

EXERCISE 19.29

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Evaluate the following integrals:

1. $\int (x + 1)\sqrt{x^2 - x + 1} dx$

Solution:

Let us assume $x + 1 = \lambda \frac{d}{dx}(x^2 - x + 1) + \mu$

$$\Rightarrow x + 1 = \lambda \left[\frac{d}{dx}(x^2) - \frac{d}{dx}(x) + \frac{d}{dx}(1) \right] + \mu$$

We know $\frac{d}{dx}(x^n) = nx^{n-1}$ and derivative of a constant is 0.

$$\Rightarrow x + 1 = \lambda (2x^{2-1} - 1 + 0) + \mu$$

$$\Rightarrow x + 1 = \lambda (2x - 1) + \mu$$

$$\Rightarrow x + 1 = 2\lambda x + \mu - \lambda$$

Comparing the coefficient of x on both sides, we get

$$2\lambda = 1 \Rightarrow \lambda = \frac{1}{2}$$

Comparing the constant on both sides, we get

$$\mu - \lambda = 1$$

$$\Rightarrow \mu - \frac{1}{2} = 1$$

$$\therefore \mu = \frac{3}{2}$$

Hence, we have $x + 1 = \frac{1}{2}(2x - 1) + \frac{3}{2}$

Substituting this value in I , we can write the integral as

$$I = \int \left[\frac{1}{2}(2x - 1) + \frac{3}{2} \right] \sqrt{x^2 - x + 1} dx$$

$$\Rightarrow I = \int \left[\frac{1}{2}(2x-1)\sqrt{x^2-x+1} + \frac{3}{2}\sqrt{x^2-x+1} \right] dx$$

$$\Rightarrow I = \int \frac{1}{2}(2x-1)\sqrt{x^2-x+1} dx + \int \frac{3}{2}\sqrt{x^2-x+1} dx$$

$$\Rightarrow I = \frac{1}{2} \int (2x-1)\sqrt{x^2-x+1} dx + \frac{3}{2} \int \sqrt{x^2-x+1} dx$$

$$\text{Let } I_1 = \frac{1}{2} \int (2x-1)\sqrt{x^2-x+1} dx$$

Now, put $x^2 - x + 1 = t$

$$\Rightarrow (2x-1) dx = dt \text{ (Differentiating both sides)}$$

Substituting this value in I_1 , we can write

$$I_1 = \frac{1}{2} \int \sqrt{t} dt$$

$$\Rightarrow I_1 = \frac{1}{2} \int t^{\frac{1}{2}} dt$$

We know that $\int x^n dx = \frac{x^{n+1}}{n+1} + c$

$$\Rightarrow I_1 = \frac{1}{2} \left(\frac{t^{\frac{1}{2}+1}}{\frac{1}{2}+1} \right) + c$$

$$\Rightarrow I_1 = \frac{1}{2} \left(\frac{t^{\frac{3}{2}}}{\frac{3}{2}} \right) + c$$

$$\Rightarrow I_1 = \frac{1}{2} \times \frac{2}{3} t^{\frac{3}{2}} + c$$

$$\Rightarrow I_1 = \frac{1}{3} t^{\frac{3}{2}} + c$$

$$\therefore I_1 = \frac{1}{3} (x^2 - x + 1)^{\frac{3}{2}} + c$$

$$\text{Let } I_2 = \frac{3}{2} \int \sqrt{x^2-x+1} dx$$

We can write $x^2 - x + 1 = x^2 - 2(x)\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2 + 1$

$$\Rightarrow x^2 - x + 1 = \left(x - \frac{1}{2}\right)^2 - \frac{1}{4} + 1$$

$$\Rightarrow x^2 - x + 1 = \left(x - \frac{1}{2}\right)^2 + \frac{3}{4}$$

$$\Rightarrow x^2 - x + 1 = \left(x - \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2$$

Hence, we can write I_2 as

$$I_2 = \frac{3}{2} \int \sqrt{\left(x - \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} dx$$

We know that $\int \sqrt{x^2 + a^2} dx = \frac{x}{2} \sqrt{x^2 + a^2} + \frac{a^2}{2} \ln|x + \sqrt{x^2 + a^2}| + c$

$$\begin{aligned} \Rightarrow I_2 &= \frac{3}{2} \left[\frac{\left(x - \frac{1}{2}\right)}{2} \sqrt{\left(x - \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} \right. \\ &\quad \left. + \frac{\left(\frac{\sqrt{3}}{2}\right)^2}{2} \ln \left| \left(x - \frac{1}{2}\right) + \sqrt{\left(x - \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} \right| \right] + c \end{aligned}$$

$$\Rightarrow I_2 = \frac{3}{2} \left[\frac{2x - 1}{4} \sqrt{x^2 - x + 1} + \frac{3}{8} \ln \left| x - \frac{1}{2} + \sqrt{x^2 - x + 1} \right| \right] + c$$

$$\therefore I_2 = \frac{3}{8} (2x - 1) \sqrt{x^2 - x + 1} + \frac{9}{16} \ln \left| x - \frac{1}{2} + \sqrt{x^2 - x + 1} \right| + c$$

Substituting I_1 and I_2 in I , we get

$$I = \frac{1}{3} (x^2 - x + 1)^{\frac{3}{2}} + \frac{3}{8} (2x - 1) \sqrt{x^2 - x + 1} + \frac{9}{16} \ln \left| x - \frac{1}{2} + \sqrt{x^2 - x + 1} \right| + c$$

Thus,

$$\begin{aligned} \int (x + 1) \sqrt{x^2 - x + 1} dx &= \frac{1}{3} (x^2 - x + 1)^{\frac{3}{2}} + \frac{3}{8} (2x - 1) \sqrt{x^2 - x + 1} + \\ &\frac{9}{16} \ln \left| x - \frac{1}{2} + \sqrt{x^2 - x + 1} \right| + c \end{aligned}$$

$$2. \int (x + 1)\sqrt{2x^2 + 3} dx$$

Solution:

$$\text{Let } I = \int (x + 1)\sqrt{2x^2 + 3} dx$$

$$\text{Let us assume } x + 1 = \lambda \frac{d}{dx}(2x^2 + 3) + \mu$$

$$\Rightarrow x + 1 = \lambda \left[\frac{d}{dx}(2x^2) + \frac{d}{dx}(3) \right] + \mu$$

$$\Rightarrow x + 1 = \lambda \left[2 \frac{d}{dx}(x^2) + \frac{d}{dx}(3) \right] + \mu$$

We know $\frac{d}{dx}(x^n) = nx^{n-1}$ and derivative of a constant is 0.

$$\Rightarrow x + 1 = \lambda (2 \times 2x^{2-1} + 0) + \mu$$

$$\Rightarrow x + 1 = \lambda (4x) + \mu$$

$$\Rightarrow x + 1 = 4\lambda x + \mu$$

Comparing the coefficient of x on both sides, we get

$$4\lambda = 1 \Rightarrow \lambda = \frac{1}{4}$$

Comparing the constant on both sides, we get

$$4\lambda = 1 \Rightarrow \lambda = \frac{1}{4}$$

Comparing the constant on both sides, we get

$$\mu = 1$$

$$\text{Hence, we have } x + 1 = \frac{1}{4}(4x) + 1$$

Substituting this value in I , we can write the integral as

$$I = \int \left[\frac{1}{4}(4x) + 1 \right] \sqrt{2x^2 + 3} dx$$

$$\Rightarrow I = \int \left[\frac{1}{4}(4x)\sqrt{2x^2 + 3} + \sqrt{2x^2 + 3} \right] dx$$

$$\Rightarrow I = \int \frac{1}{4} (4x) \sqrt{2x^2 + 3} dx + \int \sqrt{2x^2 + 3} dx$$

$$\Rightarrow I = \frac{1}{4} \int (4x) \sqrt{2x^2 + 3} dx + \int \sqrt{2x^2 + 3} dx$$

$$\text{Let } I_1 = \frac{1}{4} \int (4x) \sqrt{2x^2 + 3} dx$$

Now, put $2x^2 + 3 = t$

$$\Rightarrow (4x) dx = dt \text{ (Differentiating both sides)}$$

Substituting this value in I_1 , we can write

$$I_1 = \frac{1}{4} \int \sqrt{t} dt$$

$$\Rightarrow I_1 = \frac{1}{4} \int t^{\frac{1}{2}} dt$$

$$\text{We know that } \int x^n dx = \frac{x^{n+1}}{n+1} + c$$

$$\Rightarrow I_1 = \frac{1}{4} \left(\frac{t^{\frac{1}{2}+1}}{\frac{1}{2}+1} \right) + c$$

$$\Rightarrow I_1 = \frac{1}{4} \left(\frac{t^{\frac{3}{2}}}{\frac{3}{2}} \right) + c$$

$$\Rightarrow I_1 = \frac{1}{4} \times \frac{2}{3} t^{\frac{3}{2}} + c$$

$$\Rightarrow I_1 = \frac{1}{6} t^{\frac{3}{2}} + c$$

$$\therefore I_1 = \frac{1}{6} (2x^2 + 3)^{\frac{3}{2}} + c$$

$$\text{Let } I_2 = \int \sqrt{2x^2 + 3} dx$$

$$\text{We can write } 2x^2 + 3 = 2 \left(x^2 + \frac{3}{2} \right)$$

$$\Rightarrow 2x^2 + 3 = 2 \left[x^2 + \left(\sqrt{\frac{3}{2}} \right)^2 \right]$$

Hence, we can write I_2 as

$$I_2 = \int \sqrt{2 \left[x^2 + \left(\sqrt{\frac{3}{2}} \right)^2 \right]} dx \Rightarrow I_2 = \sqrt{2} \int \sqrt{x^2 + \left(\sqrt{\frac{3}{2}} \right)^2} dx$$

We know that $\int \sqrt{x^2 + a^2} dx = \frac{x}{2} \sqrt{x^2 + a^2} + \frac{a^2}{2} \ln|x + \sqrt{x^2 + a^2}| + c$

$$\Rightarrow I_2 = \sqrt{2} \left[\frac{x}{2} \sqrt{x^2 + \left(\sqrt{\frac{3}{2}} \right)^2} + \frac{\left(\sqrt{\frac{3}{2}} \right)^2}{2} \ln \left| x + \sqrt{x^2 + \left(\sqrt{\frac{3}{2}} \right)^2} \right| \right] + c$$

$$\Rightarrow I_2 = \sqrt{2} \left[\frac{x}{2} \sqrt{x^2 + \frac{3}{2}} + \frac{3}{4} \ln \left| x + \sqrt{x^2 + \frac{3}{2}} \right| \right] + c$$

$$\Rightarrow I_2 = \sqrt{2} \left[\frac{x}{2\sqrt{2}} \sqrt{2x^2 + 3} + \frac{3}{2 \times 2} \ln \left| x + \sqrt{x^2 + \frac{3}{2}} \right| \right] + c$$

$$\therefore I_2 = \frac{x}{2} \sqrt{2x^2 + 3} + \frac{3}{2\sqrt{2}} \ln \left| x + \sqrt{x^2 + \frac{3}{2}} \right| + c$$

Substituting I_1 and I_2 in I , we get

$$I = \frac{1}{6} (2x^2 + 3)^{\frac{3}{2}} + \frac{x}{2} \sqrt{2x^2 + 3} + \frac{3}{2\sqrt{2}} \ln \left| x + \sqrt{x^2 + \frac{3}{2}} \right| + c$$

Thus,

$$\int (x+1)\sqrt{2x^2+3} dx = \frac{1}{6} (2x^2+3)^{\frac{3}{2}} + \frac{x}{2} \sqrt{2x^2+3} + \frac{3}{2\sqrt{2}} \ln \left| x + \sqrt{x^2 + \frac{3}{2}} \right| + c$$

$$3. \int (2x - 5)\sqrt{2 + 3x - x^2} dx$$

Solution:

$$\text{Let } I = \int (2x - 5)\sqrt{2 + 3x - x^2} dx$$

$$\text{Let us assume } 2x - 5 = \lambda \frac{d}{dx}(2 + 3x - x^2) + \mu$$

$$\Rightarrow 2x - 5 = \lambda \left[\frac{d}{dx}(2) + \frac{d}{dx}(3x) - \frac{d}{dx}(x^2) \right] + \mu$$

$$\Rightarrow 2x - 5 = \lambda \left[\frac{d}{dx}(2) + 3 \frac{d}{dx}(x) - \frac{d}{dx}(x^2) \right] + \mu$$

We know $\frac{d}{dx}(x^n) = nx^{n-1}$ and derivative of a constant is 0.

$$\Rightarrow 2x - 5 = \lambda (0 + 3 - 2x^{2-1}) + \mu$$

$$\Rightarrow 2x - 5 = \lambda (3 - 2x) + \mu$$

$$\Rightarrow 2x - 5 = -2\lambda x + 3\lambda + \mu$$

Comparing the coefficient of x on both sides, we get

$$-2\lambda = 2 \Rightarrow \lambda = -1$$

Comparing the constant on both sides, we get

$$3\lambda + \mu = -5$$

$$\Rightarrow 3(-1) + \mu = -5$$

$$\Rightarrow -3 + \mu = -5$$

$$\therefore \mu = -2$$

Hence, we have $2x - 5 = -(3 - 2x) - 2$

Substituting this value in I , we can write the integral as

$$I = \int [-(3 - 2x) - 2]\sqrt{2 + 3x - x^2} dx$$

$$\Rightarrow I = \int \left[-(3 - 2x)\sqrt{2 + 3x - x^2} - 2\sqrt{2 + 3x - x^2} \right] dx$$

$$\Rightarrow I = - \int (3 - 2x)\sqrt{2 + 3x - x^2} dx - \int 2\sqrt{2 + 3x - x^2} dx$$

$$\Rightarrow I = - \int (3 - 2x)\sqrt{2 + 3x - x^2} dx - 2 \int \sqrt{2 + 3x - x^2} dx$$

$$\text{Let } I_1 = - \int (3 - 2x)\sqrt{2 + 3x - x^2} dx$$

Now, put $2 + 3x - x^2 = t$

$$\Rightarrow (3 - 2x) dx = dt \text{ (Differentiating both sides)}$$

Substituting this value in I_1 , we can write

$$I_1 = - \int \sqrt{t} dt$$

$$\Rightarrow I_1 = - \int t^{\frac{1}{2}} dt$$

$$\text{We know that } \int x^n dx = \frac{x^{n+1}}{n+1} + c$$

$$\Rightarrow I_1 = - \frac{t^{\frac{1}{2}+1}}{\frac{1}{2}+1} + c$$

$$\Rightarrow I_1 = - \frac{t^{\frac{3}{2}}}{\frac{3}{2}} + c$$

$$\Rightarrow I_1 = - \frac{2}{3} t^{\frac{3}{2}} + c$$

$$\Rightarrow I_1 = - \frac{2}{3} t^{\frac{3}{2}} + c$$

$$\therefore I_1 = - \frac{2}{3} (2 + 3x - x^2)^{\frac{3}{2}} + c$$

$$\text{Let } I_2 = -2 \int \sqrt{2 + 3x - x^2} dx$$

$$\text{We can write } 2 + 3x - x^2 = -(x^2 - 3x - 2)$$

$$\Rightarrow 2 + 3x - x^2 = - \left[x^2 - 2(x) \left(\frac{3}{2} \right) + \left(\frac{3}{2} \right)^2 - \left(\frac{3}{2} \right)^2 - 2 \right]$$

$$\Rightarrow 2 + 3x - x^2 = - \left[\left(x - \frac{3}{2} \right)^2 - \frac{9}{4} - 2 \right]$$

$$\Rightarrow 2 + 3x - x^2 = - \left[\left(x - \frac{3}{2} \right)^2 - \frac{17}{4} \right]$$

$$\Rightarrow 2 + 3x - x^2 = \frac{17}{4} - \left(x - \frac{3}{2} \right)^2$$

$$\Rightarrow 2 + 3x - x^2 = \left(\frac{\sqrt{17}}{2} \right)^2 - \left(x - \frac{3}{2} \right)^2$$

Hence, we can write I_2 as

$$I_2 = -2 \int \sqrt{\left(\frac{\sqrt{17}}{2} \right)^2 - \left(x - \frac{3}{2} \right)^2} dx$$

We have $\int \sqrt{a^2 - x^2} dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} + c$

$$\Rightarrow I_2 = -2 \left[\frac{\left(x - \frac{3}{2} \right)}{2} \sqrt{\left(\frac{\sqrt{17}}{2} \right)^2 - \left(x - \frac{3}{2} \right)^2} + \frac{\left(\frac{\sqrt{17}}{2} \right)^2}{2} \sin^{-1} \left(\frac{x - \frac{3}{2}}{\frac{\sqrt{17}}{2}} \right) \right] + c$$

$$\Rightarrow I_2 = -2 \left[\frac{2x - 3}{4} \sqrt{2 + 3x - x^2} + \frac{17}{8} \sin^{-1} \left(\frac{2x - 3}{\sqrt{17}} \right) \right] + c$$

$$\therefore I_2 = -\frac{1}{2} (2x - 3) \sqrt{2 + 3x - x^2} - \frac{17}{4} \sin^{-1} \left(\frac{2x - 3}{\sqrt{17}} \right) + c$$

Substituting I_1 and I_2 in I , we get

$$I = -\frac{2}{3} (2 + 3x - x^2)^{\frac{3}{2}} - \frac{1}{2} (2x - 3) \sqrt{2 + 3x - x^2} - \frac{17}{4} \sin^{-1} \left(\frac{2x - 3}{\sqrt{17}} \right) + c$$

Thus,

$$\int (2x - 5)\sqrt{2 + 3x - x^2} dx = -\frac{2}{3}(2 + 3x - x^2)^{\frac{3}{2}} - \frac{1}{2}(2x - 3)\sqrt{2 + 3x - x^2} - \frac{17}{4} \sin^{-1}\left(\frac{2x-3}{\sqrt{17}}\right) + c$$

4. $\int (x + 2)\sqrt{x^2 + x + 1} dx$

Solution:

Let $I = \int (x + 2)\sqrt{x^2 + x + 1} dx$

Let us assume $x + 2 = \lambda \frac{d}{dx}(x^2 + x + 1) + \mu$

$$\Rightarrow x + 2 = \lambda \left[\frac{d}{dx}(x^2) + \frac{d}{dx}(x) + \frac{d}{dx}(1) \right] + \mu$$

We know $\frac{d}{dx}(x^n) = nx^{n-1}$ and derivative of a constant is 0.

$$\Rightarrow x + 2 = \lambda (2x^{2-1} + 1 + 0) + \mu$$

$$\Rightarrow x + 2 = \lambda (2x + 1) + \mu$$

$$\Rightarrow x + 2 = 2\lambda x + \lambda + \mu$$

Comparing the coefficient of x on both sides, we get

$$2\lambda = 1 \Rightarrow \lambda = \frac{1}{2}$$

Comparing the constant on both sides, we get

$$\lambda + \mu = 2$$

$$\Rightarrow \frac{1}{2} + \mu = 2$$

$$\therefore \mu = \frac{3}{2}$$

Hence, we have $x + 2 = \frac{1}{2}(2x + 1) + \frac{3}{2}$

Substituting this value in I , we can write the integral as

$$I = \int \left[\frac{1}{2}(2x+1) + \frac{3}{2} \right] \sqrt{x^2+x+1} dx$$

$$\Rightarrow I = \int \left[\frac{1}{2}(2x+1)\sqrt{x^2+x+1} + \frac{3}{2}\sqrt{x^2+x+1} \right] dx$$

$$\Rightarrow I = \int \frac{1}{2}(2x+1)\sqrt{x^2+x+1} dx + \int \frac{3}{2}\sqrt{x^2+x+1} dx$$

$$\Rightarrow I = \frac{1}{2} \int (2x+1)\sqrt{x^2+x+1} dx + \frac{3}{2} \int \sqrt{x^2+x+1} dx$$

$$\text{Let } I_1 = \frac{1}{2} \int (2x+1)\sqrt{x^2+x+1} dx$$

Now, put $x^2+x+1 = t$

$$\Rightarrow (2x+1) dx = dt \text{ (Differentiating both sides)}$$

Substituting this value in I_1 , we can write

$$I_1 = \frac{1}{2} \int \sqrt{t} dt$$

$$\Rightarrow I_1 = \frac{1}{2} \int t^{\frac{1}{2}} dt$$

$$\text{We know that } \int x^n dx = \frac{x^{n+1}}{n+1} + c$$

$$\Rightarrow I_1 = \frac{1}{2} \left(\frac{t^{\frac{1}{2}+1}}{\frac{1}{2}+1} \right) + c$$

$$\Rightarrow I_1 = \frac{1}{2} \left(\frac{t^{\frac{3}{2}}}{\frac{3}{2}} \right) + c$$

$$\Rightarrow I_1 = \frac{1}{2} \times \frac{2}{3} t^{\frac{3}{2}} + c$$

$$\Rightarrow I_1 = \frac{1}{3} t^{\frac{3}{2}} + c$$

$$\therefore I_1 = \frac{1}{3} (x^2+x+1)^{\frac{3}{2}} + c$$

$$\text{Let } I_2 = \frac{3}{2} \int \sqrt{x^2 + x + 1} dx$$

$$\text{We can write } x^2 + x + 1 = x^2 + 2(x) \left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2 + 1$$

$$\Rightarrow x^2 + x + 1 = \left(x + \frac{1}{2}\right)^2 - \frac{1}{4} + 1$$

$$\Rightarrow x^2 + x + 1 = \left(x + \frac{1}{2}\right)^2 + \frac{3}{4}$$

$$\Rightarrow x^2 + x + 1 = \left(x + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2$$

Hence, we can write I_2 as

$$I_2 = \frac{3}{2} \int \sqrt{\left(x + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} dx$$

$$\text{We know that } \int \sqrt{x^2 + a^2} dx = \frac{x}{2} \sqrt{x^2 + a^2} + \frac{a^2}{2} \ln|x + \sqrt{x^2 + a^2}| + c$$

$$\begin{aligned} \Rightarrow I_2 &= \frac{3}{2} \left[\frac{\left(x + \frac{1}{2}\right)}{2} \sqrt{\left(x + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} \right. \\ &\quad \left. + \frac{\left(\frac{\sqrt{3}}{2}\right)^2}{2} \ln \left| \left(x + \frac{1}{2}\right) + \sqrt{\left(x + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} \right| \right] + c \end{aligned}$$

$$\Rightarrow I_2 = \frac{3}{2} \left[\frac{2x + 1}{4} \sqrt{x^2 + x + 1} + \frac{3}{8} \ln \left| x + \frac{1}{2} + \sqrt{x^2 + x + 1} \right| \right] + c$$

$$\therefore I_2 = \frac{3}{8} (2x + 1) \sqrt{x^2 + x + 1} + \frac{9}{16} \ln \left| x + \frac{1}{2} + \sqrt{x^2 + x + 1} \right| + c$$

Substituting I_1 and I_2 in I , we get

$$I = \frac{1}{3}(x^2 + x + 1)^{\frac{3}{2}} + \frac{3}{8}(2x + 1)\sqrt{x^2 + x + 1} + \frac{9}{16} \ln \left| x + \frac{1}{2} + \sqrt{x^2 + x + 1} \right| + c$$

Thus,

$$\int (x + 2)\sqrt{x^2 + x + 1} dx = \frac{1}{3}(x^2 + x + 1)^{\frac{3}{2}} + \frac{3}{8}(2x + 1)\sqrt{x^2 + x + 1} + \frac{9}{16} \ln \left| x + \frac{1}{2} + \sqrt{x^2 + x + 1} \right| + c$$

