

EXERCISE 19.22

PAGE NO: 19.114

Evaluate the following integrals:

$$1. \int \frac{1}{4 \cos^2 x + 9 \sin^2 x} dx$$

Solution:

$$\text{Given } I = \int \frac{1}{4 \cos^2 x + 9 \sin^2 x} dx$$

Dividing the numerator and denominator of the given integrand by $\cos^2 x$, we get

$$\Rightarrow I = \int \frac{1}{4 \cos^2 x + 9 \sin^2 x} dx = \int \frac{\sec^2 x}{4 + 9 \tan^2 x} dx$$

Putting $\tan x = t$ and $\sec^2 x dx = dt$, we get

$$\Rightarrow I = \int \frac{dt}{4 + 9t^2} = \frac{1}{9} \int \frac{dt}{\frac{4}{9} + t^2}$$

We know that $\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right) + c$

$$\Rightarrow \frac{1}{9} \int \frac{dt}{\frac{4}{9} + t^2} = \frac{1}{9} \times \frac{1}{\frac{2}{3}} \tan^{-1} \left(\frac{t}{\frac{2}{3}} \right) + c$$

$$= \frac{1}{6} \tan^{-1} \left(\frac{3t}{2} \right) + c$$

$$= \frac{1}{6} \tan^{-1} \left(\frac{3 \tan x}{2} \right) + c$$

$$\therefore I = \int \frac{1}{4 \cos^2 x + 9 \sin^2 x} dx = \frac{1}{6} \tan^{-1} \left(\frac{3 \tan x}{2} \right) + c$$

$$2. \int \frac{1}{4 \sin^2 x + 5 \cos^2 x} dx$$

Solution:

$$\text{Given } I = \int \frac{1}{4 \sin^2 x + 5 \cos^2 x} dx$$

Dividing the numerator and denominator of the given integrand by $\cos^2 x$, we get

$$\Rightarrow I = \int \frac{1}{4 \sin^2 x + 5 \cos^2 x} dx = \int \frac{\sec^2 x}{4 \tan^2 x + 5} dx$$

Putting $\tan x = t$ and $\sec^2 x dx = dt$, we get

$$\Rightarrow I = \int \frac{dt}{4t^2 + 5} = \frac{1}{4} \int \frac{dt}{t^2 + \left(\frac{\sqrt{5}}{2}\right)^2}$$

We know that $\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right) + c$

$$\Rightarrow \frac{1}{4} \int \frac{dt}{t^2 + \left(\frac{\sqrt{5}}{2}\right)^2} = \frac{1}{4} \times \frac{1}{\frac{\sqrt{5}}{2}} \tan^{-1} \left(\frac{t}{\frac{\sqrt{5}}{2}} \right) + c$$

$$= \frac{1}{2\sqrt{5}} \tan^{-1} \left(\frac{2t}{\sqrt{5}} \right) + c$$

$$= \frac{1}{2\sqrt{5}} \tan^{-1} \left(\frac{2 \tan x}{\sqrt{5}} \right) + c$$

$$\therefore I = \int \frac{1}{4 \sin^2 x + 5 \cos^2 x} dx = \frac{1}{2\sqrt{5}} \tan^{-1} \left(\frac{2 \tan x}{\sqrt{5}} \right) + c$$

$$3. \int \frac{2}{2 + \sin 2x} dx$$

Solution:

$$\text{Given } I = \int \frac{2}{2 + \sin 2x} dx$$

We know that $\sin 2x = 2 \sin x \cos x$

$$\Rightarrow \int \frac{2}{2 + \sin 2x} dx = \int \frac{2}{2 + 2 \sin x \cos x} dx$$

$$= \int \frac{1}{1 + \sin x \cos x} dx$$

Dividing the numerator and denominator by $\cos^2 x$,

$$\Rightarrow \int \frac{1}{1 + \sin x \cos x} dx = \int \frac{\sec^2 x}{\sec^2 x + \tan x} dx$$

Replacing $\sec^2 x$ in denominator by $1 + \tan^2 x$,

$$\Rightarrow \int \frac{\sec^2 x}{\sec^2 x + \tan x} dx = \int \frac{\sec^2 x}{1 + \tan^2 x + \tan x} dx$$

Putting $\tan x = t$ so that $\sec^2 x dx = dt$,

$$\Rightarrow \int \frac{\sec^2 x}{\tan^2 x + \tan x + 1} dx = \int \frac{dt}{t^2 + t + 1}$$

$$= \int \frac{dt}{\left(t + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

We know that $\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \left(\frac{x}{a}\right) + c$

$$\Rightarrow \int \frac{dt}{\left(t + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \frac{1}{\frac{\sqrt{3}}{2}} \tan^{-1} \left(\frac{t + \frac{1}{2}}{\frac{\sqrt{3}}{2}}\right) + c$$

$$= \frac{2}{\sqrt{3}} \tan^{-1} \left(\frac{2t + 1}{\sqrt{3}}\right) + c$$

$$\therefore I = \int \frac{2}{2 + \sin 2x} dx = \frac{2}{\sqrt{3}} \tan^{-1} \left(\frac{2t + 1}{\sqrt{3}}\right) + c$$

$$= \frac{2}{\sqrt{3}} \tan^{-1} \left(\frac{2 \tan x + 1}{\sqrt{3}}\right) + C$$

4. $\int \frac{\cos x}{\cos 3x} dx$

Solution:

Given $I = \int \frac{\cos x}{\cos 3x} dx$

$$\Rightarrow \int \frac{\cos x}{\cos 3x} dx = \int \frac{\cos x}{4 \cos^3 x - 3 \cos x} dx$$

$$= \int \frac{1}{4 \cos^2 x - 3} dx$$

Dividing numerator and denominator by $\cos^2 x$,

$$\Rightarrow \int \frac{1}{4 \cos^2 x - 3} dx = \int \frac{\sec^2 x}{4 - 3 \sec^2 x} dx$$

Replacing $\sec^2 x$ by $1 + \tan^2 x$ in denominator,

$$\Rightarrow \int \frac{\sec^2 x}{4 - 3 \sec^2 x} dx = \int \frac{\sec^2 x}{4 - 3 - 3 \tan^2 x} dx$$

$$= \int \frac{\sec^2 x}{1 - 3 \tan^2 x} dx$$

Putting $\tan x = t$ and $\sec^2 x dx = dt$, we get

$$I = \int \frac{dt}{1 - 3t^2} = \frac{1}{3} \int \frac{1}{\frac{1}{3} - t^2} dt$$

We know that $\int \frac{1}{a^2 - x^2} dx = \frac{1}{2a} \log \left| \frac{a+x}{a-x} \right| + c$

$$\Rightarrow \frac{1}{3} \int \frac{1}{\frac{1}{3} - t^2} dt = \frac{1}{3} \times \frac{1}{2 \left(\frac{1}{\sqrt{3}}\right)} \log \left| \frac{\frac{1}{\sqrt{3}} + t}{\frac{1}{\sqrt{3}} - t} \right| + c$$

$$= \frac{1}{2\sqrt{3}} \log \left| \frac{1 + \sqrt{3}t}{1 - \sqrt{3}t} \right| + c$$

$$= \frac{1}{2\sqrt{3}} \log \left| \frac{1 + \sqrt{3} \tan x}{1 - \sqrt{3} \tan x} \right| + c$$

$$\therefore I = \int \frac{\cos x}{\cos 3x} dx = \frac{1}{2\sqrt{3}} \log \left| \frac{1 + \sqrt{3} \tan x}{1 - \sqrt{3} \tan x} \right| + c$$