EXERCISE 19.30 PAGE NO: 19.176

Evaluate the following integrals:

1. $\int \frac{2x+1}{(x+1)(x-2)}\,dx$

Solution:

Here the denominator is already factored.

So let

$$\frac{2x+1}{(x+1)(x-2)} = \frac{A}{x+1} + \frac{B}{x-2} \ldots\ldots (i)$$

$$\Rightarrow \frac{2x+1}{(x+1)(x-2)} = \frac{A(x-2)+B(x+1)}{(x+1)(x-2)}$$

$\Rightarrow 2x + 1 = A(x - 2) + B(x + 1)$...... (ii)

We need to solve for A and B. One way to do this is to pick values for x which will cancel each variable.

Put x = 2 in the above equation, we get

$\Rightarrow 2(2) + 1 = A(2 - 2) + B(2 + 1)$

$\Rightarrow 3B = 5$

$\Rightarrow B = \frac{5}{3}$

Now put x = – 1 in equation (ii), we get

$\Rightarrow 2(-1) + 1 = A((-1) - 2) + B((-1) + 1)$

$\Rightarrow -3A = -1$

$\Rightarrow A = \frac{1}{3}$

We put the values of A and B values back into our partial fractions in equation

(i) Now replace this as the integrand. We get

$$\int\left[\frac{A}{x+1}+\frac{B}{x-2}\right]dx$$

$$\Rightarrow \int\left[\frac{\frac{1}{3}}{x+1}+\frac{\frac{5}{3}}{x-2}\right]dx$$

Split up the integral,

$$\Rightarrow \frac{1}{3}\int\left[\frac{1}{x+1}\right]dx+\frac{5}{3}\int\left[\frac{1}{x-2}\right]dx$$

Let substitute u = x + 1 ⇒ du = dx and z = x – 2 ⇒ dz = dx, so the above equation becomes,

$$\Rightarrow \frac{1}{3}\int\left[\frac{1}{u}\right]du+\frac{5}{3}\int\left[\frac{1}{z}\right]dz$$

On integrating we get

$$\Rightarrow \frac{1}{3}\log|u|+\frac{5}{3}\log|z|+C$$

Substituting back, we get

$$\Rightarrow \frac{1}{3}\log|x+1|+\frac{5}{3}\log|x-2|+C$$

The absolute value signs account for the domain of the natural log function (x>0).

Hence,

$$\int\frac{2x+1}{(x+1)(x-2)}dx=\frac{1}{3}\log|x+1|+\frac{5}{3}\log|x-2|+C$$

2. $\int\frac{1}{x(x-2)(x-4)}dx$

Solution:

In the given equation the denominator is already factored.

So let

$$\frac{1}{x(x-2)(x-4)} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x-4} \ldots\ldots(i)$$

$$\Rightarrow \frac{1}{x(x-2)(x-4)} = \frac{A(x-2)(x-4) + Bx(x-4) + Cx(x-2)}{x(x-2)(x-4)}$$

$\Rightarrow 1 = A(x - 2)(x - 4) + Bx(x - 4) + Cx(x - 2) \ldots\ldots (ii)$

We need to solve for A, B and C. One way to do this is to pick values for x which will cancel each variable.

Put x = 0 in the above equation, we get

$\Rightarrow 1 = A(0 - 2)(0 - 4) + B(0)(0 - 4) + C(0)(0 - 2)$

$\Rightarrow 1 = 8A + 0 + 0$

$$\Rightarrow A = \frac{1}{8}$$

Now put x = 2 in equation (ii), we get

$\Rightarrow 1 = A(2 - 2)(2 - 4) + B(2)(2 - 4) + C(2)(2 - 2)$

$\Rightarrow 1 = 0 - 4B + 0$

$$\Rightarrow B = -\frac{1}{4}$$

Now put x = 4 in equation (ii), we get

$\Rightarrow 1 = A(4 - 2)(4 - 4) + B(4)(4 - 4) + C(4)(4 - 2)$

$\Rightarrow 1 = 0 + 0 + 8C$

$$\Rightarrow C = \frac{1}{8}$$

We put the values of A, B, and C values back into our partial fractions in equation (i) and replace this as the integrand. We get

$$\int \left[\frac{A}{x} + \frac{B}{x-2} + \frac{C}{x-4}\right] dx$$

$$\Rightarrow \int \left[\frac{\frac{1}{8}}{x} + \frac{-\frac{1}{4}}{x-2} + \frac{\frac{1}{8}}{x-4}\right] dx$$

Split up the integral,

$$\Rightarrow \frac{1}{8}\int \left[\frac{1}{x}\right] dx - \frac{1}{4}\int \left[\frac{1}{x-2}\right] dx + \frac{1}{8}\int \left[\frac{1}{x-4}\right] dx$$

Let substitute u = x – 4 ⇒ du = dx and z = x – 2 ⇒ dz = dx, so the above equation becomes,

$$\Rightarrow \frac{1}{8}\int \left[\frac{1}{x}\right] dx - \frac{1}{4}\int \left[\frac{1}{z}\right] dz + \frac{1}{8}\int \left[\frac{1}{u}\right] du$$

On integrating we get

$$\Rightarrow \frac{1}{8}\log|x| - \frac{1}{4}\log|z| + \frac{1}{8}\log|u| + C$$

Substituting back, we get

$$\Rightarrow \frac{1}{8}\log|x| - \frac{1}{4}\log|x-2| + \frac{1}{8}\log|x-4| + C$$

We will take $\frac{1}{8}$ common, we get

$$\Rightarrow \frac{1}{8}[\log|x| - 2\log|x-2| + \log|x-4| + C]$$

Applying the logarithm rule we can rewrite the above equation as

$$\Rightarrow \frac{1}{8}\left[\log\left|\frac{x}{(x-2)^2}\right| + \log|x-4| + C\right]$$

$$\Rightarrow \frac{1}{8}\left[\log\left|\frac{x(x-4)}{(x-2)^2}\right|\right] + C$$

The absolute value signs account for the domain of the natural log function (x>0).

Hence,

$$\int \frac{1}{x(x-2)(x-4)} dx = \frac{1}{8}\left[\log\left|\frac{x(x-4)}{(x-2)^2}\right|\right] + C$$

3. $\int \frac{x^2+x-1}{x^2+x-6} dx$

Solution:

First we have to simplify numerator, we get

$$\frac{x^2 + x - 1}{x^2 + x - 6}$$

$$= \frac{x^2 + x - 6 + 5}{x^2 + x - 6}$$

$$= \frac{x^2 + x - 6}{x^2 + x - 6} + \frac{5}{x^2 + x - 6}$$

$$= 1 + \frac{5}{x^2 + x - 6}$$

Now we will factorize denominator by splitting the middle term, we get

$$1 + \frac{5}{x^2 + x - 6}$$

The above equation can be written as

$$= 1 + \frac{5}{x^2 + 3x - 2x - 6}$$

By taking factors common

$$= 1 + \frac{5}{x(x + 3) - 2(x + 3)}$$

$$= 1 + \frac{5}{(x + 3)(x - 2)}$$

Now the denominator is factorized, so let separate the fraction through partial fraction, hence let

$$\frac{5}{(x + 3)(x - 2)} = \frac{A}{x + 3} + \frac{B}{x - 2} \dots\dots (i)$$

$$\Rightarrow \frac{5}{(x + 3)(x - 2)} = \frac{A(x - 2) + B(x + 3)}{(x + 3)(x - 2)}$$

$\Rightarrow 5 = A(x - 2) + B(x + 3)$...... (ii)

We need to solve for A and B. One way to do this is to pick values for x which will cancel each variable.

Put $x = 2$ in the above equation, we get

$\Rightarrow 5 = A (2 - 2) + B (2 + 3)$

$\Rightarrow 5 = 0 + 5B$

$\Rightarrow B = 1$

Now put $x = -3$ in equation (ii), we get

$\Rightarrow 5 = A ((-3) - 2) + B ((-3) + 3)$

$\Rightarrow 5 = -5A$
$\Rightarrow A = -1$

We put the values of A and B values back into our partial fractions in equation (i) and replace this as the integrand. We get

$$\int \left[1 + \frac{A}{x + 3} + \frac{B}{x - 2}\right] dx$$

$$\Rightarrow \int \left[1 + \frac{-1}{x + 3} + \frac{1}{x - 2}\right] dx$$

Split up the integral,

$$\Rightarrow \int 1 dx - \int \left[\frac{1}{x+3}\right] dx + \int \left[\frac{1}{x-2}\right] dx$$

Let substitute u = x + 3 ⇒ du = dx and z = x – 2 ⇒ dz = dx, so the above equation becomes,

$$\Rightarrow \int 1 dx - \int \left[\frac{1}{u}\right] du + \int \left[\frac{1}{z}\right] dz$$

On integrating we get

$\Rightarrow x - \log |u| + \log |z| + C$

Substituting back, we get

$\Rightarrow x - \log |x + 3| + \log |x - 2| + C$

Applying the logarithm rule, we can rewrite the above equation as

$$\Rightarrow x + \log\left|\frac{x-2}{x+3}\right| + C$$

The absolute value signs account for the domain of the natural log function (x>0).

Hence,

$$\int \frac{x^2 + x - 1}{x^2 + x - 6} dx = x + \log\left|\frac{x-2}{x+3}\right| + C$$

4. $\int \frac{3 + 4x - x^2}{(x+2)(x-1)} dx$

Solution:

First we simplify numerator, we get

$$\frac{3 + 4x - x^2}{(x+2)(x-1)}$$

$$= \frac{-(x^2 - 4x - 3)}{x^2 + x - 2}$$

$$= \frac{-(x^2 + x - 5x - 2 - 1)}{x^2 + x - 2}$$

$$= \frac{-(x^2 + x - 2)}{x^2 + x - 2} + \frac{5x + 1}{x^2 + x - 2}$$

$$= -1 + \frac{5x + 1}{(x + 2)(x - 1)}$$

Now the denominator is factorized, so let separate the fraction through partial fraction, hence let

$$\frac{5x + 1}{(x + 2)(x - 1)} = \frac{A}{x + 2} + \frac{B}{x - 1} \ldots\ldots (i)$$

$$\Rightarrow \frac{5x + 1}{(x + 2)(x - 1)} = \frac{A(x - 1) + B(x + 2)}{(x + 2)(x - 1)}$$

$\Rightarrow 5x + 1 = A(x - 1) + B(x + 2)$...... (ii)

We need to solve for A and B. One way to do this is to pick values for x which will cancel each variable.

Put x = 1 in the above equation, we get

$\Rightarrow 5(1) + 1 = A\ 1 - 1) + B\ (1 + 2)$

$\Rightarrow 6 = 0 + 3B$

$\Rightarrow B = 2$

Now put x = – 2 in equation (ii), we get

$\Rightarrow 5(-2) + 1 = A\ ((-2) - 1) + B\ ((-2) + 2)$

$\Rightarrow -9 = -3A + 0$

$\Rightarrow A = 3$

We put the values of A and B values back into our partial fractions in equation (i) and replace this as the integrand. We get

$$\int \left[-1 + \frac{5x + 1}{(x + 2)(x - 1)}\right] dx$$

$$\Rightarrow \int \left[-1 + \frac{A}{x + 2} + \frac{B}{x - 1}\right] dx$$

$$\Rightarrow \int \left[-1 + \frac{3}{x + 2} + \frac{2}{x - 1}\right] dx$$

Split up the integral,

$$\Rightarrow -\int 1dx + 3\int \left[\frac{1}{x + 2}\right] dx + 2\int \left[\frac{1}{x - 1}\right] dx$$

Let substitute u = x + 2 ⇒ du = dx and z = x – 1 ⇒ dz = dx, so the above equation becomes,

$$\Rightarrow -\int 1dx + 3\int \left[\frac{1}{u}\right] du + 2\int \left[\frac{1}{z}\right] dz$$

On integrating we get

$$\Rightarrow -x + 3\log|u| + 2\log|z| + C$$

Substituting back, we get

$$\Rightarrow -x + 3\log|x + 2| + 2\log|x - 1| + C$$

The absolute value signs account for the domain of the natural log function (x>0).

Hence,

$$\int \frac{3 + 4x - x^2}{(x + 2)(x - 1)} dx = -x + 3\log|x + 2| + 2\log|x - 1| + C$$

5. $\int \frac{x^2 + 1}{x^2 - 1} dx$

Solution:
First we simplify numerator, we get

$$\frac{x^2 + 1}{x^2 - 1}$$

$$= \frac{x^2 - 1 + 2}{x^2 - 1}$$

$$= \frac{x^2 - 1}{x^2 - 1} + \frac{2}{x^2 - 1}$$

$$= 1 + \frac{2}{(x-1)(x+1)}$$

Now the denominator is factorized, so let separate the fraction through partial fraction, hence let

$$\frac{2}{(x+1)(x-1)} = \frac{A}{x+1} + \frac{B}{x-1} \ldots\ldots (i)$$

$$\Rightarrow \frac{2}{(x+1)(x-1)} = \frac{A(x-1) + B(x+1)}{(x+1)(x-1)}$$

$$\Rightarrow 2 = A(x-1) + B(x+1) \ldots\ldots (ii)$$

We need to solve for A and B. One way to do this is to pick values for x which will cancel each variable.

Put x = 1 in the above equation, we get

$\Rightarrow 2 = A(1 - 1) + B(1 + 1)$

$\Rightarrow 2 = 0 + 2B$

$\Rightarrow B = 1$

Now put x = – 1 in equation (ii), we get

$\Rightarrow 2 = A((-1) - 1) + B((-1) + 1)$

$\Rightarrow 2 = -2A + 0$

$\Rightarrow A = -1$

We put the values of A and B values back into our partial fractions in equation (i) and replace this as the integrand. We get

$$\int \left[1 + \frac{2}{(x-1)(x+1)}\right] dx$$

$$\Rightarrow \int \left[1 + \frac{A}{x + 1} + \frac{B}{x - 1}\right] dx$$

$$\Rightarrow \int \left[1 + \frac{-1}{x + 1} + \frac{1}{x - 1}\right] dx$$

Split up the integral,

$$\Rightarrow \int 1dx - \int \left[\frac{1}{x + 1}\right] dx + \int \left[\frac{1}{x - 1}\right] dx$$

Let substitute u = x + 1 ⇒ du = dx and z = x – 1 ⇒ dz = dx, so the above equation becomes,

$$\Rightarrow \int 1dx - \int \left[\frac{1}{u}\right] du + \int \left[\frac{1}{z}\right] dz$$

On integrating we get

$$\Rightarrow x - \log|u| + \log|z| + C$$

Substituting back, we get

$$\Rightarrow x - \log|x + 1| + \log|x - 1| + C$$

Applying the logarithm rule we get

$$\Rightarrow x + \log\left|\frac{x - 1}{x + 1}\right| + C$$

The absolute value signs account for the domain of the natural log function (x>0).

Hence,

$$\int \frac{x^2 + 1}{x^2 - 1} dx = x + \log\left|\frac{x - 1}{x + 1}\right| + C$$