f'(c) = 0

$\Rightarrow 3c^2 - 10c + 8 = 0$

$\Rightarrow c = \frac{10 \pm \sqrt{(-10)^2 - (4\times3\times8)}}{2\times3}$

$\Rightarrow c = \frac{10 \pm \sqrt{100-96}}{6}$

$\Rightarrow c = \frac{10 \pm 2}{6}$

$\Rightarrow c = \frac{12}{6}$ or $c = \frac{8}{6}$

$\Rightarrow c = \frac{4}{3} \in (1, 2)$ (neglecting the value 2)

∴ Rolle's Theorem is verified.

(iv) f (x) = x (x – 1)^2 on [0, 1]

Solution:

Given function is f (x) = x(x – 1)^2 on [0, 1]

Since, given function f is a polynomial it is continuous and differentiable everywhere that is, on R.

Let us find the values at extremes

$\Rightarrow f(0) = 0(0 - 1)^2$

$\Rightarrow f(0) = 0$

$\Rightarrow f(1) = 1(1 - 1)^2$

$\Rightarrow f(1) = 0^2$

$\Rightarrow f(1) = 0$

∴ f (0) = f (1), Rolle's theorem applicable for function 'f' on [0,1].

Let's find the derivative of f(x)

$\Rightarrow f'(x) = \frac{d(x(x-1)^2)}{dx}$

Differentiating using product rule:

$\Rightarrow f'(x) = (x-1)^2 \times \frac{d(x)}{dx} + x\frac{d((x-1)^2)}{dx}$

$\Rightarrow f'(x) = ((x-1)^2\times1) + (x\times2\times(x-1))$

⇒ $f'(x) = (x - 1)^2 + 2(x^2 - x)$

⇒ $f'(x) = x^2 - 2x + 1 + 2x^2 - 2x$

⇒ $f'(x) = 3x^2 - 4x + 1$

We have f'(c) = 0 c ϵ (0, 1), from the definition given above.

⇒ $f'(c) = 0$

⇒ $3c^2 - 4c + 1 = 0$

⇒ $c = \frac{4 \pm \sqrt{(-4)^2 - (4 \times 3 \times 1)}}{2 \times 3}$

⇒ $c = \frac{4 \pm \sqrt{16 - 12}}{6}$

⇒ $c = \frac{4 \pm \sqrt{4}}{6}$

⇒ $c = \frac{6}{6}$ or $c = \frac{2}{6}$

⇒ $c = \frac{1}{3} \in (0, 1)$

∴ Rolle's Theorem is verified.

(v) f (x) = $(x^2 - 1)(x - 2)$ on [-1, 2]

Solution:

Given function is f (x) = $(x^2 - 1)(x - 2)$ on [– 1, 2]

Since, given function f is a polynomial it is continuous and differentiable everywhere that is on R.

Let us find the values at extremes:

⇒ $f(-1) = ((-1)^2 - 1)(-1 - 2)$

⇒ $f(-1) = (1 - 1)(-3)$

⇒ $f(-1) = (0)(-3)$

⇒ $f(-1) = 0$

⇒ $f(2) = (2^2 - 1)(2 - 2)$

⇒ $f(2) = (4 - 1)(0)$

⇒ $f(2) = 0$

∴ f (– 1) = f (2), Rolle's theorem applicable for function f on [– 1,2].
Let's find the derivative of f(x)

$$\Rightarrow f'(x) = \frac{d((x^2-1)(x-2))}{dx}$$

Differentiating using product rule,

$$\Rightarrow f'(x) = (x-2) \times \frac{d(x^2-1)}{dx} + (x^2-1)\frac{d(x-2)}{dx}$$

$\Rightarrow f'(x) = ((x-2) \times 2x) + ((x^2-1) \times 1)$

$\Rightarrow f'(x) = 2x^2 - 4x + x^2 - 1$

$f'(x) = 3x^2 - 4x - 1$

We have f'(c) = 0 c ∈ (-1, 2), from the definition of Rolle's Theorem

f'(c) = 0

$3c^2 - 4c - 1 = 0$

c = 4 ± √[(-4)^2 – (4 x 3 x -1)]/ (2 x 3)   [Using the Quadratic Formula]

c = 4 ± √[16 + 12]/ 6

c = (4 ± √28)/ 6

c = (4 ± 2√7)/ 6

c = (2 ±√7)/ 3 = 1.5 ± √7/3

c = 1.5 + √7/3 or 1.5 - √7/3

So,

c = 1.5 - √7/3 since c ∈ (-1, 2)

∴ Rolle's Theorem is verified.

(vi) f (x) = x (x – 4)^2 on [0, 4]

Solution:

Given function is f (x) = x (x – 4)^2 on [0, 4]

Since, given function f is a polynomial it is continuous and differentiable everywhere i.e., on R.

Let us find the values at extremes:

⇒ f (0) = 0(0 – 4)^2

⇒ f (0) = 0

⇒ f (4) = 4(4 – 4)^2

⇒ f (4) = 4(0)^2

⇒ f (4) = 0

∴ f (0) = f (4), Rolle's theorem applicable for function 'f' on [0,4].
Let's find the derivative of f(x):

$\Rightarrow f'(x) = \frac{d(x(x-4)^2)}{dx}$

Differentiating using product rule

$\Rightarrow f'(x) = (x-4)^2 \times \frac{d(x)}{dx} + x\frac{d((x-4)^2)}{dx}$

$\Rightarrow f'(x) = ((x-4)^2 \times 1) + (x \times 2 \times (x-4))$

$\Rightarrow f'(x) = (x-4)^2 + 2(x^2 - 4x)$

$\Rightarrow f'(x) = x^2 - 8x + 16 + 2x^2 - 8x$

$\Rightarrow f'(x) = 3x^2 - 16x + 16$

We have f'(c) = 0 c ϵ (0, 4), from the definition of Rolle's Theorem.

$\Rightarrow f'(c) = 0$

$\Rightarrow 3c^2 - 16c + 16 = 0$

$\Rightarrow c = \frac{16 \pm \sqrt{(-16)^2 - (4 \times 3 \times 16)}}{2 \times 3}$

$\Rightarrow c = \frac{16 \pm \sqrt{256 - 192}}{6}$

$\Rightarrow c = \frac{16 \pm \sqrt{64}}{6}$

$\Rightarrow c = \frac{8}{6} \text{ or } c = \frac{24}{6}$

$\Rightarrow c = \frac{8}{6} \in (0, 4)$

∴ Rolle's Theorem is verified.

(vii) f (x) = x (x – 2)^2 on [0, 2]

Solution:
Given function is f (x) = x (x – 2)^2 on [0, 2]

Since, given function f is a polynomial it is continuous and differentiable everywhere that is on R.

Let us find the values at extremes:

$\Rightarrow f(0) = 0(0-2)^2$

$\Rightarrow f(0) = 0$

$\Rightarrow f(2) = 2(2-2)^2$

$\Rightarrow f(2) = 2(0)^2$

$\Rightarrow f(2) = 0$

f (0) = f(2), Rolle's theorem applicable for function f on [0,2].

Let's find the derivative of f(x)

$$\Rightarrow f'(x) = \frac{d(x(x-2)^2)}{dx}$$

Differentiating using UV rule,

$$\Rightarrow f'(x) = (x-2)^2 \times \frac{d(x)}{dx} + x\frac{d((x-2)^2)}{dx}$$

$\Rightarrow f'(x) = ((x-2)^2 \times 1) + (x \times 2 \times (x-2))$

$\Rightarrow f'(x) = (x-2)^2 + 2(x^2 - 2x)$

$\Rightarrow f'(x) = x^2 - 4x + 4 + 2x^2 - 4x$

$\Rightarrow f'(x) = 3x^2 - 8x + 4$

We have f'(c) = 0 c ϵ (0, 1), from the definition of Rolle's Theorem.

$\Rightarrow f'(c) = 0$

$\Rightarrow 3c^2 - 8c + 4 = 0$

$$\Rightarrow c = \frac{8 \pm \sqrt{(-8)^2 - (4\times3\times4)}}{2\times3}$$

$$\Rightarrow c = \frac{8 \pm \sqrt{64-48}}{6}$$

$$\Rightarrow c = \frac{8 \pm \sqrt{16}}{6}$$

c = 12/6 or 4/6

c = 2 or 2/3

So,

c = 2/3 since c ∈ (0, 2)

∴ Rolle's Theorem is verified.

(viii) f (x) = x^2 + 5x + 6 on [-3, -2]

Solution:

Given function is f (x) = $x^2 + 5x + 6$ on [– 3, – 2]

Since, given function f is a polynomial it is continuous and differentiable everywhere i.e., on R. Let us find the values at extremes:

⇒ $f(-3) = (-3)^2 + 5(-3) + 6$

⇒ $f(-3) = 9 - 15 + 6$

⇒ $f(-3) = 0$

⇒ $f(-2) = (-2)^2 + 5(-2) + 6$

⇒ $f(-2) = 4 - 10 + 6$

⇒ $f(-2) = 0$

∴ f (– 3) = f(– 2), Rolle's theorem applicable for function f on [– 3, – 2].

Let's find the derivative of f(x):

⇒ $f'(x) = \frac{d(x^2 + 5x + 6)}{dx}$

⇒ $f'(x) = \frac{d(x^2)}{dx} + \frac{d(5x)}{dx} + \frac{d(6)}{dx}$

⇒ $f'(x) = 2x + 5 + 0$

⇒ $f'(x) = 2x + 5$

We have f'(c) = 0 c ϵ (– 3, – 2), from the definition of Rolle's Theorem

⇒ $f'(c) = 0$

⇒ $2c + 5 = 0$

⇒ $2c = -5$

⇒ $c = -\frac{5}{2}$

⇒ $C = -2.5 \in (-3, -2)$

∴ Rolle's Theorem is verified.

3. Verify the Rolle's Theorem for each of the following functions on the indicated

intervals:
(i) f (x) = cos 2 (x – π/4) on [0, π/2]

Solution:

Given function is $f(x) = \cos 2\left(x - \frac{\pi}{4}\right)$ on $\left[0, \frac{\pi}{2}\right]$

We know that cosine function is continuous and differentiable on R.

Let's find the values of the function at an extreme,

$\Rightarrow f(0) = \cos 2\left(0 - \frac{\pi}{4}\right)$

$\Rightarrow f(0) = \cos 2\left(-\frac{\pi}{4}\right)$

$\Rightarrow f(0) = \cos\left(-\frac{\pi}{2}\right)$

We know that cos (– x) = cos x

⇒ f (0) = 0

$\Rightarrow f\left(\frac{\pi}{2}\right) = \cos 2\left(\frac{\pi}{2} - \frac{\pi}{4}\right)$

$\Rightarrow f\left(\frac{\pi}{2}\right) = \cos 2\left(\frac{\pi}{4}\right)$

$\Rightarrow f\left(\frac{\pi}{2}\right) = \cos\left(\frac{\pi}{2}\right)$

$\Rightarrow f\left(\frac{\pi}{2}\right) = 0$

We get $f(0) = f\left(\frac{\pi}{2}\right)$, so there exist a $c \in \left(0, \frac{\pi}{2}\right)$ such that f'(c) = 0.

Let's find the derivative of f(x)

$\Rightarrow f'(x) = \frac{d\left(\cos 2\left(x - \frac{\pi}{4}\right)\right)}{dx}$

$\Rightarrow f'(x) = -\sin\left(2\left(x - \frac{\pi}{4}\right)\right)\frac{d\left(2\left(x - \frac{\pi}{4}\right)\right)}{dx}$

$\Rightarrow f'(x) = -2\sin 2\left(x - \frac{\pi}{4}\right)$

We have f'(c) = 0,

$\Rightarrow -2\sin 2\left(c - \frac{\pi}{4}\right) = 0$

$\Rightarrow c - \frac{\pi}{4} = 0$

$\Rightarrow c = \frac{\pi}{4} \in \left(0, \frac{\pi}{2}\right)$

∴ Rolle's Theorem is verified.

(ii) f (x) = sin 2x on [0, π/2]

Solution:

Given function is f (x) = sin2x on $\left[0, \frac{\pi}{2}\right]$

We know that sine function is continuous and differentiable on R. Let's find the values of function at extreme,

⇒ f (0) = sin2 (0)

⇒ f (0) = sin0

⇒ f (0) = 0

⇒ $f\left(\frac{\pi}{2}\right) = \sin 2\left(\frac{\pi}{2}\right)$

⇒ $f\left(\frac{\pi}{2}\right) = \sin(\pi)$

⇒ $f\left(\frac{\pi}{2}\right) = 0$

We have $f(0) = f\left(\frac{\pi}{2}\right)$, so there exist a $c \in \left(0, \frac{\pi}{2}\right)$ such that f'(c) = 0.

Let's find the derivative of f(x)

⇒ $f'(x) = \frac{d(\sin 2x)}{dx}$

⇒ $f'(x) = \cos 2x \frac{d(2x)}{dx}$

⇒ f'(x) = 2cos2x

We have f'(c) = 0,

⇒ 2 cos 2c = 0

⇒ $2c = \frac{\pi}{2}$

⇒ $c = \frac{\pi}{4} \in \left(0, \frac{\pi}{2}\right)$

∴ Rolle's Theorem is verified.

(iii) f (x) = cos 2x on [-π/4, π/4]

Solution:

Given function is cos 2x on $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$

We know that cosine function is continuous and differentiable on R. Let's find the values of the function at an extreme,

$\Rightarrow f\left(-\frac{\pi}{4}\right) = \cos 2\left(-\frac{\pi}{4}\right)$

$\Rightarrow f(0) = \cos\left(-\frac{\pi}{2}\right)$

We know that cos (– x) = cos x

⇒ f (0) = 0

$\Rightarrow f\left(\frac{\pi}{4}\right) = \cos 2\left(\frac{\pi}{4}\right)$

$\Rightarrow f\left(\frac{\pi}{2}\right) = \cos\left(\frac{\pi}{2}\right)$

$\Rightarrow f\left(\frac{\pi}{2}\right) = 0$

We have $f\left(-\frac{\pi}{4}\right) = f\left(\frac{\pi}{4}\right)$, so there exist a $c \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$ such that f'(c) = 0.

Let's find the derivative of f(x)

$\Rightarrow f'(x) = \frac{d(\cos 2x)}{dx}$

$\Rightarrow f'(x) = -\sin 2x \frac{d(2x)}{dx}$

⇒ f'(x) = – 2sin2x

We have f'(c) = 0,

⇒ – 2sin2c = 0

sin 2c = 0

⇒ 2c = 0

So,

c = 0 as c ∈ (-π/4, π/4)

∴ Rolle's Theorem is verified.

(iv) f (x) = e^x sin x on [0, π]

Solution:

Given function is f (x) = e^x sin x on [0, π]

We know that exponential and sine functions are continuous and differentiable on R.
Let's find the values of the function at an extreme,

$\Rightarrow$ f (0) = $e^0 \sin(0)$

$\Rightarrow$ f (0) = 1×0

$\Rightarrow$ f (0) = 0

$\Rightarrow f(\pi) = e^{\pi}\sin(\pi)$

$\Rightarrow f(\pi) = e^{\pi} \times 0$

$\Rightarrow f(\pi) = 0$

We have $f(0) = f(\pi)$, so there exist a $c \epsilon (0, \pi)$ such that f'(c) = 0.

Let's find the derivative of f(x)

$\Rightarrow f'(x) = \frac{d(e^x \sin x)}{dx}$

$\Rightarrow f'(x) = \sin x \frac{d(e^x)}{dx} + e^x \frac{d(\sin x)}{dx}$

$\Rightarrow$ f'(x) = e^x (sin x + cos x)

We have f'(c) = 0,

$\Rightarrow$ e^c (sin c + cos c) = 0

$\Rightarrow$ sin c + cos c = 0

$\Rightarrow \frac{1}{\sqrt{2}}\sin c + \frac{1}{\sqrt{2}}\cos c = 0$

$\Rightarrow \sin\left(\frac{\pi}{4}\right)\sin c + \cos\left(\frac{\pi}{4}\right)\cos c = 0$

$\Rightarrow \cos\left(c - \frac{\pi}{4}\right) = 0$

$\Rightarrow c - \frac{\pi}{4} = \frac{\pi}{2}$

$\Rightarrow c = \frac{3\pi}{4} \epsilon (0, \pi)$

∴ Rolle's Theorem is verified.

(v) f (x) = e^x cos x on [-π/2, π/2]

Solution:

Given function is f(x) = e^x cos x on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

We know that exponential and cosine functions are continuous and differentiable on R. Let's find the values of the function at an extreme,

$\Rightarrow f\left(-\frac{\pi}{2}\right) = e^{-\frac{\pi}{2}}\cos\left(-\frac{\pi}{2}\right)$

$\Rightarrow f\left(-\frac{\pi}{2}\right) = e^{-\frac{\pi}{2}} \times 0$

$\Rightarrow f\left(-\frac{\pi}{2}\right) = 0$

$\Rightarrow f\left(\frac{\pi}{2}\right) = e^{\frac{\pi}{2}}\cos\left(\frac{\pi}{2}\right)$

$\Rightarrow f(\pi) = e^{\frac{\pi}{2}} \times 0$

$\Rightarrow f(\pi) = 0$

We have $f\left(-\frac{\pi}{2}\right) = f\left(\frac{\pi}{2}\right)$, so there exist a $c \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ such that f'(c) = 0.

Let's find the derivative of f(x)

$\Rightarrow f'(x) = \frac{d(e^x \cos x)}{dx}$

$\Rightarrow f'(x) = \cos x\frac{d(e^x)}{dx} + e^x\frac{d(\cos x)}{dx}$

$\Rightarrow f'(x) = e^x(-\sin x + \cos x)$

We have f'(c) = 0,

$\Rightarrow e^c(-\sin c + \cos c) = 0$

$\Rightarrow -\sin c + \cos c = 0$

$\Rightarrow \frac{-1}{\sqrt{2}}\sin c + \frac{1}{\sqrt{2}}\cos c = 0$

$\Rightarrow -\sin\left(\frac{\pi}{4}\right)\sin c + \cos\left(\frac{\pi}{4}\right)\cos c = 0$

$\Rightarrow \cos\left(c + \frac{\pi}{4}\right) = 0$

$\Rightarrow c + \frac{\pi}{4} = \frac{\pi}{2}$

$\Rightarrow c = \frac{\pi}{4} \epsilon \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

∴ Rolle's Theorem is verified.

(vi) f (x) = cos 2x on [0, π]

Solution:

Given function is f (x) = cos 2x on [0, π]

We know that cosine function is continuous and differentiable on R. Let's find the values of function at extreme,

⇒ f (0) = cos2(0)

⇒ f (0) = cos(0)

⇒ f (0) = 1

⇒ f (π) = cos2(π)

⇒ f (π) = cos(2 π)

⇒ f (π) = 1

We have f (0) = f (π), so there exist a c belongs to (0, π) such that f'(c) = 0.

Let's find the derivative of f(x)

$\Rightarrow f'(x) = \frac{d(\cos 2x)}{dx}$

$\Rightarrow f'(x) = -\sin 2x \frac{d(2x)}{dx}$

⇒ f'(x) = – 2sin2x

We have f'(c) = 0,

⇒ – 2sin2c = 0

sin 2c = 0

So, 2c = 0 or π

c = 0 or π/2

But,
c = π/2 as c ∈ (0, π)
Hence, Rolle's Theorem is verified.

(vii) $f(x) = \frac{\sin x}{e^x}$ on $0 \le x \le \pi$

Solution:

Given function is $f(x) = \frac{\sin x}{e^x}$ on $[0, \pi]$

This can be written as

$\Rightarrow f(x) = e^{-x} \sin x$ on $[0, \pi]$

We know that exponential and sine functions are continuous and differentiable on R. Let's find the values of the function at an extreme,

$\Rightarrow f(0) = e^{-0}\sin(0)$

$\Rightarrow f(0) = 1 \times 0$

$\Rightarrow f(0) = 0$

$\Rightarrow f(\pi) = e^{-\pi}\sin(\pi)$

$\Rightarrow f(\pi) = e^{-\pi} \times 0$

$\Rightarrow f(\pi) = 0$

We have f (0) = f (π), so there exist a c belongs to (0, π) such that f'(c) = 0.

Let's find the derivative of f(x)

$\Rightarrow f'(x) = \frac{d(e^{-x}\sin x)}{dx}$

$\Rightarrow f'(x) = \sin x \frac{d(e^{-x})}{dx} + e^{-x}\frac{d(\sin x)}{dx}$

$\Rightarrow f'(x) = \sin x(-e^{-x}) + e^{-x}(\cos x)$

⇒ $f'(x) = e^{-x}(-\sin x + \cos x)$

We have $f'(c) = 0$,

⇒ $e^{-c}(-\sin c + \cos c) = 0$

⇒ $-\sin c + \cos c = 0$

⇒ $-\frac{1}{\sqrt{2}}\sin c + \frac{1}{\sqrt{2}}\cos c = 0$

⇒ $-\sin\left(\frac{\pi}{4}\right)\sin c + \cos\left(\frac{\pi}{4}\right)\cos c = 0$

⇒ $\cos\left(c + \frac{\pi}{4}\right) = 0$

⇒ $c + \frac{\pi}{4} = \frac{\pi}{2}$

⇒ $c = \frac{\pi}{4} \in (0, \pi)$

∴ Rolle's Theorem is verified.

(viii) f (x) = sin 3x on [0, π]

Solution:

Given function is f (x) = sin3x on [0, π]

We know that sine function is continuous and differentiable on R. Let's find the values of function at extreme,

⇒ f (0) = sin3(0)

⇒ f (0) = sin0

⇒ f (0) = 0

⇒ f (π) = sin3(π)

⇒ f (π) = sin(3 π)

⇒ f (π) = 0

We have f (0) = f (π), so there exist a c belongs to (0, π) such that f'(c) = 0.

Let's find the derivative of f(x)

$\Rightarrow f'(x) = \frac{d(\sin 3x)}{dx}$

$\Rightarrow f'(x) = \cos 3x \frac{d(3x)}{dx}$

$\Rightarrow f'(x) = 3\cos 3x$

We have $f'(c) = 0$,

$\Rightarrow 3\cos 3c = 0$

$\Rightarrow 3c = \frac{\pi}{2}$

$\Rightarrow c = \frac{\pi}{6} \epsilon (0, \pi)$

∴ Rolle's Theorem is verified.

(ix) $f(x) = e^{1-x^2}$ on $[-1, 1]$

Solution:

Given function is $f(x) = e^{1-x^2}$ on [– 1, 1]

We know that exponential function is continuous and differentiable over R.
Let's find the value of function f at extremes,

$\Rightarrow f(-1) = e^{1-(-1)^2}$

$\Rightarrow f(-1) = e^{1-1}$

$\Rightarrow f(-1) = e^0$

$\Rightarrow f(-1) = 1$

$\Rightarrow f(1) = e^{1-1^2}$

$\Rightarrow f(1) = e^{1-1}$

$\Rightarrow f(1) = e^0$

$\Rightarrow f(1) = 1$

We got f (– 1) = f (1) so, there exists a c ϵ (– 1, 1) such that f'(c) = 0.

Let's find the derivative of the function f:

$\Rightarrow f'(x) = \frac{d\left(e^{1-x^2}\right)}{dx}$

$\Rightarrow f'(x) = e^{1-x^2}\frac{d(1-x^2)}{dx}$

$\Rightarrow f'(x) = e^{1-x^2}(-2x)$

We have f'(c) = 0

$\Rightarrow e^{1-c^2}(-2c) = 0$

$\Rightarrow 2c = 0$

$\Rightarrow c = 0 \in [-1, 1]$

∴ Rolle's Theorem is verified.

(x) f (x) = log (x^2 + 2) – log 3 on [-1, 1]

Solution:

Given function is f (x) = log(x^2 + 2) – log3 on [– 1, 1]

We know that logarithmic function is continuous and differentiable in its own domain.

We check the values of the function at the extreme,

⇒ f (– 1) = log(($-1)^2$ + 2) – log 3

⇒ f (– 1) = log (1 + 2) – log 3

⇒ f (– 1) = log 3 – log 3

⇒ f (– 1) = 0

⇒ f (1) = log (1^2 + 2) – log 3

⇒ f (1) = log (1 + 2) – log 3

⇒ f (1) = log 3 – log 3

⇒ f (1) = 0

We have got f (– 1) = f (1). So, there exists a c such that c ϵ (– 1, 1) such that f'(c) = 0.

Let's find the derivative of the function f,

$$\Rightarrow f'(x) = \frac{d(\log(x^2+2)-\log 3)}{dx}$$

$$\Rightarrow f'(x) = \frac{1}{x^2+2}\frac{d(x^2+2)}{dx} - 0$$

$$\Rightarrow f'(x) = \frac{2x}{x^2+2}$$

We have f'(c) = 0

$$\Rightarrow \frac{2c}{c^2+2} = 0$$

⇒ 2c = 0

⇒ c = 0 ϵ (– 1, 1)

∴ Rolle's Theorem is verified.

(xi) f (x) = sin x + cos x on [0, π/2]

Solution:

Given function is f(x) = sin x + cos x on $\left[0, \frac{\pi}{2}\right]$

We know that sine and cosine functions are continuous and differentiable on R. Let's the value of function f at extremes:

⇒ f (0) = sin (0) + cos (0)

⇒ f (0) = 0 + 1

⇒ f (0) = 1

⇒ $f\left(\frac{\pi}{2}\right) = \sin\left(\frac{\pi}{2}\right) + \cos\left(\frac{\pi}{2}\right)$

⇒ $f\left(\frac{\pi}{2}\right) = 1 + 0$

⇒ $f\left(\frac{\pi}{2}\right) = 1$

We have $f(0) = f\left(\frac{\pi}{2}\right)$. So, there exists a c ϵ $\left(0, \frac{\pi}{2}\right)$ such that f'(c) = 0.

Let's find the derivative of the function f.

⇒ $f'(x) = \frac{d(\sin x + \cos x)}{dx}$

⇒ f'(x) = cos x – sin x

We have f’(c) = 0

⇒ Cos c – sin c = 0

$\Rightarrow \frac{1}{\sqrt{2}}\cos c - \frac{1}{\sqrt{2}}\sin c = 0$

$\Rightarrow \sin\left(\frac{\pi}{4}\right)\cos c - \cos\left(\frac{\pi}{4}\right)\sin c = 0$

$\Rightarrow \sin\left(\frac{\pi}{4} - c\right) = 0$

$\Rightarrow \frac{\pi}{4} - c = 0$

$\Rightarrow c = \frac{\pi}{4} \in \left(0, \frac{\pi}{2}\right)$

∴ Rolle’s Theorem is verified.

(xii) f (x) = 2 sin x + sin 2x on [0, π]

Solution:

Given function is f (x) = 2sinx + sin2x on [0, π]
We know that sine function continuous and differentiable over R.
Let’s check the values of function f at the extremes
⇒ f (0) = 2sin(0) + sin2(0)
⇒ f (0) = 2(0) + 0
⇒ f (0) = 0
⇒ f (π) = 2sin(π) + sin2(π)
⇒ f (π) = 2(0) + 0
⇒ f (π) = 0
We have f (0) = f (π), so there exist a c belongs to (0, π) such that f’(c) = 0.
Let’s find the derivative of function f.

$\Rightarrow f'(x) = \frac{d(2\sin x + \sin 2x)}{dx}$

$\Rightarrow f'(x) = 2\cos x + \cos 2x \frac{d(2x)}{dx}$

⇒ f’(x) = 2cosx + 2cos2x
⇒ f’(x) = 2cosx + 2(2cos^{2}x – 1)

$\Rightarrow f'(x) = 4\cos^2 x + 2\cos x - 2$

We have $f'(c) = 0$,

$\Rightarrow 4\cos^2 c + 2\cos c - 2 = 0$

$\Rightarrow 2\cos^2 c + \cos c - 1 = 0$

$\Rightarrow 2\cos^2 c + 2\cos c - \cos c - 1 = 0$

$\Rightarrow 2\cos c(\cos c + 1) - 1(\cos c + 1) = 0$

$\Rightarrow (2\cos c - 1)(\cos c + 1) = 0$

$\Rightarrow \cos c = \frac{1}{2} \text{ or } \cos c = -1$

$\Rightarrow c = \frac{\pi}{3} \in (0, \pi)$

∴ Rolle's Theorem is verified.

(xiii) $f(x) = \frac{x}{2} - \sin\frac{\pi x}{6}$ **on** $[-1, 0]$

Solution:

Given function is $f(x) = \frac{x}{2} - \sin\left(\frac{\pi x}{6}\right)$ on $[-1, 0]$

We know that sine function is continuous and differentiable over R.

Now we have to check the values of 'f' at an extreme

$\Rightarrow f(-1) = \frac{-1}{2} - \sin\left(\frac{\pi(-1)}{6}\right)$

$\Rightarrow f(-1) = -\frac{1}{2} - \sin\left(\frac{-\pi}{6}\right)$

$\Rightarrow f(-1) = -\frac{1}{2} - \left(-\frac{1}{2}\right)$

$\Rightarrow f(-1) = 0$

$\Rightarrow f(0) = \frac{0}{2} - \sin\left(\frac{\pi(0)}{6}\right)$

$\Rightarrow f(0) = 0 - \sin(0)$

$\Rightarrow f(0) = 0 - 0$

$\Rightarrow$ f (0) = 0

We have got f (– 1) = f (0). So, there exists a c ϵ (– 1, 0) such that f'(c) = 0.

Now we have to find the derivative of the function 'f'

$\Rightarrow f'(x) = \frac{d\left(\frac{x}{2}-\sin\left(\frac{\pi x}{6}\right)\right)}{dx}$

$\Rightarrow f'(x) = \frac{1}{2} - \cos\left(\frac{\pi x}{6}\right)\frac{d\left(\frac{\pi x}{6}\right)}{dx}$

$\Rightarrow f'(x) = \frac{1}{2} - \frac{\pi}{6}\cos\left(\frac{\pi x}{6}\right)$

We have f'(c) = 0

$\Rightarrow \frac{1}{2} - \frac{\pi}{6}\cos\left(\frac{\pi c}{6}\right) = 0$

$\Rightarrow \frac{\pi}{6}\cos\left(\frac{\pi c}{6}\right) = \frac{1}{2}$

$\Rightarrow \cos\left(\frac{\pi c}{6}\right) = \frac{1}{2} \times \frac{6}{\pi}$

$\Rightarrow \cos\left(\frac{\pi c}{6}\right) = \frac{3}{\pi}$

$\Rightarrow \frac{\pi c}{6} = \cos^{-1}\left(\frac{3}{\pi}\right)$

$\Rightarrow c = \frac{6}{\pi}\cos^{-1}\left(\frac{3}{\pi}\right)$

Cosine is positive between $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, for our convenience we take the interval to be $-\frac{\pi}{2} \le \theta \le 0$, since the values of the cosine repeats.

We know that $\frac{3}{\pi}$ value is nearly equal to 1. So, the value of the c nearly equal to 0.

So, we can clearly say that cϵ (– 1, 0).

∴ Rolle's Theorem is verified.

(xiv). $f(x) = \frac{6x}{\pi} - 4\sin^2 x$ **on** $[0, \frac{\pi}{6}]$

Solution:

Given function is $f(x) = \frac{6x}{\pi} - 4\sin^2 x$ on $\left[0, \frac{\pi}{6}\right]$

We know that sine function is continuous and differentiable over R.

Now we have to check the values of function 'f' at the extremes,

$\Rightarrow f(0) = \frac{6(0)}{\pi} - 4\sin^2(0)$

$\Rightarrow f(0) = 0 - 4(0)$

$\Rightarrow f(0) = 0$

$\Rightarrow f\left(\frac{\pi}{6}\right) = \frac{6\left(\frac{\pi}{6}\right)}{\pi} - 4\sin^2\left(\frac{\pi}{6}\right)$

$\Rightarrow f\left(\frac{\pi}{6}\right) = \frac{\pi}{\pi} - 4\left(\frac{1}{2}\right)^2$

$\Rightarrow f\left(\frac{\pi}{6}\right) = 1 - 4\left(\frac{1}{4}\right)$

$\Rightarrow f\left(\frac{\pi}{6}\right) = 1 - 1$

$\Rightarrow f\left(\frac{\pi}{6}\right) = 0$.

We have $f(0) = f\left(\frac{\pi}{6}\right)$. So, there exists a c $\in \left(0, \frac{\pi}{6}\right)$ such that f'(c) = 0.

We have to find the derivative of function 'f.'

$\Rightarrow f'(x) = \frac{d\left(\frac{6x}{\pi} - 4\sin^2 x\right)}{dx}$

$\Rightarrow f'(x) = \frac{6}{\pi} - 4 \times 2\sin x \times \frac{d(\sin x)}{dx}$

$\Rightarrow f'(x) = \frac{6}{\pi} - 8\sin x(\cos x)$

$\Rightarrow f'(x) = \frac{6}{\pi} - 4(2\sin x\cos x)$

$\Rightarrow f'(x) = \frac{6}{\pi} - 4\sin 2x$

We have f'(c) = 0

$\Rightarrow \frac{6}{\pi} - 4\sin 2c = 0$

$\Rightarrow 4\sin 2c = \frac{6}{\pi}$

$\Rightarrow \sin 2c = \frac{6}{4\pi}$

We know $\frac{6}{4\pi} < \frac{1}{2}$

$\Rightarrow \sin 2c < \frac{1}{2}$

$\Rightarrow 2c < \sin^{-1}\left(\frac{1}{2}\right)$

$\Rightarrow 2c < \frac{\pi}{6}$

$\Rightarrow c < \frac{\pi}{12} \epsilon \left(0, \frac{\pi}{6}\right)$

∴ Rolle's Theorem is verified.

(xv) f (x) = $4^{\sin x}$ on [0, π]

Solution:

Given function is f (x) = $4^{\sin x}$ on [0, π]

We that sine function is continuous and differentiable over R.

Now we have to check the values of function 'f' at extremes

⇒ f (0) = $4^{\sin (0)}$

⇒ f (0) = 4^0

⇒ f (0) = 1

⇒ f (π) = $4^{\sin\pi}$

⇒ f (π) = 4^0

⇒ f (π) = 1

We have f (0) = f (π). So, there exists a c ϵ (0, π) such that f'(c) = 0.

Now we have to find the derivative of 'f'

$\Rightarrow f'(x) = \frac{d(4^{\sin x})}{dx}$

$\Rightarrow f'(x) = 4^{\sin x}\log 4\frac{d(\sin x)}{dx}$

$\Rightarrow f'(x) = 4^{\sin x}\log 4\cos x$

We have f'(c) = 0

⇒ $4^{\sin c}\log 4\cos c = 0$

⇒ Cos c = 0

$\Rightarrow c = \frac{\pi}{2}\epsilon(0,\pi)$

∴ Rolle's Theorem is verified.

(xvi) f (x) = x^2 – 5x + 4 on [0, π/6]

Solution:
Given function is f (x) = x^2 – 5x + 4 on [1, 4]
Since, given function f is a polynomial it is continuous and differentiable everywhere i.e., on R.
Let us find the values at extremes
⇒ f (1) = 1^2 – 5(1) + 4
⇒ f (1) = 1 – 5 + 4
⇒ f (1) = 0
⇒ f (4) = 4^2 – 5(4) + 4
⇒ f (4) = 16 – 20 + 4
⇒ f (4) = 0
We have f (1) = f (4). So, there exists a c ϵ (1, 4) such that f'(c) = 0.

Let's find the derivative of f(x):

$\Rightarrow f'(x) = \frac{d(x^2 - 5x + 4)}{dx}$

$\Rightarrow f'(x) = \frac{d(x^2)}{dx} - \frac{d(5x)}{dx} + \frac{d(4)}{dx}$

$\Rightarrow f'(x) = 2x - 5 + 0$

$\Rightarrow f'(x) = 2x - 5$

We have f'(c) = 0

$\Rightarrow f'(c) = 0$

$\Rightarrow 2c - 5 = 0$

$\Rightarrow 2c = 5$

$\Rightarrow c = \frac{5}{2}$

$\Rightarrow C = 2.5 \in (1, 4)$

∴ Rolle's Theorem is verified.

(xvii) f (x) = $\sin^4 x + \cos^4 x$ on [0, π/2]

Solution:

Given function is f (x) = $\sin^4 x + \cos^4 x$ on $\left[0, \frac{\pi}{2}\right]$

We know that sine and cosine functions are continuous and differentiable functions over R.

Now we have to find the value of function 'f' at extremes

$\Rightarrow f(0) = \sin^4(0) + \cos^4(0)$

$\Rightarrow f(0) = (0)^4 + (1)^4$

$\Rightarrow f(0) = 0 + 1$

$\Rightarrow f(0) = 1$

$\Rightarrow f\left(\frac{\pi}{2}\right) = \sin^4\left(\frac{\pi}{2}\right) + \cos^4\left(\frac{\pi}{2}\right)$

$\Rightarrow f\left(\frac{\pi}{2}\right) = 1^4 + 0^4$

$\Rightarrow f\left(\frac{\pi}{2}\right) = 1 + 0$

$\Rightarrow f\left(\frac{\pi}{2}\right) = 1$

We have $f(0) = f\left(\frac{\pi}{2}\right)$. So, there exists a c $\epsilon \left(0, \frac{\pi}{2}\right)$ such that f'(c) = 0.

Now we have to find the derivative of the function 'f'.

$\Rightarrow f'(x) = \frac{d(\sin^4 x + \cos^4 x)}{dx}$

$\Rightarrow f'(x) = 4\sin^3 x\frac{d(\sin x)}{dx} + 4\cos^3 x\frac{d(\cos x)}{dx}$

$\Rightarrow f'(x) = 4\sin^3 x\cos x - 4\cos^3 x\sin x$

$\Rightarrow f'(x) = 4 \sin x \cos x (\sin^2 x - \cos^2 x)$

$\Rightarrow f'(x) = 2(2 \sin x \cos x) (-\cos 2x)$

$\Rightarrow f'(x) = -2(\sin 2x) (\cos 2x)$

$\Rightarrow f'(x) = -\sin 4x$

We have f'(c) = 0

$\Rightarrow -\sin 4c = 0$

$\Rightarrow \sin 4c = 0$

$\Rightarrow 4c = 0$ or π

$\Rightarrow c = \frac{\pi}{4} \epsilon \left(0, \frac{\pi}{2}\right)$

∴ Rolle's Theorem is verified.

(xviii) f (x) = sin x – sin 2x on [0, π]

Solution:

Given function is f (x) = sin x – sin2x on [0, π]

We know that sine function is continuous and differentiable over R.

Now we have to check the values of the function 'f' at the extremes.

⇒ f (0) = sin (0)–sin 2(0)

⇒ f (0) = 0 – sin (0)

⇒ f (0) = 0

⇒ f (π) = sin(π) – sin2(π)

⇒ f (π) = 0 – sin(2π)

⇒ f (π) = 0

We have f (0) = f (π). So, there exists a c ϵ (0, π) such that f'(c) = 0.

Now we have to find the derivative of the function 'f'

$$\Rightarrow f'(x) = \frac{d(\sin x - \sin 2x)}{dx}$$

$$\Rightarrow f'(x) = \cos x - \cos 2x \frac{d(2x)}{dx}$$

$$\Rightarrow f'(x) = \cos x - 2\cos 2x$$

$$\Rightarrow f'(x) = \cos x - 2(2\cos^2 x - 1)$$

$$\Rightarrow f'(x) = \cos x - 4\cos^2 x + 2$$

We have f'(c) = 0

$$\Rightarrow \text{Cos } c - 4\cos^2 c + 2 = 0$$

$$\Rightarrow \cos c = \frac{-1 \pm \sqrt{(1)^2 - (4 \times -4 \times 2)}}{2 \times -4}$$

$$\Rightarrow \cos c = \frac{-1 \pm \sqrt{1 + 33}}{-8}$$

$$\Rightarrow c = \cos^{-1}\left(\frac{-1 \pm \sqrt{33}}{-8}\right)$$

We can see that c ϵ (0, π)

∴ Rolle's Theorem is verified.

4. Using Rolle's Theorem, find points on the curve $y = 16 - x^2$, x ∈ [-1, 1], where tangent is parallel to x – axis.

Solution:

Given function is $y = 16 - x^2$, $x \in [-1, 1]$

We know that polynomial function is continuous and differentiable over R.

Let us check the values of 'y' at extremes

$\Rightarrow y(-1) = 16 - (-1)^2$

$\Rightarrow y(-1) = 16 - 1$

$\Rightarrow y(-1) = 15$

$\Rightarrow y(1) = 16 - (1)^2$

$\Rightarrow y(1) = 16 - 1$

$\Rightarrow y(1) = 15$

We have $y(-1) = y(1)$. So, there exists a $c \in (-1, 1)$ such that $f'(c) = 0$.

We know that for a curve g, the value of the slope of the tangent at a point r is given by $g'(r)$.

Now we have to find the derivative of curve y

$\Rightarrow y' = \frac{d(16-x^2)}{dx}$

$\Rightarrow y' = -2x$

We have $y'(c) = 0$

$\Rightarrow -2c = 0$

$\Rightarrow c = 0 \in (-1, 1)$

Value of y at x = 1 is

$\Rightarrow y = 16 - 0^2$

$\Rightarrow y = 16$

$\therefore$ The point at which the curve y has a tangent parallel to x – axis (since the slope of x – axis is 0) is (0, 16).