EXERCISE 15.2 PAGE NO: 15.17

1. Verify Lagrange's mean value theorem for the following functions on the indicated intervals. In each case find a point 'c' in the indicated interval as stated by the Lagrange's mean value theorem:
(i) f (x) = $x^2 - 1$ on [2, 3]

Solution:

Given f (x) = $x^2 - 1$ on [2, 3]

We know that every polynomial function is continuous everywhere on (−∞, ∞) and differentiable for all arguments. Here, f(x) is a polynomial function. So it is continuous in [2, 3] and differentiable in (2, 3). So both the necessary conditions of Lagrange's mean value theorem is satisfied.

Therefore, there exist a point c ∈ (2, 3) such that:

$$f'(c) = \frac{f(3) - f(2)}{3 - 2}$$

$$\Rightarrow f'(c) = \frac{f(3) - f(2)}{1}$$

f (x) = $x^2 - 1$

Differentiating with respect to x

f'(x) = 2x

For f'(c), put the value of x=c in f'(x):

f'(c) = 2c

For f (3), put the value of x=3 in f(x):

f (3) = $(3)^2 - 1$

= 9 – 1

= 8

For f (2), put the value of x=2 in f(x):

$f(2) = (2)^2 - 1$

$= 4 - 1$

$= 3$

$\therefore f'(c) = f(3) - f(2)$

$\Rightarrow 2c = 8 - 3$

$\Rightarrow 2c = 5$

$\Rightarrow c = \frac{5}{2} \in (2, 3)$

Hence, Lagrange's mean value theorem is verified.

(ii) $f(x) = x^3 - 2x^2 - x + 3$ on [0, 1]

Solution:

Given $f(x) = x^3 - 2x^2 - x + 3$ on [0, 1]
Every polynomial function is continuous everywhere on $(-\infty, \infty)$ and differentiable for all arguments. Here, f(x) is a polynomial function. So it is continuous in [0, 1] and differentiable in (0, 1). So both the necessary conditions of Lagrange's mean value theorem is satisfied.
Therefore, there exist a point $c \in (0, 1)$ such that:

$$f'(c) = \frac{f(1) - f(0)}{1 - 0}$$

$$\Rightarrow f'(c) = \frac{f(1) - f(0)}{1}$$

$f(x) = x^3 - 2x^2 - x + 3$
Differentiating with respect to x
$f'(x) = 3x^2 - 2(2x) - 1$
$= 3x^2 - 4x - 1$
For f'(c), put the value of x=c in f'(x)
$f'(c)= 3c^2 - 4c - 1$
For f (1), put the value of x = 1 in f(x)
$f(1)= (1)^3 - 2(1)^2 - (1) + 3$

= 1 – 2 – 1 + 3

= 1

For f (0), put the value of x=0 in f(x)

$f(0) = (0)^3 - 2(0)^2 - (0) + 3$

= 0 – 0 – 0 + 3

= 3

∴ f’(c) = f(1) – f(0)

$\Rightarrow 3c^2 - 4c - 1 = 1 - 3$

$\Rightarrow 3c^2 - 4c = 1 + 1 - 3$

$\Rightarrow 3c^2 - 4c = -1$

$\Rightarrow 3c^2 - 4c + 1 = 0$

$\Rightarrow 3c^2 - 3c - c + 1 = 0$

$\Rightarrow 3c(c - 1) - 1(c - 1) = 0$

$\Rightarrow (3c - 1)(c - 1) = 0$

$$\Rightarrow c = \frac{1}{3}, 1$$

$$\Rightarrow c = \frac{1}{3} \in (0, 1)$$

Hence, Lagrange’s mean value theorem is verified.

(iii) f (x) = x (x – 1) on [1, 2]

Solution:

Given f (x) = x (x – 1) on [1, 2]

$= x^2 - x$

Every polynomial function is continuous everywhere on (−∞, ∞) and differentiable for all arguments. Here, f(x) is a polynomial function. So it is continuous in [1, 2] and differentiable in (1, 2). So both the necessary conditions of Lagrange’s mean value theorem is satisfied.

Therefore, there exist a point c ∈ (1, 2) such that:

$$f'(c) = \frac{f(2) - f(1)}{2 - 1}$$

$$\Rightarrow f'(c) = \frac{f(2) - f(1)}{1}$$

$f(x) = x^2 - x$

Differentiating with respect to x
f'(x) = 2x – 1
For f'(c), put the value of x=c in f'(x):
f'(c)= 2c – 1
For f (2), put the value of x = 2 in f(x)
f (2) = $(2)^2$ – 2
= 4 – 2
= 2
For f (1), put the value of x = 1 in f(x):
f (1)= $(1)^2$ – 1
= 1 – 1
= 0
∴ f'(c) = f(2) – f(1)
⇒ 2c – 1 = 2 – 0
⇒ 2c = 2 + 1
⇒ 2c = 3
$\Rightarrow c = \frac{3}{2} \in (1, 2)$

Hence, Lagrange's mean value theorem is verified.

(iv) f (x) = x^2 – 3x + 2 on [-1, 2]

Solution:
Given f (x) = x^2 – 3x + 2 on [– 1, 2]
Every polynomial function is continuous everywhere on (−∞, ∞) and differentiable for all arguments. Here, f(x) is a polynomial function. So it is continuous in [– 1, 2] and differentiable in (– 1, 2). So both the necessary conditions of Lagrange's mean value theorem is satisfied.
Therefore, there exist a point c ∈ (-1, 2) such that:

$$f'(c) = \frac{f(2) - f(-1)}{2 - (-1)}$$

$$\Rightarrow f'(c) = \frac{f(2) - f(-1)}{2 + 1}$$

$$\Rightarrow f'(c) = \frac{f(2) - f(-1)}{3}$$

$f(x) = x^2 - 3x + 2$

Differentiating with respect to x

$f'(x) = 2x - 3$

For f'(c), put the value of x = c in f'(x):

$f'(c) = 2c - 3$

For f (2), put the value of x = 2 in f(x)

$f(2) = (2)^2 - 3(2) + 2$

$= 4 - 6 + 2$

$= 0$

For f (– 1), put the value of x = – 1 in f(x):

$f(-1) = (-1)^2 - 3(-1) + 2$

$= 1 + 3 + 2$

$= 6$

$$f'(c) = \frac{f(2) - f(-1)}{3}$$

$$\Rightarrow 2c - 3 = \frac{0 - 6}{3}$$

$$\Rightarrow 2c = \frac{-6}{3} + 3$$

$\Rightarrow 2c = -2 + 3$

$\Rightarrow 2c = 1$

$\Rightarrow c = ½ \in (-1, 2)$

Hence, Lagrange's mean value theorem is verified.

(v) f (x) = $2x^2 - 3x + 1$ on [1, 3]

Solution:

Given $f(x) = 2x^2 - 3x + 1$ on [1, 3]

Every polynomial function is continuous everywhere on (−∞, ∞) and differentiable for all arguments. Here, f(x) is a polynomial function. So it is continuous in [1, 3] and differentiable in (1, 3). So both the necessary conditions of Lagrange's mean value theorem is satisfied.

Therefore, there exist a point c ∈ (1, 3) such that:

$$f'(c) = \frac{f(3) - f(1)}{3 - 1}$$

$$\Rightarrow f'(c) = \frac{f(3) - f(1)}{2}$$

f (x) = $2x^2 - 3x + 1$
Differentiating with respect to x
f'(x) = 2(2x) – 3
= 4x – 3
For f'(c), put the value of x = c in f'(x):
f'(c)= 4c – 3
For f (3), put the value of x = 3 in f(x):
f (3) = $2(3)^2 - 3(3) + 1$
= 2 (9) – 9 + 1
= 18 – 8 = 10
For f (1), put the value of x = 1 in f(x):
f (1) = $2(1)^2 - 3(1) + 1$
= 2 (1) – 3 + 1
= 2 – 2 = 0

$$f'(c) = \frac{f(3) - f(1)}{2}$$

$$\Rightarrow 4c - 3 = \frac{10 - 0}{2}$$

$$\Rightarrow 4c = \frac{10}{2} + 3$$

$$\Rightarrow 4c = 5 + 3$$

$$\Rightarrow 4c = 8$$

$$\Rightarrow c = \frac{8}{4} = 2 \in (1, 3)$$

Hence, Lagrange's mean value theorem is verified.

(vi) f (x) = $x^2 - 2x + 4$ on [1, 5]

Solution:

Given f (x) = $x^2 - 2x + 4$ on [1, 5]

Every polynomial function is continuous everywhere on (−∞, ∞) and differentiable for all arguments. Here, f(x) is a polynomial function. So it is continuous in [1, 5] and differentiable in (1, 5). So both the necessary conditions of Lagrange's mean value theorem is satisfied.

Therefore, there exist a point c ∈ (1, 5) such that:

$$f'(c) = \frac{f(5) - f(1)}{5-1}$$

$$\Rightarrow f'(c) = \frac{f(5) - f(1)}{4}$$

f (x) = $x^2 - 2x + 4$

Differentiating with respect to x:

f'(x) = 2x – 2

For f'(c), put the value of x=c in f'(x):

f'(c)= 2c – 2

For f (5), put the value of x=5 in f(x):

f (5)= $(5)^2 - 2(5) + 4$

= 25 – 10 + 4

= 19

For f (1), put the value of x = 1 in f(x)

f (1) = $(1)^2 - 2(1) + 4$

= 1 – 2 + 4

= 3

$$f'(c) = \frac{f(5) - f(1)}{4}$$

$$\Rightarrow 2c - 2 = \frac{19-3}{4}$$

$$\Rightarrow 2c = \frac{16}{4} + 2$$

$$\Rightarrow 2c = 4 + 2$$

$$\Rightarrow 2c = 6$$

$$\Rightarrow c = \frac{6}{2} = 3 \in (1, 5)$$

Hence, Lagrange's mean value theorem is verified.

(vii) f (x) = 2x – x^2 on [0, 1]

Solution:

Given f (x) = 2x – x^2 on [0, 1]

Every polynomial function is continuous everywhere on (−∞, ∞) and differentiable for all arguments. Here, f(x) is a polynomial function. So it is continuous in [0, 1] and differentiable in (0, 1). So both the necessary conditions of Lagrange's mean value theorem is satisfied.

Therefore, there exist a point c ∈ (0, 1) such that:

$$f'(c) = \frac{f(1) - f(0)}{1 - 0}$$

⇒ f'(c) = f(1) – f(0)

f (x) = 2x – x^2

Differentiating with respect to x:

f'(x) = 2 – 2x

For f'(c), put the value of x = c in f'(x):

f'(c)= 2 – 2c

For f (1), put the value of x = 1 in f(x):

f (1)= 2(1) – $(1)^2$

= 2 – 1

= 1

For f (0), put the value of x = 0 in f(x):

f (0) = 2(0) – $(0)^2$

= 0 – 0

= 0

f'(c) = f(1) – f(0)

⇒ 2 – 2c = 1 – 0

⇒ – 2c = 1 – 2

⇒ – 2c = – 1

$$\Rightarrow c = \frac{-1}{-2} = \frac{1}{2} \in (0, 1)$$

Hence, Lagrange's mean value theorem is verified.

(viii) f (x) = (x – 1) (x – 2) (x – 3)

Solution:

Given f (x) = (x – 1) (x – 2) (x – 3) on [0, 4]

$= (x^2 - x - 2x + 2)(x - 3)$
$= (x^2 - 3x + 2)(x - 3)$
$= x^3 - 3x^2 + 2x - 3x^2 + 9x - 6$
$= x^3 - 6x^2 + 11x - 6$ on [0, 4]

Every polynomial function is continuous everywhere on (−∞, ∞) and differentiable for all arguments. Here, f(x) is a polynomial function. So it is continuous in [0, 4] and differentiable in (0, 4). So both the necessary conditions of Lagrange's mean value theorem is satisfied.

Therefore, there exist a point c ∈ (0, 4) such that:

$$f'(c) = \frac{f(4) - f(0)}{4 - 0}$$

$$\Rightarrow f'(c) = \frac{f(4) - f(0)}{4}$$

$f(x) = x^3 - 6x^2 + 11x - 6$

Differentiating with respect to x:

$f'(x) = 3x^2 - 6(2x) + 11$
$= 3x^2 - 12x + 11$

For f'(c), put the value of x = c in f'(x):

$f'(c) = 3c^2 - 12c + 11$

For f (4), put the value of x = 4 in f(x):

$f(4) = (4)^3 - 6(4)^2 + 11(4) - 6$
$= 64 - 96 + 44 - 6$
$= 6$

For f (0), put the value of x = 0 in f(x):

$f(0) = (0)^3 - 6(0)^2 + 11(0) - 6$
$= 0 - 0 + 0 - 6$
$= -6$

$$f'(c) = \frac{f(4) - f(0)}{4}$$

$3c^2 - 12c + 11 = [6 - (-6)]/4$
$3c^2 - 12c + 11 = 12/4$
$3c^2 - 12c + 11 = 3$
$3c^2 - 12c + 8 = 0$

We know that for quadratic equation, $ax^2 + bx + c = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\Rightarrow c = \frac{-(-12) \pm \sqrt{(-12)^2 - 4 \times 3 \times 8}}{2 \times 3}$$

$$\Rightarrow c = \frac{12 \pm \sqrt{144 - 96}}{6}$$

$$\Rightarrow c = \frac{12 \pm \sqrt{48}}{6}$$

$$\Rightarrow c = \frac{12 \pm 4\sqrt{3}}{6}$$

$$\Rightarrow c = \frac{12}{6} \pm \frac{4\sqrt{3}}{6}$$

$$\Rightarrow c = 2 \pm \frac{2\sqrt{3}}{3}$$

$$\Rightarrow c = 2 + \frac{2\sqrt{3}}{3}, 2 - \frac{2\sqrt{3}}{3} \in c$$

Hence, Lagrange's mean value theorem is verified.

(ix). $f(x) = \sqrt{25 - x^2}$ on $[-3, 4]$

Solution:

Given

$f(x) = \sqrt{25 - x^2}$ on $[-3, 4]$

Here, $\sqrt{25 - x^2} > 0$

$\Rightarrow 25 - x^2 > 0$

$\Rightarrow x^2 < 25$

$\Rightarrow -5 < x < 5$

$\Rightarrow \sqrt{25 - x^2}$ has unique values for all $x \in (-5, 5)$

$\therefore$ f (x) is continuous in [– 3, 4]

$$f(x) = (25 - x^2)^{\frac{1}{2}}$$

Differentiating with respect to x:

$$f'(x) = \frac{1}{2}(25 - x^2)^{\left(\frac{1}{2} - 1\right)} \frac{d(25 - x^2)}{dx}$$

$$\Rightarrow f'(x) = \frac{1}{2}(25 - x^2)^{-\frac{1}{2}}(-2x)$$

$$\Rightarrow f'(x) = \frac{-2x}{2\,(25 - x^2)^{\frac{1}{2}}}$$

$$\Rightarrow f'(x) = \frac{-2x}{2\,(25 - x^2)^{\frac{1}{2}}}$$

$$\Rightarrow f'(x) = \frac{-x}{\sqrt{25 - x^2}}$$

Here also,

$\sqrt{25 - x^2} > 0$

$\Rightarrow -5 < x < 5$

$\therefore$ f (x) is differentiable in (– 3, 4)

So both the necessary conditions of Lagrange's mean value theorem is satisfied. Therefore, there exist a point c ∈ (-3, 4) such that:

$$f'(c) = \frac{f(4) - f(-3)}{4 - (-3)}$$

$$\Rightarrow f'(c) = \frac{f(4) - f(-3)}{4 + 3}$$

$$\Rightarrow f'(c) = \frac{f(4) - f(-3)}{7}$$

$$f(x) = (25 - x^2)^{\frac{1}{2}}$$

On differentiating with respect to x:

$$f'(x) = \frac{-x}{\sqrt{25 - x^2}}$$

For f'(c), put the value of x = c in f'(x):

$$f'(c) = \frac{-c}{\sqrt{25 - c^2}}$$

For f (4), put the value of x = 4 in f(x):

$$f(4) = \left(25 - 4^2\right)^{\frac{1}{2}}$$

$$\Rightarrow f(4) = (25 - 16)^{\frac{1}{2}}$$

$$\Rightarrow f(4) = (9)^{\frac{1}{2}}$$

$$\Rightarrow f(4) = 3$$

For f (– 3), put the value of x = – 3 in f(x):

$$f(-3) = \left(25 - (-3)^2\right)^{\frac{1}{2}}$$

$$\Rightarrow f(-3) = (25 - 9)^{\frac{1}{2}}$$

$$\Rightarrow f(-3) = (16)^{\frac{1}{2}}$$

$$\Rightarrow f(-3) = 4$$

$$f'(c) = \frac{f(4) - f(-3)}{7}$$

$$\Rightarrow \frac{-c}{\sqrt{25-c^2}} = \frac{3-4}{7}$$

$$\Rightarrow \frac{-c}{\sqrt{25-c^2}} = \frac{-1}{7}$$

$$\Rightarrow -7c = -\sqrt{25-c^2}$$

Squaring on both sides:

$$\Rightarrow (-7c)^2 = (-\sqrt{25-c^2})^2$$

$$\Rightarrow 49c^2 = 25 - c^2$$

$$\Rightarrow 50c^2 = 25$$

$$\Rightarrow c^2 = \frac{25}{50}$$

$$\Rightarrow c^2 = \frac{1}{2}$$

$$\Rightarrow c = \pm\frac{1}{\sqrt{2}} \in (-3, 4)$$

Hence, Lagrange's mean value theorem is verified.

(x) f (x) = $\tan^{-1}x$ on [0, 1]

Solution:

Given f (x) = $\tan^{-1} x$ on [0, 1]

$\text{Tan}^{-1} x$ has unique value for all x between 0 and 1.

∴ f (x) is continuous in [0, 1]

f (x) = $\tan^{-1} x$

Differentiating with respect to x:

$$f'(x) = \frac{1}{1+x^2}$$

x^2 always has value greater than 0.

$\Rightarrow 1 + x^2 > 0$

∴ f (x) is differentiable in (0, 1)

So both the necessary conditions of Lagrange's mean value theorem is satisfied.
Therefore, there exist a point c ∈ (0, 1) such that:

$$f'(c) = \frac{f(1) - f(0)}{1 - 0}$$

$\Rightarrow f'(c) = f(1) - f(0)$

$f(x) = \tan^{-1} x$

Differentiating with respect to x:

$$f'(x) = \frac{1}{1+x^2}$$

For f'(c), put the value of x=c in f'(x):

$$f'(c) = \frac{1}{1+c^2}$$

For f (1), put the value of x=1 in f(x):

$f(1) = \tan^{-1} 1$

$$\Rightarrow f(1) = \frac{\pi}{4}$$

For f (0), put the value of x=0 in f(x):

$f(0) = \tan^{-1} 0$

$\Rightarrow f(0) = 0$

$f'(c) = f(1) - f(0)$

$$\Rightarrow \frac{1}{1+c^2} = \frac{\pi}{4} - 0$$

$$\Rightarrow \frac{1}{1+c^2} = \frac{\pi}{4}$$

$\Rightarrow 4 = \pi(1+c^2)$

$\Rightarrow 4 = \pi + \pi c^2$

$\Rightarrow -\pi c^2 = \pi - 4$

$\Rightarrow c^2 = \frac{\pi - 4}{-\pi}$

$\Rightarrow c^2 = \frac{4 - \pi}{\pi}$

$\Rightarrow c = \sqrt{\frac{4}{\pi} - 1} \approx 0.52 \in (0, 1)$

Hence, Lagrange's mean value theorem is verified.

(xi) $f(x) = x + \frac{1}{x}$ on $[1, 3]$

Solution:

Given

$f(x) = x + \frac{1}{x}$ on $[1, 3]$

F (x) has unique values for all x ∈ (1, 3)

∴ f (x) is continuous in [1, 3]

$f(x) = x + \frac{1}{x}$ on $[1, 3]$

Differentiating with respect to x

$f'(x) = 1 + (-1)(x)^{-2}$

$\Rightarrow f'(x) = 1 - \frac{1}{x^2}$

$\Rightarrow f'(x) = \frac{x^2 - 1}{x^2}$

Here, $x^2 \neq 0$

⇒ f'(x) exists for all values except 0

∴ f (x) is differentiable in (1, 3)

So both the necessary conditions of Lagrange's mean value theorem is satisfied. Therefore, there exist a point c ∈ (1, 3) such that:

$$f'(c)=\frac{f(3)-f(1)}{3-1}$$

$$\Rightarrow f'(c)=\frac{f(3)-f(1)}{2}$$

$$f(x)=x+\frac{1}{x}$$

On differentiating with respect to x:

$$f'(x)=\frac{x^2-1}{x^2}$$

For f'(c), put the value of x=c in f'(x):

$$f'(c)=\frac{c^2-1}{c^2}$$

For f (3), put the value of x = 3 in f(x):

$$f(3)=3+\frac{1}{3}$$

$$\Rightarrow f(3)=\frac{9+1}{3}$$

$$\Rightarrow f(3)=\frac{10}{3}$$

For f (1), put the value of x = 1 in f(x):

$$f(1)=1+\frac{1}{1}$$

$$\Rightarrow f(1)=2$$

$$\Rightarrow f'(c)=\frac{f(3)-f(1)}{2}$$

$$\Rightarrow \frac{c^2-1}{c^2}=\frac{\frac{10}{3}-2}{2}$$

$$\Rightarrow 2(c^2 - 1) = c^2\left(\frac{10}{3} - 2\right)$$

$$\Rightarrow 2(c^2 - 1) = c^2\left(\frac{10 - 6}{3}\right)$$

$$\Rightarrow 2(c^2 - 1) = c^2\left(\frac{4}{3}\right)$$

$$\Rightarrow 6(c^2 - 1) = 4c^2$$

$$\Rightarrow 6c^2 - 6 = 4c^2$$

$$\Rightarrow 6c^2 - 4c^2 = 6$$

$$\Rightarrow 2c^2 = 6$$

$$\Rightarrow c^2 = \frac{6}{2}$$

$$\Rightarrow c^2 = 3$$

$$\Rightarrow c = \pm\sqrt{3} \in (-3, 4)$$

Hence, Lagrange's mean value theorem is verified.

(xii) f (x) = x (x + 4)^2 on [0, 4]

Solution:

Given f (x) = x (x + 4)^2 on [0, 4]
= x [(x)^2 + 2 (4) (x) + (4)^2]
= x (x^2 + 8x + 16)
= x^3 + 8x^2 + 16x on [0, 4]
Every polynomial function is continuous everywhere on (−∞, ∞) and differentiable for all arguments. Here, f(x) is a polynomial function. So it is continuous in [0, 4] and differentiable in (0, 4). So both the necessary conditions of Lagrange's mean value theorem is satisfied. Therefore, there exist a point c ∈ (0, 4) such that:

$$f'(c) = \frac{f(4) - f(0)}{4 - 0}$$

$$\Rightarrow f'(c) = \frac{f(4) - f(0)}{4}$$

$f(x) = x^3 + 8x^2 + 16x$

Differentiating with respect to x:

$f'(x) = 3x^2 + 8(2x) + 16$

$= 3x^2 + 16x + 16$

For f'(c), put the value of x = c in f'(x):

$f'(c) = 3c^2 + 16c + 16$

For f (4), put the value of x = 4 in f(x):

$f(4) = (4)^3 + 8(4)^2 + 16(4)$

$= 64 + 128 + 64$

$= 256$

For f (0), put the value of x = 0 in f(x):

$f(0) = (0)^3 + 8(0)^2 + 16(0)$

$= 0 + 0 + 0$

$= 0$

$$f'(c) = \frac{f(4) - f(0)}{4}$$

$$\Rightarrow 3c^2 + 16c + 16 = \frac{256 - 0}{4}$$

$$\Rightarrow 3c^2 + 16c + 16 = \frac{256}{4}$$

$$\Rightarrow 3c^2 + 16c + 16 = 64$$

$$\Rightarrow 3c^2 + 16c + 16 - 64 = 0$$

$$\Rightarrow 3c^2 + 16c - 48 = 0$$

For quadratic equation, $ax^2 + bx + c = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\Rightarrow c = \frac{-(16) \pm \sqrt{(16)^2 - 4 \times 3 \times (-48)}}{2 \times 3}$$

$$\Rightarrow c = \frac{-16 \pm \sqrt{256 + 576}}{6}$$

$$\Rightarrow c = \frac{-16 \pm \sqrt{832}}{6}$$

$$\Rightarrow c = \frac{-16 \pm 8\sqrt{13}}{6}$$

$$\Rightarrow c = \frac{-16}{6} \pm \frac{8\sqrt{13}}{6}$$

$$\Rightarrow c = \frac{-8}{3} \pm \frac{4\sqrt{13}}{3}$$

$$\Rightarrow c = \frac{-8}{3} + \frac{4\sqrt{13}}{3}, \frac{-8}{3} - \frac{4\sqrt{13}}{3} \in c$$

Hence, Lagrange's mean value theorem is verified.

(xiii) $f(x) = \sqrt{x^2 - 4}$ on [2, 4]

Solution:

Given

$f(x) = \sqrt{x^2 - 4}$ on [2, 4]

Here,

$\sqrt{x^2 - 4} > 0$

$\Rightarrow x^2 - 4 > 0$

$\Rightarrow x^2 > 4$

$\Rightarrow$ f (x) exists for all values expect (– 2, 2)

∴ f (x) is continuous in [2, 4]

$f(x) = \sqrt{x^2 - 4}$

Differentiating with respect to x:

$$f'(x) = \frac{1}{2}(x^2 - 4)^{\left(\frac{1}{2} - 1\right)} \frac{d(x^2 - 4)}{dx}$$

$$\Rightarrow f'(x)=\frac{1}{2}(x^2-4)^{-\frac{1}{2}}(2x)$$

$$\Rightarrow f'(x)=\frac{2x}{2\,(x^2-4)^{\frac{1}{2}}}$$

$$\Rightarrow f'(x)=\frac{x}{\sqrt{x^2-4}}$$

Here also, $\sqrt{x^2-4}>0$

$\Rightarrow$ f'(x) exists for all values of x except (2, – 2)

∴ f (x) is differentiable in (2, 4)

So both the necessary conditions of Lagrange's mean value theorem is satisfied.

Therefore, there exist a point c ∈ (2, 4) such that:

$$f'(c)=\frac{f(4)\ -\ f(2)}{4-2}$$

$$\Rightarrow f'(c)=\frac{f(4)-f(2)}{2}$$

$$f(x)=\sqrt{x^2-4}$$

On differentiating with respect to x:

$$f'(x)=\frac{x}{\sqrt{x^2-4}}$$

For f'(c), put the value of x=c in f'(x):

$$f'(c)=\frac{c}{\sqrt{c^2-4}}$$

For f (4), put the value of x = 4 in f(x):

$$f(4)=\sqrt{4^2-4}$$

$\Rightarrow f(4) = (16-4)^{\frac{1}{2}}$

$\Rightarrow f(4) = \sqrt{12}$

$\Rightarrow f(4) = 2\sqrt{3}$

For f (2), put the value of x = 2 in f(x):

$f(2) = \sqrt{2^2 - 4}$

$\Rightarrow f(2) = (4-4)^{\frac{1}{2}}$

$\Rightarrow f(2) = 0$

$\Rightarrow f'(c) = \frac{f(4) - f(2)}{2}$

$\Rightarrow \frac{c}{\sqrt{c^2 - 4}} = \frac{2\sqrt{3} - 0}{2}$

$\Rightarrow \frac{c}{\sqrt{c^2 - 4}} = \sqrt{3}$

$\Rightarrow c = (\sqrt{3})\sqrt{c^2 - 4}$

Squaring both sides:

$\Rightarrow (c)^2 = (\,(\sqrt{3})\sqrt{c^2 - 4})^2$

$\Rightarrow c^2 = 3(c^2 - 4)$

$\Rightarrow c^2 = 3c^2 - 12$

$\Rightarrow -2c^2 = -12$

$\Rightarrow c^2 = \frac{-12}{-2}$

$\Rightarrow c^2 = 6$

$\Rightarrow c = \pm\sqrt{6}$

$\Rightarrow c = \sqrt{6} \in (2, 4)$

Hence, Lagrange's mean value theorem is verified.

(xiv) f (x) = x^2 + x – 1 on [0, 4]

Solution:

Given f (x) = x^2 + x – 1 on [0, 4]

Every polynomial function is continuous everywhere on (−∞, ∞) and differentiable for all arguments. Here, f(x) is a polynomial function. So it is continuous in [0, 4] and differentiable in (0, 4). So both the necessary conditions of Lagrange's mean value theorem is satisfied. Therefore, there exist a point c ∈ (0, 4) such that:

$$f'(c) = \frac{f(4) - f(0)}{4 - 0}$$

$$\Rightarrow f'(c) = \frac{f(4) - f(0)}{4}$$

f (x) = x^2 + x – 1

Differentiating with respect to x:

f'(x) = 2x + 1

For f'(c), put the value of x = c in f'(x):

f'(c) = 2c + 1

For f (4), put the value of x = 4 in f(x):

f (4)= $(4)^2$ + 4 – 1

= 16 + 4 – 1

= 19

For f (0), put the value of x = 0 in f(x):

f (0) = $(0)^2$ + 0 – 1

= 0 + 0 – 1

= – 1

$$f'(c) = \frac{f(4) - f(0)}{4}$$

$$\Rightarrow 2c + 1 = \frac{19 - (-1)}{4}$$

$$\Rightarrow 2c + 1 = \frac{20}{4}$$

$\Rightarrow 2c + 1 = 5$

$\Rightarrow 2c = 5 - 1$

$\Rightarrow 2c = 4$

$\Rightarrow c = \frac{4}{2} = 2 \in (0, 4)$

Hence, Lagrange's mean value theorem is verified.

(xv) f (x) = sin x – sin 2x – x on [0, π]

Solution:

Given f (x) = sin x – sin 2x – x on [0, π]

Sin x and cos x functions are continuous everywhere on (−∞, ∞) and differentiable for all arguments. So both the necessary conditions of Lagrange's mean value theorem is satisfied. Therefore, there exist a point c ∈ (0, π) such that:

$$f'(c) = \frac{f(\pi) - f(0)}{\pi - 0}$$

$$\Rightarrow f'(c) = \frac{f(\pi) - f(0)}{\pi}$$

f (x) = sin x – sin 2x – x

Differentiating with respect to x:

f (x) = sin x – sin 2x – x

$$\Rightarrow f'(x) = \cos x - \cos 2x \frac{d(2x)}{dx} - 1$$

$$\Rightarrow f'(x) = \cos x - 2\cos 2x - 1$$

For f'(c), put the value of x=c in f'(x):

f'(c) = cos c – 2cos 2c – 1

For f (π), put the value of x = π in f(x):

f (π) = sin π – sin 2π – π

= 0 – 0 – π

$= -\pi$

For f (0), put the value of x=0 in f(x):

f (0) = sin 0 – sin 2(0) – 0

= sin 0 – sin 0 – 0

= 0 – 0 – 0

= 0

$$f'(c) = \frac{f(\pi) - f(0)}{\pi}$$

$$\Rightarrow \cos c - 2\cos 2c - 1 = \frac{-\pi - 0}{\pi}$$

$\Rightarrow$ Cos c – 2cos 2c – 1 = – 1

$\Rightarrow$ Cos c – 2(2cos^2 c – 1) = – 1 + 1

$\Rightarrow$ Cos c – 4cos^2 c + 2 = 0

$\Rightarrow$ 4cos^2 c – cos c – 2 = 0

For quadratic equation, $ax^2 + bx + c = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\Rightarrow \cos c = \frac{-(-1) \pm \sqrt{(-1)^2 - 4 \times 4 \times (-2)}}{2 \times 4}$$

$$\Rightarrow \cos c = \frac{1 \pm \sqrt{1 + 32}}{8}$$

$$\Rightarrow \cos c = \frac{1 \pm \sqrt{33}}{8}$$

$$\Rightarrow c = \cos^{-1}\left(\frac{1 \pm \sqrt{33}}{8}\right) \in (0, \pi)$$

Hence, Lagrange's mean value theorem is verified.

(xvi) f (x) = $x^3 - 5x^2 - 3x$ on [1, 3]

Solution:

Given f (x) = $x^3 - 5x^2 - 3x$ on [1, 3]

Every polynomial function is continuous everywhere on (−∞, ∞) and differentiable for all arguments. Here, f(x) is a polynomial function. So it is continuous in [1, 3] and differentiable in (1, 3). So both the necessary conditions of Lagrange's mean value theorem is satisfied.

Therefore, there exist a point c ∈ (1, 3) such that:

$$f'(c) = \frac{f(3) - f(1)}{3 - 1}$$

$$\Rightarrow f'(c) = \frac{f(3) - f(1)}{2}$$

f (x) = $x^3 - 5x^2 - 3x$

Differentiating with respect to x:

f'(x) = $3x^2 - 5(2x) - 3$

= $3x^2 - 10x - 3$

For f'(c), put the value of x=c in f'(x):

f'(c)= $3c^2 - 10c - 3$

For f (3), put the value of x = 3 in f(x):

f (3)= $(3)^3 - 5(3)^2 - 3(3)$

= 27 – 45 – 9

= – 27

For f (1), put the value of x = 1 in f(x):

f (1)= $(1)^3 - 5(1)^2 - 3(1)$

= 1 – 5 – 3

= – 7

$$f'(c) = \frac{f(3) - f(1)}{2}$$

$$\Rightarrow 3c^2 - 10c - 3 = \frac{(-27) - (-7)}{2}$$

$$\Rightarrow 3c^2 - 10c - 3 = \frac{-27 + 7}{2}$$

$$\Rightarrow 3c^2 - 10c - 3 = \frac{-20}{2}$$

$\Rightarrow 3c^2 - 10c - 3 = -10$
$\Rightarrow 3c^2 - 10c - 3 + 10 = 0$
$\Rightarrow 3c^2 - 10c + 7 = 0$
$\Rightarrow 3c^2 - 7c - 3c + 7 = 0$
$\Rightarrow c(3c - 7) - 1(3c - 7) = 0$
$\Rightarrow (3c - 7)(c - 1) = 0$

$$\Rightarrow c = \frac{7}{3}, 1$$

$$\Rightarrow c = \frac{7}{3} \in (1, 3)$$

Hence, Lagrange's mean value theorem is verified.

2. Discuss the applicability of Lagrange's mean value theorem for the function f(x) = |x| on [– 1, 1].

Solution:

Given f (x) = |x| on [– 1, 1]

So f(x) can be defined as $= \begin{cases} -x, & x<0 \\ x, & x\geq 0 \end{cases}$

For differentiability at x = 0,

$$\text{LHD} = \lim_{x\to 0^-} \frac{f(0-h) - f(0)}{-h}$$

{Since f(x) = – x, x<0}

$$= \lim_{x\to 0^-} \frac{-(0-h) - 0}{-h}$$

$$= \lim_{x\to 0^-} \frac{h-0}{-h}$$

$$= \lim_{x\to 0^-} \frac{h}{-h}$$

$= -1$

$$\text{RHD} = \lim_{x\to 0^+} \frac{f(0-h) - f(0)}{-h}$$

{Since f(x) = x, x>0}

$$= \lim_{x \to 0^-} \frac{(0-h)-0}{-h}$$

$$= \lim_{x \to 0^-} \frac{-h-0}{-h}$$

$$= \lim_{x \to 0^-} \frac{-h}{-h}$$

= 1

LHD ≠ RHD

⇒ f (x) is not differential at x=0

∴ Lagrange's mean value theorem is not applicable for the function f(x) = |x| on [– 1, 1].

3. Show that the Lagrange's mean value theorem is not applicable to the function f(x) = 1/x on [–1, 1].

Solution:

Given $f(x) = \frac{1}{x}$ on $[-1, 1]$

Here, x ≠ 0

⇒ f (x) exists for all values of x except 0

⇒ f (x) is discontinuous at x=0

∴ f (x) is not continuous in [– 1, 1]

Hence the Lagrange's mean value theorem is not applicable to the function f (x) = 1/x on [-1, 1]

4. Verify the hypothesis and conclusion of Lagrange's mean value theorem for the function

$$f(x) = \frac{1}{4x-1}, 1 \le x \le 4.$$

Solution:

Given

$f(x)=\frac{1}{4x-1}$ on $[1, 4]$

Where $4x - 1>0$

f'(x) has unique values for all x except ¼

∴ f (x) is continuous in [1, 4]

$f(x)=\frac{1}{4x-1}$

Differentiating with respect to x:

$f'(x)=(-1)(4x-1)^{-2}(4)$

$\Rightarrow f'(x) = -\frac{4}{(4x-1)^2}$

Here, $4x - 1>0$

f'(x) has unique values for all x except ¼

∴ f (x) is differentiable in (1, 4)

So both the necessary conditions of Lagrange's mean value theorem is satisfied. Therefore, there exist a point c ∈ (1, 4) such that:

$f'(c)=\frac{f(4)\ -\ f(1)}{4-1}$

$\Rightarrow f'(c)=\frac{f(4)-f(1)}{3}$

$f(x)=\frac{1}{4x-1}$

On differentiating with respect to x:

$f'(x) = -\frac{4}{(4x-1)^2}$

For f'(c), put the value of x=c in f'(x):

$$f'(c) = -\frac{4}{(4c-1)^2}$$

For f (4), put the value of x = 4 in f (x):

$$f(4) = \frac{1}{4(4)-1}$$

$$\Rightarrow f(4) = \frac{1}{16-1}$$

$$\Rightarrow f(4) = \frac{1}{15}$$

For f (1), put the value of x = 1 in f(x):

$$f(1) = \frac{1}{4(1)-1}$$

$$\Rightarrow f(1) = \frac{1}{4-1}$$

$$\Rightarrow f(1) = \frac{1}{3}$$

$$\Rightarrow f'(c) = \frac{f(4)-f(1)}{3}$$

$$\Rightarrow -\frac{4}{(4c-1)^2} = \frac{\frac{1}{15}-\frac{1}{3}}{3}$$

$$\Rightarrow -3(4) = (4c-1)^2\left(\frac{1}{15}-\frac{1}{3}\right)$$

$$\Rightarrow -12 = (4c-1)^2\left(\frac{3-15}{45}\right)$$

$$\Rightarrow -12 = (4c-1)^2\left(\frac{-12}{45}\right)$$

$$\Rightarrow -12 \times \frac{45}{-12} = (4c-1)^2$$

$$\Rightarrow -12\times\frac{45}{-12}=(4c-1)^2$$

$$\Rightarrow (4c-1)^2 = 45$$

$$\Rightarrow (4c-1) = \pm\sqrt{45}$$

$$\Rightarrow (4c-1) = \pm 3\sqrt{5}$$

$$\Rightarrow c = \frac{\pm 3\sqrt{5}+1}{4}$$

$$\Rightarrow c = \frac{3\sqrt{5}+1}{4} \approx 1.92 \in (1, 4)$$

Hence, Lagrange's mean value theorem is verified.

5. Find a point on the parabola $y = (x - 4)^2$, where the tangent is parallel to the chord joining (4, 0) and (5, 1).

Solution:

Given $f(x) = (x - 4)^2$ on [4, 5]

This interval [a, b] is obtained by x – coordinates of the points of the chord.

Every polynomial function is continuous everywhere on $(-\infty, \infty)$ and differentiable for all arguments. Here, f(x) is a polynomial function. So it is continuous in [4, 5] and differentiable in (4, 5). So both the necessary conditions of Lagrange's mean value theorem is satisfied.

Therefore, there exist a point c ∈ (4, 5) such that:

$$f'(c) = \frac{f(5) - f(4)}{5-4}$$

$$\Rightarrow f'(c) = \frac{f(5) - f(4)}{1}$$

$$f(x) = (x-4)^2$$

Differentiating with respect to x:

$$f'(x) = 2(x-4)\frac{d(x-4)}{dx}$$

$$\Rightarrow f'(x) = 2(x-4)(1)$$

$\Rightarrow f'(x) = 2(x - 4)$
For $f'(c)$, put the value of $x=c$ in $f'(x)$:
$f'(c) = 2(c - 4)$
For f (5), put the value of $x=5$ in f(x):
$f(5) = (5 - 4)^2$
$= (1)^2$
$= 1$
For f (4), put the value of $x=4$ in f(x):
$f(4) = (4 - 4)^2$
$= (0)^2$
$= 0$
$f'(c) = f(5) - f(4)$
$\Rightarrow 2(c - 4) = 1 - 0$
$\Rightarrow 2c - 8 = 1$
$\Rightarrow 2c = 1 + 8$
$\Rightarrow c = \frac{9}{2} = 4.5 \in (4, 5)$
We know that, the value of c obtained in Lagrange's Mean value Theorem is nothing but the value of x – coordinate of the point of the contact of the tangent to the curve which is parallel to the chord joining the points (4, 0) and (5, 1).
Now, put this value of x in f(x) to obtain y:
$y = (x - 4)^2$

$$\Rightarrow y = \left(\frac{9}{2} - 4\right)^2$$

$$\Rightarrow y = \left(\frac{9-8}{2}\right)^2$$

$$\Rightarrow y = \left(\frac{1}{2}\right)^2$$

$$\Rightarrow y = \frac{1}{4}$$

Hence, the required point is $\left(\frac{9}{2}, \frac{1}{4}\right)$