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1. Give an example of a function
(i) Which is one-one but not onto.
(ii) Which is not one-one but onto.
(iii) Which is neither one-one nor onto.

Solution:
(i) Let f: Z → Z given by f(x) = 3x + 2
Let us check one-one condition on f(x) = 3x + 2
Injectivity:
Let x and y be any two elements in the domain (Z), such that f(x) = f(y).
f (x) = f(y)
⇒ 3x + 2 =3y + 2
⇒ 3x = 3y
⇒ x = y
⇒ f(x) = f(y)
⇒ x = y
So, f is one-one.
Surjectivity:
Let y be any element in the co-domain (Z), such that f(x) = y for some element x in Z(domain).
Let f(x) = y
⇒ 3x + 2 = y
⇒ 3x = y - 2
⇒ x = (y - 2)/3. It may not be in the domain (Z)
Because if we take y = 3,
x = (y - 2)/3 = (3-2)/3 = 1/3 ∉ domain Z.
So, for every element in the co domain there need not be any element in the domain such that f(x) = y.
Thus, f is not onto.

(ii) Example for the function which is not one-one but onto
Let f: Z → N ∪ {0} given by f(x) = |x|
Injectivity:
Let x and y be any two elements in the domain (Z),

Such that f(x) = f(y).
⇒ |x| = |y|
⇒ x = ± y
So, different elements of domain f may give the same image.
So, f is not one-one.
Surjectivity:
Let y be any element in the co domain (Z), such that f(x) = y for some element x in Z (domain).
f(x) = y
⇒ |x| = y
⇒ x = ± y
Which is an element in Z (domain).
So, for every element in the co-domain, there exists a pre-image in the domain.
Thus, f is onto.

(iii) Example for the function which is neither one-one nor onto.
Let f: Z → Z given by f(x) = $2x^2 + 1$
Injectivity:
Let x and y be any two elements in the domain (Z), such that f(x) = f(y).
f(x) = f(y)
⇒ $2x^2+1 = 2y^2+1$
⇒ $2x^2 = 2y^2$
⇒ $x^2 = y^2$
⇒ x = ± y
So, different elements of domain f may give the same image.
Thus, f is not one-one.
Surjectivity:
Let y be any element in the co-domain (Z), such that f(x) = y for some element x in Z (domain).
f (x) = y
⇒ $2x^2+1=y$
⇒ $2x^2= y - 1$
⇒ $x^2 = (y-1)/2$
⇒ x = √ ((y-1)/2) ∉ Z always.
For example, if we take, y = 4,
x = ± √ ((y-1)/2)
= ± √ ((4-1)/2)

$= \pm \sqrt{(3/2)} \notin Z$
So, x may not be in Z (domain).
Thus, f is not onto.

2. Which of the following functions from A to B are one-one and onto?
(i) $f_1 = \{(1, 3), (2, 5), (3, 7)\}$; A = {1, 2, 3}, B = {3, 5, 7}
(ii) $f_2 = \{(2, a), (3, b), (4, c)\}$; A = {2, 3, 4}, B = {a, b, c}
(iii) $f_3 = \{(a, x), (b, x), (c, z), (d, z)\}$; A = {a, b, c, d,}, B = {x, y, z}.

Solution:
(i) Consider $f_1 = \{(1, 3), (2, 5), (3, 7)\}$; A = {1, 2, 3}, B = {3, 5, 7}
Injectivity:
$f_1(1) = 3$
$f_1(2) = 5$
$f_1(3) = 7$
⇒ Every element of A has different images in B.
So, f_1 is one-one.
Surjectivity:
Co-domain of f_1 = {3, 5, 7}
Range of f_1 =set of images = {3, 5, 7}
⇒ Co-domain = range
So, f_1 is onto.

(ii) Consider $f_2 = \{(2, a), (3, b), (4, c)\}$; A = {2, 3, 4}, B = {a, b, c}
Injectivity:
$f_2(2) = a$
$f_2(3) = b$
$f_2(4) = c$
⇒ Every element of A has different images in B.
So, f_2 is one-one.
Surjectivity:
Co-domain of f_2 = {a, b, c}
Range of f_2 = set of images = {a, b, c}
⇒ Co-domain = range
So, f_2 is onto.

(iii) Consider $f_3 = \{(a, x), (b, x), (c, z), (d, z)\}$; A = {a, b, c, d,}, B = {x, y, z}

Injectivity:
f_3 (a) = x
f_3 (b) = x
f_3 (c) = z
f_3 (d) = z
⇒ a and b have the same image x.
Also c and d have the same image z
So, f_3 is not one-one.
Surjectivity:
Co-domain of f_3 ={x, y, z}
Range of f_3 =set of images = {x, z}
So, the co-domain is not same as the range.
So, f_3 is not onto.

3. Prove that the function f: N → N, defined by $f(x) = x^2 + x + 1$, is one-one but not onto

Solution:
Given f: N → N, defined by $f(x) = x^2 + x + 1$
Now we have to prove that given function is one-one
Injectivity:
Let x and y be any two elements in the domain (N), such that f(x) = f(y).
⇒ $x^2 + x + 1 = y^2 + y + 1$
⇒ $(x^2 - y^2) + (x - y) = 0$ `
⇒ $(x + y)(x - y) + (x - y) = 0$
⇒ $(x - y)(x + y + 1) = 0$
⇒ x - y = 0 [x + y + 1 cannot be zero because x and y are natural numbers
⇒ x = y
So, f is one-one.
Surjectivity:
When x = 1
$x^2 + x + 1 = 1 + 1 + 1 = 3$
⇒ $x^2 + x + 1 \geq 3$, for every x in N.
⇒ f(x) will not assume the values 1 and 2.
So, f is not onto.

4. Let A = {−1, 0, 1} and f = {(x, x^2) : x ∈ A}. Show that f : A → A is neither one-one nor onto.

Solution:

Given A = {−1, 0, 1} and f = {(x, x^2): x ∈ A}

Also given that, $f(x) = x^2$

Now we have to prove that given function neither one-one or nor onto.

Injectivity:

Let x = 1

Therefore $f(1) = 1^2 = 1$ and

$f(-1) = (-1)^2 = 1$

⇒ 1 and -1 have the same images.

So, f is not one-one.

Surjectivity:

Co-domain of f = {-1, 0, 1}

$f(1) = 1^2 = 1$,

$f(-1) = (-1)^2 = 1$ and

f(0) = 0

⇒ Range of f = {0, 1}

So, both are not same.

Hence, f is not onto

5. Classify the following function as injection, surjection or bijection:

(i) f: N → N given by $f(x) = x^2$

(ii) f: Z → Z given by $f(x) = x^2$

(iii) f: N → N given by $f(x) = x^3$

(iv) f: Z → Z given by $f(x) = x^3$

(v) f: R → R, defined by $f(x) = |x|$

(vi) f: Z → Z, defined by $f(x) = x^2 + x$

(vii) f: Z → Z, defined by $f(x) = x - 5$

(viii) f: R → R, defined by $f(x) = \sin x$

(ix) f: R → R, defined by $f(x) = x^3 + 1$

(x) f: R → R, defined by $f(x) = x^3 - x$

(xi) f: R → R, defined by $f(x) = \sin^2 x + \cos^2 x$

(xii) f: Q − {3} → Q, defined by $f(x) = (2x + 3)/(x - 3)$

(xiii) f: Q → Q, defined by $f(x) = x^3 + 1$

(xiv) f: R → R, defined by $f(x) = 5x^3 + 4$

(xv) f: R → R, defined by $f(x) = 5x^3 + 4$

(xvi) f: R → R, defined by $f(x) = 1 + x^2$

(xvii) f: R → R, defined by f(x) = x/(x^2 + 1)

Solution:

(i) Given f: N → N, given by f(x) = x^2

Now we have to check for the given function is injection, surjection and bijection condition.

Injection condition:

Let x and y be any two elements in the domain (N), such that f(x) = f(y).

f(x) = f(y)

$x^2 = y^2$

x = y (We do not get ± because x and y are in N that is natural numbers)

So, f is an injection.

Surjection condition:

Let y be any element in the co-domain (N), such that f(x) = y for some element x in N (domain).

f(x) = y

x^2= y

x = √y, which may not be in N.

For example, if y = 3,

x = √3 is not in N.

So, f is not a surjection.

Also f is not a bijection.

(ii) Given f: Z → Z, given by f(x) = x^2

Now we have to check for the given function is injection, surjection and bijection condition.

Injection condition:

Let x and y be any two elements in the domain (Z), such that f(x) = f(y).

f(x) = f(y)

$x^2 = y^2$

x = ±y

So, f is not an injection.

Surjection test:

Let y be any element in the co-domain (Z), such that f(x) = y for some element x in Z (domain).

f(x) = y

x^2 = y

$x = \pm \sqrt{y}$ which may not be in Z.
For example, if y = 3,
$x = \pm \sqrt{3}$ is not in Z.
So, f is not a surjection.
Also f is not bijection.

(iii) Given f: N → N given by $f(x) = x^3$
Now we have to check for the given function is injection, surjection and bijection condition.
Injection condition:
Let x and y be any two elements in the domain (N), such that f(x) = f(y).
f(x) = f(y)
$x^3 = y^3$
x = y
So, f is an injection
Surjection condition:
Let y be any element in the co-domain (N), such that f(x) = y for some element x in N (domain).
f(x) = y
$x^3 = y$
$x = \sqrt[3]{y}$ which may not be in N.
For example, if y = 3,
$X = \sqrt[3]{3}$ is not in N.
So, f is not a surjection and f is not a bijection.

(iv) Given f: Z → Z given by $f(x) = x^3$
Now we have to check for the given function is injection, surjection and bijection condition.
Injection condition:
Let x and y be any two elements in the domain (Z), such that f(x) = f(y)
f(x) = f(y)
$x^3 = y^3$
x = y
So, f is an injection.
Surjection condition:
Let y be any element in the co-domain (Z), such that f(x) = y for some element x in Z (domain).

$f(x) = y$
$x^3 = y$
$x = \sqrt[3]{y}$ which may not be in Z.
For example, if y = 3,
$x = \sqrt[3]{3}$ is not in Z.
So, f is not a surjection and f is not a bijection.

(v) Given f: R → R, defined by $f(x) = |x|$
Now we have to check for the given function is injection, surjection and bijection condition.
Injection test:
Let x and y be any two elements in the domain (R), such that f(x) = f(y)
$f(x) = f(y)$
$|x|=|y|$
$x = \pm y$
So, f is not an injection.
Surjection test:
Let y be any element in the co-domain (R), such that f(x) = y for some element x in R (domain).
$f(x) = y$
$|x|=y$
$x = \pm y \in Z$
So, f is a surjection and f is not a bijection.

(vi) Given f: Z → Z, defined by $f(x) = x^2 + x$
Now we have to check for the given function is injection, surjection and bijection condition.
Injection test:
Let x and y be any two elements in the domain (Z), such that f(x) = f(y).
$f(x) = f(y)$
$x^2+ x = y^2 + y$
Here, we cannot say that x = y.
For example, x = 2 and y = - 3
Then,
$x^2 + x = 2^2 + 2 = 6$
$y^2 + y = (-3)^2 - 3 = 6$
So, we have two numbers 2 and -3 in the domain Z whose image is same as 6.

So, f is not an injection.
Surjection test:
Let y be any element in the co-domain (Z),
such that f(x) = y for some element x in Z (domain).
f(x) = y
$x^2 + x = y$
Here, we cannot say $x \in Z$.
For example, y = - 4.
$x^2 + x = -4$
$x^2 + x + 4 = 0$
$x = (-1 \pm \sqrt{-5})/2 = (-1 \pm i\sqrt{5})/2$ which is not in Z.
So, f is not a surjection and f is not a bijection.

(vii) Given f: Z → Z, defined by f(x) = x – 5
Now we have to check for the given function is injection, surjection and bijection condition.
Injection test:
Let x and y be any two elements in the domain (Z), such that f(x) = f(y).
f(x) = f(y)
x - 5 = y - 5
x = y
So, f is an injection.
Surjection test:
Let y be any element in the co-domain (Z), such that f(x) = y for some element x in Z (domain).
f(x) = y
x - 5 = y
x = y + 5, which is in Z.
So, f is a surjection and f is a bijection

(viii) Given f: R → R, defined by f(x) = sin x
Now we have to check for the given function is injection, surjection and bijection condition.
Injection test:
Let x and y be any two elements in the domain (R), such that f(x) = f(y).
f(x) = f(y)
Sin x = sin y

Here, x may not be equal to y because sin 0 = sin π.
So, 0 and π have the same image 0.
So, f is not an injection.
Surjection test:
Range of f = [-1, 1]
Co-domain of f = R
Both are not same.
So, f is not a surjection and f is not a bijection.

(ix) Given f: R → R, defined by $f(x) = x^3 + 1$
Now we have to check for the given function is injection, surjection and bijection condition.
Injection test:
Let x and y be any two elements in the domain (R), such that f(x) = f(y).
$f(x) = f(y)$
$x^3+1 = y^3+ 1$
$x^3= y^3$
$x = y$
So, f is an injection.
Surjection test:
Let y be any element in the co-domain (R), such that f(x) = y for some element x in R (domain).
$f(x) = y$
$x^3+1=y$
$x = \sqrt[3]{(y - 1)} \in R$
So, f is a surjection.
So, f is a bijection.

(x) Given f: R → R, defined by $f(x) = x^3 - x$
Now we have to check for the given function is injection, surjection and bijection condition.
Injection test:
Let x and y be any two elements in the domain (R), such that f(x) = f(y).
$f(x) = f(y)$
$x^3 - x = y^3 - y$
Here, we cannot say x = y.
For example, x = 1 and y = -1

$x^3 - x = 1 - 1 = 0$
$y^3 - y = (-1)^3 - (-1) - 1 + 1 = 0$
So, 1 and -1 have the same image 0.
So, f is not an injection.
Surjection test:
Let y be any element in the co-domain (R), such that f(x) = y for some element x in R (domain).
f(x) = y
$x^3 - x = y$
By observation we can say that there exist some x in R, such that $x^3 - x = y$.
So, f is a surjection and f is not a bijection.

(xi) Given f: R → R, defined by $f(x) = \sin^2 x + \cos^2 x$
Now we have to check for the given function is injection, surjection and bijection condition.
Injection condition:
$f(x) = \sin^2 x + \cos^2 x$
We know that $\sin^2 x + \cos^2 x = 1$
So, f(x) = 1 for every x in R.
So, for all elements in the domain, the image is 1.
So, f is not an injection.
Surjection condition:
Range of f = {1}
Co-domain of f = R
Both are not same.
So, f is not a surjection and f is not a bijection.

(xii) Given f: Q − {3} → Q, defined by f (x) = (2x +3)/(x-3)
Now we have to check for the given function is injection, surjection and bijection condition.
Injection test:
Let x and y be any two elements in the domain (Q − {3}), such that f(x) = f(y).
f(x) = f(y)
(2x + 3)/(x - 3) = (2y + 3)/(y - 3)
(2x + 3) (y − 3) = (2y + 3) (x − 3)
2xy − 6x + 3y − 9 = 2xy − 6y + 3x − 9
9x = 9y

$x = y$
So, f is an injection.
Surjection test:
Let y be any element in the co-domain (Q − {3}), such that f(x) = y for some element x in Q (domain).
$f(x) = y$
$(2x + 3)/(x - 3) = y$
$2x + 3 = x y - 3y$
$2x - x y = -3y - 3$
$x (2-y) = -3 (y + 1)$
$x = -3(y + 1)/(2 - y)$ which is not defined at y = 2.
So, f is not a surjection and f is not a bijection.

(xiii) Given f: Q → Q, defined by $f(x) = x^3 + 1$
Now we have to check for the given function is injection, surjection and bijection condition.
Injection test:
Let x and y be any two elements in the domain (Q), such that f(x) = f(y).
$f(x) = f(y)$
$x^3 + 1 = y^3 + 1$
$x^3 = y^3$
$x = y$
So, f is an injection.
Surjection test:
Let y be any element in the co-domain (Q), such that f(x) = y for some element x in Q (domain).
$f(x) = y$
$x^3 + 1 = y$
$x = \sqrt[3]{(y-1)}$, which may not be in Q.
For example, if y= 8,
$x^3 + 1 = 8$
$x^3 = 7$
$x = \sqrt[3]{7}$, which is not in Q.
So, f is not a surjection and f is not a bijection.

(xiv) Given f: R → R, defined by $f(x) = 5x^3 + 4$
Now we have to check for the given function is injection, surjection and bijection

condition.
Injection test:
Let x and y be any two elements in the domain (R), such that f(x) = f(y).
f(x) = f(y)
$5x^3 + 4 = 5y^3 + 4$
$5x^3 = 5y^3$
$x^3 = y^3$
x = y
So, f is an injection.
Surjection test:
Let y be any element in the co-domain (R), such that f(x) = y for some element x in R (domain).
f(x) = y
$5x^3 + 4 = y$
$x^3 = (y - 4)/5 \in R$
So, f is a surjection and f is a bijection.

(xv) Given f: R → R, defined by $f(x) = 5x^3 + 4$
Now we have to check for the given function is injection, surjection and bijection condition.
Injection condition:
Let x and y be any two elements in the domain (R), such that f(x) = f(y).
f(x) = f(y)
$5x^3 + 4 = 5y^3 + 4$
$5x^3 = 5y^3$
$x^3 = y^3$
x = y
So, f is an injection.
Surjection test:
Let y be any element in the co-domain (R), such that f(x) = y for some element x in R (domain).
f(x) = y
$5x^3 + 4 = y$
$x^3 = (y - 4)/5 \in R$
So, f is a surjection and f is a bijection.

(xvi) Given f: R → R, defined by $f(x) = 1 + x^2$

Now we have to check for the given function is injection, surjection and bijection condition.
Injection condition:
Let x and y be any two elements in the domain (R), such that f(x) = f(y).
f(x) = f(y)
$1 + x^2 = 1 + y^2$
$x^2 = y^2$
$x = \pm y$
So, f is not an injection.
Surjection test:
Let y be any element in the co-domain (R), such that f(x) = y for some element x in R (domain).
f(x) = y
$1 + x^2 = y$
$x^2 = y - 1$
x = ± √-1 = ± i` is not in R.
So, f is not a surjection and f is not a bijection.

(xvii) Given f: R → R, defined by $f(x) = x/(x^2 + 1)$
Now we have to check for the given function is injection, surjection and bijection condition.
Injection condition:
Let x and y be any two elements in the domain (R), such that f(x) = f(y).
f(x) = f(y)
$x /(x^2 + 1) = y /(y^2 + 1)$
$x y^2 + x = x^2y + y$
$xy^2 - x^2y + x - y = 0$
$-x y (-y + x) + 1 (x - y) = 0$
$(x - y) (1 - x y) = 0$
x = y or x = 1/y
So, f is not an injection.
Surjection test:
Let y be any element in the co-domain (R), such that f(x) = y for some element x in R (domain).
f(x) = y
$x /(x^2 + 1) = y$
$y x^2 - x + y = 0$

x = (-(-1) ± √ (1-4y^2))/(2y) if y ≠ 0
= (1 ± √ (1-4y^2))/ (2y), which may not be in R
For example, if y=1, then
(1 ± √ (1-4)) / (2y) = (1 ± i √3)/2, which is not in R
So, f is not surjection and f is not bijection.

6. If f: A → B is an injection, such that range of f = {a}, determine the number of elements in A.

Solution:
Given f: A → B is an injection
And also given that range of f = {a}
So, the number of images of f = 1
Since, f is an injection, there will be exactly one image for each element of f .
So, number of elements in A = 1.

7. Show that the function f: R − {3} → R − {2} given by f(x) = (x-2)/(x-3) is a bijection.

Solution:
Given that f: R − {3} → R − {2} given by f (x) = (x-2)/(x-3)
Now we have to show that the given function is one-one and on-to
Injectivity:
Let x and y be any two elements in the domain (R − {3}), such that f(x) = f(y).
f(x) = f(y)
⇒ (x - 2) /(x - 3) = (y - 2) /(y - 3)
⇒ (x - 2) (y - 3) = (y - 2) (x - 3)
⇒ x y – 3 x – 2 y + 6 = x y - 3y - 2x + 6
⇒ x = y
So, f is one-one.
Surjectivity:
Let y be any element in the co-domain (R − {2}), such that f(x) = y for some element x in R − {3} (domain).
f(x) = y
⇒ (x - 2) /(x - 3) = y
⇒ x - 2 = x y - 3y
⇒ x y - x = 3y - 2
⇒ x (y - 1) = 3y - 2

⇒ x = (3y - 2)/ (y - 1), which is in R - {3}
So, for every element in the co-domain, there exists some pre-image in the domain.
⇒ f is onto.
Since, f is both one-one and onto, it is a bijection.

8. Let *A* = [-1, 1]. Then, discuss whether the following function from *A* to itself is one-one, onto or bijective:
(i) f (x) = x/2
(ii) g (x) = |x|
(iii) h (x) = x^2

Solution:
(i) Given f: A → A, given by f (x) = x/2
Now we have to show that the given function is one-one and on-to
Injection test:
Let x and y be any two elements in the domain (A), such that f(x) = f(y).
f(x) = f(y)
x/2 = y/2
x = y
So, f is one-one.
Surjection test:
Let y be any element in the co-domain (A), such that f(x) = y for some element x in A (domain)
f(x) = y
x/2 = y
x = 2y, which may not be in A.
For example, if y = 1, then
x = 2, which is not in A.
So, f is not onto.
So, f is not bijective.

(ii) Given g: A → A, given by g (x) = |x|
Now we have to show that the given function is one-one and on-to
Injection test:
Let x and y be any two elements in the domain (A), such that f(x) = f(y).
g(x) = g(y)
|x| = |y|

$x = \pm y$
So, f is not one-one.
Surjection test:
For y = -1, there is no value of x in A.
So, g is not onto.
So, g is not bijective.

(iii) Given h: A → A, given by h (x) = x^2
Now we have to show that the given function is one-one and on-to
Injection test:
Let x and y be any two elements in the domain (A), such that h(x) = h(y).
h(x) = h(y)
$x^2 = y^2$
$x = \pm y$
So, f is not one-one.
Surjection test:
For y = - 1, there is no value of x in A.
So, h is not onto.
So, h is not bijective.

9. Are the following set of ordered pair of a function? If so, examine whether the mapping is injective or surjective:
(i) {(x, y): x is a person, y is the mother of x}
(ii) {(a, b): a is a person, b is an ancestor of a}

Solution:
Let f = {(x, y): x is a person, y is the mother of x}
As, for each element x in domain set, there is a unique related element y in co-domain set.
So, f is the function.
Injection test:
As, y can be mother of two or more persons
So, f is not injective.
Surjection test:
For every mother y defined by (x, y), there exists a person x for whom y is mother.
So, f is surjective.
Therefore, f is surjective function.

(ii) Let g = {(a, b): a is a person, b is an ancestor of a}
Since, the ordered map (a, b) does not map 'a' - a person to a living person.
So, g is not a function.

10. Let *A* = {1, 2, 3}. Write all one-one from *A* to itself.

Solution:
Given A = {1, 2, 3}
Number of elements in A = 3
Number of one-one functions = number of ways of arranging 3 elements = 3! = 6
(i) {(1, 1), (2, 2), (3, 3)}
(ii) {(1, 1), (2, 3), (3, 2)}
(iii) {(1, 2), (2, 2), (3, 3)}
(iv) {(1, 2), (2, 1), (3, 3)}
(v) {(1, 3), (2, 2), (3, 1)}
(vi) {(1, 3), (2, 1), (3,2)}

11. If f: R → R be the function defined by f(x) = $4x^3 + 7$, show that f is a bijection.

Solution:
Given f: R → R is a function defined by f(x) = $4x^3 + 7$
Injectivity:
Let x and y be any two elements in the domain (R), such that f(x) = f(y)
$\Rightarrow 4x^3 + 7 = 4y^3 + 7$
$\Rightarrow 4x^3 = 4y^3$
$\Rightarrow x^3 = y^3$
$\Rightarrow x = y$
So, f is one-one.
Surjectivity:
Let y be any element in the co-domain (R), such that f(x) = y for some element x in R (domain)
f(x) = y
$\Rightarrow 4x^3 + 7 = y$
$\Rightarrow 4x^3 = y - 7$
$\Rightarrow x^3 = (y - 7)/4$
$\Rightarrow x = \sqrt[3]{(y-7)/4}$ in R

So, for every element in the co-domain, there exists some pre-image in the domain. f is onto.

Since, f is both one-to-one and onto, it is a bijection.

EXERCISE 2.2 PAGE NO: 2.46

1. Find gof and fog when f: R → R and g : R → R is defined by
(i) f(x) = 2x + 3 and g(x) = x^2 + 5.
(ii) f(x) = 2x + x^2 and g(x) = x^3
(iii) f (x) = x^2 + 8 and g(x) = 3x^3 + 1
(iv) f (x) = x and g(x) = |x|
(v) f(x) = x^2 + 2x − 3 and g(x) = 3x − 4
(vi) f(x) = 8x^3 and g(x) = $x^{1/3}$

Solution:
(i) Given, f: R → R and g: R → R
So, gof: R → R and fog: R → R
Also given that f(x) = 2x + 3 and g(x) = x^2 + 5
Now, (gof) (x) = g (f (x))
= g (2x +3)
= $(2x + 3)^2 + 5$
= $4x^2 + 9 + 12x + 5$
= $4x^2 + 12x + 14$
Now, (fog) (x) = f (g (x))
= f (x^2 + 5)
= 2 (x^2 + 5) +3
= 2 x^2 + 10 + 3
= $2x^2 + 13$

(ii) Given, f: R → R and g: R → R
so, gof: R → R and fog: R → R
f(x) = 2x + x^2 and g(x) = x^3
(gof) (x)= g (f (x))
= g ($2x+x^2$)
= $(2x+x^2)^3$
Now, (fog) (x) = f (g (x))
= f (x^3)
= 2 (x^3) + $(x^3)^2$
= $2x^3 + x^6$

(iii) Given, f: R → R and g: R → R
So, gof: R → R and fog: R → R
$f(x) = x^2 + 8$ and $g(x) = 3x^3 + 1$
(gof) (x) = g (f (x))
$= g (x^2 + 8)$
$= 3 (x^2+8)^3 + 1$
Now, (fog) (x) = f (g (x))
$= f (3x^3 + 1)$
$= (3x^3+1)^2 + 8$
$= 9x^6 + 6x^3 + 1 + 8$
$= 9x^6 + 6x^3 + 9$

(iv) Given, f: R → R and g: R → R
So, gof: R → R and fog: R → R
f(x) = x and g(x) = $|x|$
(gof) (x) = g (f (x))
= g (x)
$= |x|$
Now (fog) (x) = f (g (x))
$= f (|x|)$
$= |x|$

(v) Given, f: R → R and g: R → R
So, gof: R → R and fog: R → R
$f(x) = x^2 + 2x - 3$ and $g(x) = 3x - 4$
(gof) (x) = g (f(x))
$= g (x^2 + 2x - 3)$
$= 3 (x^2 + 2x - 3) - 4$
$= 3x^2 + 6x - 9 - 4$
$= 3x^2 + 6x - 13$
Now, (fog) (x) = f (g (x))
$= f (3x - 4)$
$= (3x - 4)^2 + 2 (3x - 4) -3$
$= 9x^2 + 16 - 24x + 6x - 8 - 3$
$= 9x^2 - 18x + 5$

(vi) Given, f: R → R and g: R → R

So, gof: R → R and fog: R → R
$f(x) = 8x^3$ and $g(x) = x^{1/3}$
(gof) (x) = g (f (x))
$= g(8x^3)$
$= (8x^3)^{1/3}$
$= [(2x)^3]^{1/3}$
= 2x
Now, (fog) (x) = f (g (x))
$= f(x^{1/3})$
$= 8(x^{1/3})^3$
= 8x

2. Let f = {(3, 1), (9, 3), (12, 4)} and g = {(1, 3), (3, 3) (4, 9) (5, 9)}. Show that gof and fog are both defined. Also, find fog and gof.

Solution:
Given f = {(3, 1), (9, 3), (12, 4)} and g = {(1, 3), (3, 3) (4, 9) (5, 9)}
f : {3, 9, 12} → {1, 3, 4} and g : {1, 3, 4, 5} → {3, 9}
Co-domain of f is a subset of the domain of g.
So, gof exists and gof: {3, 9, 12} → {3, 9}
(gof) (3) = g (f (3)) = g (1) = 3
(gof) (9) = g (f (9)) = g (3) = 3
(gof) (12) = g (f (12)) = g (4) = 9
⇒ gof = {(3, 3), (9, 3), (12, 9)}
Co-domain of g is a subset of the domain of f.
So, fog exists and fog: {1, 3, 4, 5} → {3, 9, 12}
(fog) (1) = f (g (1)) = f (3) = 1
(fog) (3) = f (g (3)) = f (3) = 1
(fog) (4) = f (g (4)) = f (9) = 3
(fog) (5) = f (g (5)) = f (9) = 3
⇒ fog = {(1, 1), (3, 1), (4, 3), (5, 3)}

3. Let f = {(1, −1), (4, −2), (9, −3), (16, 4)} and g = {(−1, −2), (−2, −4), (−3, −6), (4, 8)}. Show that gof is defined while fog is not defined. Also, find gof.

Solution:
Given f = {(1, −1), (4, −2), (9, −3), (16, 4)} and g = {(−1, −2), (−2, −4), (−3, −6), (4, 8)}

f: {1, 4, 9, 16} → {-1, -2, -3, 4} and g: {-1, -2, -3, 4} → {-2, -4, -6, 8}
Co-domain of f = domain of g
So, gof exists and gof: {1, 4, 9, 16} → {-2, -4, -6, 8}
(gof) (1) = g (f (1)) = g (−1) = −2
(gof) (4) = g (f (4)) = g (−2) = −4
(gof) (9) = g (f (9)) = g (−3) = −6
(gof) (16) = g (f (16)) = g (4) = 8
So, gof = {(1, −2), (4, −4), (9, −6), (16, 8)}
But the co-domain of g is not same as the domain of f.
So, fog does not exist.

4. Let A = {a, b, c}, B = {u, v, w} and let f and g be two functions from A to B and from B to A, respectively, defined as: f = {(a, v), (b, u), (c, w)}, g = {(u, b), (v, a), (w, c)}. Show that f and g both are bijections and find fog and gof.

Solution:
Given f = {(a, v), (b, u), (c, w)}, g = {(u, b), (v, a), (w, c)}.
Also given that A = {a, b, c}, B = {u, v, w}
Now we have to show f and g both are bijective.
Consider f = {(a, v), (b, u), (c, w)} and f: A → B
Injectivity of f: No two elements of A have the same image in B.
So, f is one-one.
Surjectivity of f: Co-domain of f = {u, v, w}
Range of f = {u, v, w}
Both are same.
So, f is onto.
Hence, f is a bijection.
Now consider g = {(u, b), (v, a), (w, c)} and g: B → A
Injectivity of g: No two elements of B have the same image in A.
So, g is one-one.
Surjectivity of g: Co-domain of g = {a, b, c}
Range of g = {a, b, c}
Both are the same.
So, g is onto.
Hence, g is a bijection.
Now we have to find fog,
we know that Co-domain of g is same as the domain of f.

So, fog exists and fog: {u v, w} → {u, v, w}
(fog) (u) = f (g (u)) = f (b) = u
(fog) (v) = f (g (v)) = f (a) = v
(fog) (w) = f (g (w)) = f (c) = w
So, fog = {(u, u), (v, v), (w, w)}
Now we have to find gof,
Co-domain of f is same as the domain of g.
So, fog exists and gof: {a, b, c} → {a, b, c}
(gof) (a) = g (f (a)) = g (v) = a
(gof) (b) = g (f (b)) = g (u) = b
(gof) (c) = g (f (c)) = g (w) = c
So, gof = {(a, a), (b, b), (c, c)}

5. Find fog (2) and gof (1) when f: R → R; f(x) = x^2 + 8 and g: R → R; g(x) = $3x^3$ + 1.

Solution:
Given f: R → R; f(x) = x^2 + 8 and g: R → R; g(x) = $3x^3$ + 1.
Consider (fog) (2) = f (g (2))
= f (3 × 2^3 + 1)
= f(3 × 8 + 1)
= f (25)
= 25^2 + 8
= 633
(gof) (1) = g (f (1))
= g (1^2 + 8)
= g (9)
= 3 × 9^3 + 1
= 2188

6. Let R^+ be the set of all non-negative real numbers. If f: R^+ → R^+ and g : R^+ → R^+ are defined as f(x)=x^2 and g(x)=+ √x, find fog and gof. Are they equal functions.

Solution:
Given f: R^+ → R^+ and g: R^+ → R^+
So, fog: R^+ → R^+ and gof: R^+ → R^+
Domains of fog and gof are the same.
Now we have to find fog and gof also we have to check whether they are equal or not,

Consider (fog) (x) = f (g (x))
= f ($\sqrt{x}$)
= $\sqrt{x^2}$
= x
Now consider (gof) (x) = g (f (x))
= g (x^2)
= $\sqrt{x^2}$
= x
So, (fog) (x) = (gof) (x), $\forall x \in R^+$
Hence, fog = gof

7. Let f: R → R and g: R → R be defined by f(x) = x^2 and g(x) = x + 1. Show that fog ≠ gof.

Solution:
Given f: R → R and g: R → R.
So, the domains of f and g are the same.
Consider (fog) (x) = f (g (x))
= f (x + 1) = $(x + 1)^2$
= $x^2 + 1 + 2x$
Again consider (gof) (x) = g (f (x))
= g (x^2) = $x^2 + 1$
So, fog ≠ gof

EXERCISE 2.3 PAGE NO: 2.54

1. Find fog and gof, if
(i) f (x) = e^x, g (x) = $\log_e x$
(ii) f (x) = x^2, g (x) = cos x
(iii) f (x) = |x|, g (x) = sin x
(iv) f (x) = x+1, g(x) = e^x
(v) f (x) = $\sin^{-1} x$, g(x) = x^2
(vi) f (x) = x+1, g (x) = sin x
(vii) f(x)= x + 1, g (x) = 2x + 3
(viii) f(x) = c, c ∈ R, g(x) = sin x^2
(ix) f(x) = x^2 + 2 , g (x) = 1 − 1/ (1-x)

Solution:
(i) Given f (x) = e^x, g(x) = $\log_e x$
Let f: R → (0, ∞); and g: (0, ∞) → R
Now we have to calculate fog,
Clearly, the range of g is a subset of the domain of f.
fog: (0, ∞) → R
(fog) (x) = f (g (x))
= f ($\log_e x$)
= $\log_e e^x$
= x
Now we have to calculate gof,
Clearly, the range of f is a subset of the domain of g.
⇒ fog: R→ R
(gof) (x) = g (f (x))
= g (e^x)
= $\log_e e^x$
= x

(ii) f (x) = x^2, g(x) = cos x
f: R→ [0, ∞) ; g: R→[−1, 1]
Now we have to calculate fog,
Clearly, the range of g is not a subset of the domain of f.
⇒ Domain (fog) = {x: x ∈ domain of g and g (x) ∈ domain of f}
⇒ Domain (fog) = x: x ∈ R and cos x ∈ R}

⇒ Domain of (fog) = R
(fog): R→ R
(fog) (x) = f (g (x))
= f (cos x)
= $\cos^2 x$
Now we have to calculate gof,
Clearly, the range of f is a subset of the domain of g.
⇒ fog: R→R
(gof) (x) = g (f (x))
= g (x^2)
= cos x^2

(iii) Given f (x) = |x|, g(x) = sin x
f: R → (0, ∞) ; g : R→[−1, 1]
Now we have to calculate fog,
Clearly, the range of g is a subset of the domain of f.
⇒ fog: R→R
(fog) (x) = f (g (x))
= f (sin x)
= |sin x|
Now we have to calculate gof,
Clearly, the range of f is a subset of the domain of g.
⇒ fog : R→ R
(gof) (x) = g (f (x))
= g (|x|)
= sin |x|

(iv) Given f (x) = x + 1, g(x) = e^x
f: R→R ; g: R → [1, ∞)
Now we have calculate fog:
Clearly, range of g is a subset of domain of f.
⇒ fog: R→R
(fog) (x) = f (g (x))
= f (e^x)
= $e^x + 1$
Now we have to compute gof,
Clearly, range of f is a subset of domain of g.

$\Rightarrow$ fog: R→R
(gof) (x) = g (f (x))
= g (x+1)
= e^{x+1}

(v) Given f (x) = $\sin^{-1} x$, g(x) = x^2
f: [−1,1]→ [(-π)/2 ,π/2]; g : R → [0, ∞)
Now we have to compute fog:
Clearly, the range of g is not a subset of the domain of f.
Domain (fog) = {x: x ∈ domain of g and g (x) ∈ domain of f}
Domain (fog) = {x: x ∈ R and x^2 ∈ [−1, 1]}
Domain (fog) = {x: x ∈ R and x ∈ [−1, 1]}
Domain of (fog) = [−1, 1]
fog: [−1,1] → R
(fog) (x) = f (g (x))
= f (x^2)
= $\sin^{-1} (x^2)$
Now we have to compute gof:
Clearly, the range of f is a subset of the domain of g.
fog: [−1, 1] → R
(gof) (x) = g (f (x))
= g ($\sin^{-1} x$)
= $(\sin^{-1} x)^2$

(vi) Given f(x) = x+1, g(x) = sin x
f: R→R ; g: R→[−1, 1]
Now we have to compute fog
Clearly, the range of g is a subset of the domain of f.
Set of the domain of f.
$\Rightarrow$ fog: R→ R
(fog) (x) = f (g (x))
= f (sin x)
= sin x + 1
Now we have to compute gof,
Clearly, the range of f is a subset of the domain of g.
$\Rightarrow$ fog: R → R
(gof) (x) = g (f (x))

= g (x+1)
= sin (x+1)

(vii) Given f (x) = x+1, g (x) = 2x + 3
f: R→R ; g: R → R
Now we have to compute fog
Clearly, the range of g is a subset of the domain of f.
⇒ fog: R→ R
(fog) (x) = f (g (x))
= f (2x+3)
= 2x + 3 + 1
= 2x + 4
Now we have to compute gof
Clearly, the range of f is a subset of the domain of g.
⇒ fog: R → R
(gof) (x) = g (f (x))
= g (x+1)
= 2 (x + 1) + 3
= 2x + 5

(viii) Given f (x) = c, g (x) = sin x^2
f: R → {c} ; g: R→ [0, 1]
Now we have to compute fog
Clearly, the range of g is a subset of the domain of f.
fog: R→R
(fog) (x) = f (g (x))
= f (sin x^2)
= c
Now we have to compute gof,
Clearly, the range of f is a subset of the domain of g.
⇒ fog: R→ R
(gof) (x) = g (f (x))
= g (c)
= sin c^2

(ix) Given f (x) = x^2+ 2 and g (x) = 1 - 1 / (1 - x)
f: R → [2, ∞)

For domain of g: $1 - x \neq 0$

$\Rightarrow x \neq 1$

$\Rightarrow$ Domain of g = R – {1}

$g(x) = 1 - [1/(1 - x)] = (1 - x - 1)/(1 - x) = -x/(1 - x)$

For range of g

$y = (-x)/(1 - x)$

$\Rightarrow y - xy = -x$

$\Rightarrow y = xy - x$

$\Rightarrow y = x(y - 1)$

$\Rightarrow x = y/(y - 1)$

Range of g = R – {1}

So, g: R – {1} → R – {1}

Now we have to compute fog

Clearly, the range of g is a subset of the domain of f.

$\Rightarrow$ fog: R – {1} → R

$(fog)(x) = f(g(x))$

$= f(-x/(1 - x))$

$= ((-x)/(1 - x))^2 + 2$

$= (x^2 + 2x^2 + 2 - 4x) / (1 - x)^2$

$= (3x^2 - 4x + 2)/(1 - x)^2$

Now we have to compute gof

Clearly, the range of f is a subset of the domain of g.

$\Rightarrow$ gof: R→R

$(gof)(x) = g(f(x))$

$= g(x^2 + 2)$

$= 1 - 1 / (1 - (x^2 + 2))$

$= - 1/(1 - (x^2 + 2))$

$= (x^2 + 2)/(x^2 + 1)$

2. Let $f(x) = x^2 + x + 1$ and $g(x) = \sin x$. Show that fog ≠ gof.

Solution:

Given $f(x) = x^2 + x + 1$ and $g(x) = \sin x$

Now we have to prove fog ≠ gof

$(fog)(x) = f(g(x))$

$= f(\sin x)$

$= \sin^2 x + \sin x + 1$

And (gof) (x) = g (f (x))
= g (x^2+ x + 1)
= sin (x^2+ x + 1)
So, fog ≠ gof.

3. If f(x) = |x|, prove that fof = f.

Solution:
Given f(x) = |x|,
Now we have to prove that fof = f.
Consider (fof) (x) = f (f (x))
= f (|x|)
= ||x||
= |x|
= f (x)
So,
(fof) (x) = f (x), ∀x ∈ R
Hence, fof = f

4. If f(x) = 2x + 5 and g(x) = x^2 + 1 be two real functions, then describe each of the following functions:
(i) fog
(ii) gof
(iii) fof
(iv) f^2
Also, show that fof ≠ f^2

Solution:
f(x) and g(x) are polynomials.
⇒ f: R → R and g: R → R.
So, fog: R → R and gof: R → R.
(i) (fog) (x) = f (g (x))
= f (x^2 + 1)
= 2 (x^2 + 1) + 5
=2x^2 + 2 + 5
= 2x^2 +7

(ii) (gof) (x) = g (f (x))
= g (2x +5)
= $(2x + 5)^2 + 1$
= $4x^2 + 20x + 26$

(iii) (fof) (x) = f (f (x))
= f (2x +5)
= 2 (2x + 5) + 5
= 4x + 10 + 5
= 4x + 15

(iv) f^2 (x) = f (x) x f (x)
= (2x + 5) (2x + 5)
= $(2x + 5)^2$
= $4x^2 + 20x + 25$
Hence, from (iii) and (iv) clearly fof ≠ f^2

5. If f(x) = sin x and g(x) = 2x be two real functions, then describe gof and fog. Are these equal functions?

Solution:
Given f(x) = sin x and g(x) = 2x
We know that
f: R→ [−1, 1] and g: R→ R
Clearly, the range of f is a subset of the domain of g.
gof: R→ R
(gof) (x) = g (f (x))
= g (sin x)
= 2 sin x
Clearly, the range of g is a subset of the domain of f.
fog: R → R
So, (fog) (x) = f (g (x))
= f (2x)
= sin (2x)
Clearly, fog ≠ gof
Hence they are not equal functions.

6. Let f, g, h be real functions given by f(x) = sin x, g (x) = 2x and h (x) = cos x. Prove that fog = go (f h).

Solution:

Given that f(x) = sin x, g (x) = 2x and h (x) = cos x
We know that f: R→ [−1, 1] and g: R→ R
Clearly, the range of g is a subset of the domain of f.
fog: R → R
Now, (f h) (x) = f (x) h (x) = (sin x) (cos x) = ½ sin (2x)
Domain of f h is R.
Since range of sin x is [-1, 1], −1 ≤ sin 2x ≤ 1
⇒ -1/2 ≤ sin x/2 ≤ 1/2
Range of f h = [-1/2, 1/2]
So, (f h): R → [(-1)/2, 1/2]
Clearly, range of f h is a subset of g.
⇒ go (f h): R → R
⇒ Domains of fog and go (f h) are the same.
So, (fog) (x) = f (g (x))
= f (2x)
= sin (2x)
And (go (f h)) (x) = g ((f(x). h(x))
= g (sin x cos x)
= 2sin x cos x
= sin (2x)
⇒ (fog) (x) = (go (f h)) (x), ∀x ∈ R
Hence, fog = go (f h)

EXERCISE 2.4 PAGE NO: 2.68

1. State with reason whether the following functions have inverse:
(i) f: {1, 2, 3, 4} → {10} with f = {(1, 10), (2, 10), (3, 10), (4, 10)}
(ii) g: {5, 6, 7, 8} → {1, 2, 3, 4} with g = {(5, 4), (6, 3), (7, 4), (8, 2)}
(iii) h: {2, 3, 4, 5} → {7, 9, 11, 13} with h = {(2, 7), (3, 9), (4, 11), (5, 13)}

Solution:
(i) Given f: {1, 2, 3, 4} → {10} with f = {(1, 10), (2, 10), (3, 10), (4, 10)}
We have:
f (1) = f (2) = f (3) = f (4) = 10
⇒ f is not one-one.
⇒ f is not a bijection.
So, f does not have an inverse.

(ii) Given g: {5, 6, 7, 8} → {1, 2, 3, 4} with g = {(5, 4), (6, 3), (7, 4), (8, 2)}
from the question it is clear that g (5) = g (7) = 4
⇒ f is not one-one.
⇒ f is not a bijection.
So, f does not have an inverse.

(iii) Given h: {2, 3, 4, 5} → {7, 9, 11, 13} with h = {(2, 7), (3, 9), (4, 11), (5, 13)}
Here, different elements of the domain have different images in the co-domain.
⇒ h is one-one.
Also, each element in the co-domain has a pre-image in the domain.
⇒ h is onto.
⇒ h is a bijection.
Therefore h inverse exists.
⇒ h has an inverse and it is given by
h^{-1} = {(7, 2), (9, 3), (11, 4), (13, 5)}

2. Find f^{-1} if it exists: f: A → B, where
(i) A = {0, −1, −3, 2}; B = {−9, −3, 0, 6} and f(x) = 3 x.
(ii) A = {1, 3, 5, 7, 9}; B = {0, 1, 9, 25, 49, 81} and f(x) = x^2

Solution:
(i) Given A = {0, −1, −3, 2}; B = {−9, −3, 0, 6} and f(x) = 3 x.
So, f = {(0, 0), (-1, -3), (-3, -9), (2, 6)}

Here, different elements of the domain have different images in the co-domain.
Clearly, this is one-one.
Range of f = Range of f = B
so, f is a bijection and,
Thus, f^{-1} exists.
Hence, f^{-1}= {(0, 0), (-3, -1), (-9, -3), (6, 2)}

(ii) Given A = {1, 3, 5, 7, 9}; B = {0, 1, 9, 25, 49, 81} and f(x) = x^2
So, f = {(1, 1), (3, 9), (5, 25), (7, 49), (9, 81)}
Here, different elements of the domain have different images in the co-domain.
Clearly, f is one-one.
But this is not onto because the element 0 in the co-domain (B) has no pre-image in the domain (A)
⇒ f is not a bijection.
So, f^{-1}does not exist.

3. Consider f: {1, 2, 3} → {a, b, c} and g: {a, b, c} → {apple, ball, cat} defined as f (1) = a, f (2) = b, f (3) = c, g (a) = apple, g (b) = ball and g (c) = cat. Show that f, g and gof are invertible. Find f^{-1}, g^{-1} and $gof^{-1}$and show that $(gof)^{-1}$ = f^{-1}o g^{-1}

Solution:
Given f = {(1, a), (2, b), (c , 3)} and g = {(a , apple) , (b , ball) , (c , cat)} Clearly , f and g are bijections.
So, f and g are invertible.
Now,
f^{-1} = {(a ,1) , (b , 2) , (3,c)} and g^{-1} = {(apple, a), (ball , b), (cat , c)}
So, f^{-1} o g^{-1}= {apple, 1), (ball, 2), (cat, 3)}......... (1)
f: {1,2,3,} → {a, b, c} and g: {a, b, c} → {apple, ball, cat}
So, gof: {1, 2, 3} → {apple, ball, cat}
⇒ (gof) (1) = g (f (1)) = g (a) = apple
(gof) (2) = g (f (2))
= g (b)
= ball,
And (gof) (3) = g (f (3))
= g (c)
= cat
∴ gof = {(1, apple), (2, ball), (3, cat)}

Clearly, gof is a bijection.
So, gof is invertible.
$(gof)^{-1}$ = {(apple, 1), (ball, 2), (cat, 3)}....... (2)
Form (1) and (2), we get
$(gof)^{-1} = f^{-1} \text{ o } g^{-1}$

4. Let A = {1, 2, 3, 4}; B = {3, 5, 7, 9}; C = {7, 23, 47, 79} and f: A → B, g: B → C be defined as f(x) = 2x + 1 and g(x) = $x^2 - 2$. Express $(gof)^{-1}$ and $f^{-1} og^{-1}$ as the sets of ordered pairs and verify that $(gof)^{-1} = f^{-1} og^{-1}$.

Solution:
Given that f (x) = 2x + 1
⇒ f= {(1, 2(1) + 1), (2, 2(2) + 1), (3, 2(3) + 1), (4, 2(4) + 1)}
= {(1, 3), (2, 5), (3, 7), (4, 9)}
Also given that g(x) = x^2-2
⇒ g = {(3, 3^2-2), (5, 5^2-2), (7, 7^2-2), (9, 9^2-2)}
= {(3, 7), (5, 23), (7, 47), (9, 79)}
Clearly f and g are bijections and, hence, f^{-1}: B→ A and g^{-1}: C→ B exist.
So, f^{-1}= {(3, 1), (5, 2), (7, 3), (9, 4)}
And g^{-1}= {(7, 3), (23, 5), (47, 7), (79, 9)}
Now, (f^{-1} o g^{-1}): C→ A
f^{-1} o g^{-1} = {(7, 1), (23, 2), (47, 3), (79, 4)}.........(1)
Also, f: A→B and g: B → C,
⇒ gof: A → C, $(gof)^{-1}$: C→ A
So, f^{-1} o $g^{-1}$and $(gof)^{-1}$ have same domains.
(gof) (x) = g (f (x))
=g (2x + 1)
=$(2x +1)^2 - 2$
⇒ (gof) (x) = $4x^2 + 4x + 1 - 2$
⇒ (gof) (x) = $4x^2 + 4x - 1$
Then, (gof) (1) = g (f (1))
= 4 + 4 − 1
=7,
(gof) (2) = g (f (2))
= $4(2)^2 + 4(2) - 1 = 23$,
(gof) (3) = g (f (3))
= $4(3)^2 + 4(3) - 1 = 47$ and

(gof) (4) = g (f (4))
= $4(4)^2 + 4(4) - 1 = 79$
So, gof = {(1, 7), (2, 23), (3, 47), (4, 79)}
⇒ $(gof)^{-1}$ = {(7, 1), (23, 2), (47, 3), (79, 4)}...... (2)
From (1) and (2), we get:
$(gof)^{-1} = f^{-1} \circ g^{-1}$

5. Show that the function f: Q → Q, defined by f(x) = 3x + 5, is invertible. Also, find f^{-1}

Solution:
Given function f: Q → Q, defined by f(x) = 3x + 5
Now we have to show that the given function is invertible.
Injection of f:
Let x and y be two elements of the domain (Q),
Such that f(x) = f(y)
⇒ 3x + 5 = 3y + 5
⇒ 3x = 3y
⇒ x = y
so, f is one-one.
Surjection of f:
Let y be in the co-domain (Q),
Such that f(x) = y
⇒ 3x +5 = y
⇒ 3x = y - 5
⇒ x = (y -5)/3 belongs to Q domain
⇒ f is onto.
So, f is a bijection and, hence, it is invertible.
Now we have to find f^{-1}:
Let $f^{-1}(x)$ = y...... (1)
⇒ x = f(y)
⇒ x = 3y + 5
⇒ x −5 = 3y
⇒ y = (x - 5)/3
Now substituting this value in (1) we get
So, $f^{-1}(x) = (x - 5)/3$

6. Consider f: R → R given by f(x) = 4x + 3. Show that f is invertible. Find the inverse

of f.

Solution:

Given f: R → R given by f(x) = 4x + 3

Now we have to show that the given function is invertible.

Consider injection of f:

Let x and y be two elements of domain (R),

Such that f(x) = f(y)

⇒ 4x + 3 = 4y + 3

⇒ 4x = 4y

⇒ x = y

So, f is one-one.

Now surjection of f:

Let y be in the co-domain (R),

Such that f(x) = y.

⇒ 4x + 3 = y

⇒ 4x = y -3

⇒ x = (y-3)/4 in R (domain)

⇒ f is onto.

So, f is a bijection and, hence, it is invertible.

Now we have to find f^{-1}

Let $f^{-1}(x)$ = y....... (1)

⇒ x = f (y)

⇒ x = 4y + 3

⇒ x − 3 = 4y

⇒ y = (x -3)/4

Now substituting this value in (1) we get

So, $f^{-1}(x)$ = (x-3)/4

7. Consider f: R → R^+ → [4, ∞) given by f(x) = x^2 + 4. Show that f is invertible with inverse f^{-1} of f given by $f^{-1}(x)$ = √ (x-4) where R^+ is the set of all non-negative real numbers.

Solution:

Given f: R → R^+ → [4, ∞) given by f(x) = x^2 + 4.

Now we have to show that f is invertible,

Consider injection of f:

Let x and y be two elements of the domain (Q),
Such that f(x) = f(y)
⇒ $x^2 + 4 = y^2 + 4$
⇒ $x^2 = y^2$
⇒ x = y (as co-domain as R+)
So, f is one-one
Now surjection of f:
Let y be in the co-domain (Q),
Such that f(x) = y
⇒ $x^2 + 4 = y$
⇒ $x^2 = y - 4$
⇒ x = √ (y-4) in R
⇒ f is onto.
So, f is a bijection and, hence, it is invertible.
Now we have to find f^{-1}:
Let f^{-1} (x) = y...... (1)
⇒ x = f (y)
⇒ $x = y^2 + 4$
⇒ $x - 4 = y^2$
⇒ y = √ (x-4)
So, $f^{-1}(x)$ = √ (x-4)
Now substituting this value in (1) we get,
So, $f^{-1}(x)$ = √ (x-4)

8. If f(x) = (4x + 3)/ (6x - 4), x ≠ (2/3) show that fof(x) = x, for all x ≠ (2/3). What is the inverse of f?

Solution:
It is given that f(x) = (4x + 3)/ (6x - 4), x ≠ 2/3
Now we have to show fof(x) = x
(fof)(x) = f (f(x))
= f ((4x+ 3)/ (6x - 4))
= (4((4x + 3)/ (6x -4)) + 3)/ (6 ((4x +3)/ (6x - 4)) - 4)
= (16x + 12 + 18x - 12)/ (24x + 18 - 24x + 16)
= (34x)/ (34)
= x
Therefore, fof(x) = x for all x ≠ 2/3

=> fof = 1
Hence, the given function f is invertible and the inverse of f is f itself.

9. Consider f: $R^+ \to [-5, \infty)$ given by $f(x) = 9x^2 + 6x - 5$. Show that f is invertible with $f^{-1}(x) = (\sqrt{(x+6)}-1)/3$

Solution:
Given f: $R^+ \to [-5, \infty)$ given by $f(x) = 9x^2 + 6x - 5$
We have to show that f is invertible.
Injectivity of f:
Let x and y be two elements of domain (R^+),
Such that f(x) = f(y)
$\Rightarrow 9x^2 + 6x - 5 = 9y^2 + 6y - 5$
$\Rightarrow 9x^2 + 6x = 9y^2 + 6y$
$\Rightarrow x = y$ (As, x, y $\in R^+$)
So, f is one-one.
Surjectivity of f:
Let y is in the co domain (Q)
Such that f(x) = y
$\Rightarrow 9x^2 + 6x - 5 = y$
$\Rightarrow 9x^2 + 6x = y + 5$
$\Rightarrow 9x^2 + 6x + 1 = y + 6$ (By adding 1 on both sides)
$\Rightarrow (3x + 1)^2 = y + 6$
$\Rightarrow 3x + 1 = \sqrt{(y + 6)}$
$\Rightarrow 3x = \sqrt{(y + 6)} - 1$
$\Rightarrow x = (\sqrt{(y + 6)}-1)/3$ in R^+ (domain)
f is onto.
So, f is a bijection and hence, it is invertible.
Now we have to find f^{-1}
Let $f^{-1}(x) = y$..... (1)
$\Rightarrow x = f(y)$
$\Rightarrow x = 9y^2 + 6y - 5$
$\Rightarrow x + 5 = 9y^2 + 6y$
$\Rightarrow x + 6 = 9y^2 + 6y + 1$ (adding 1 on both sides)
$\Rightarrow x + 6 = (3y + 1)^2$
$\Rightarrow 3y + 1 = \sqrt{(x + 6)}$
$\Rightarrow 3y = \sqrt{(x + 6)} - 1$

$\Rightarrow y = (\sqrt{(x+6)}-1)/3$
Now substituting this value in (1) we get,
So, $f^{-1}(x) = (\sqrt{(x+6)}-1)/3$

10. If f: R → R be defined by $f(x) = x^3 - 3$, then prove that f^{-1} exists and find a formula for f^{-1}. Hence, find $f^{-1}(24)$ and $f^{-1}(5)$.

Solution:
Given f: R → R be defined by $f(x) = x^3 - 3$
Now we have to prove that f^{-1} exists
Injectivity of f:
Let x and y be two elements in domain (R),
Such that, $x^3 - 3 = y^3 - 3$
$\Rightarrow x^3 = y^3$
$\Rightarrow x = y$
So, f is one-one.
Surjectivity of f:
Let y be in the co-domain (R)
Such that f(x) = y
$\Rightarrow x^3 - 3 = y$
$\Rightarrow x^3 = y + 3$
$\Rightarrow x = \sqrt[3]{(y+3)}$ in R
$\Rightarrow$ f is onto.
So, f is a bijection and, hence, it is invertible.
Finding f^{-1}:
Let $f^{-1}(x) = y$........ (1)
$\Rightarrow x = f(y)$
$\Rightarrow x = y^3 - 3$
$\Rightarrow x + 3 = y^3$
$\Rightarrow y = \sqrt[3]{(x + 3)} = f^{-1}(x)$ [from (1)]
So, $f^{-1}(x) = \sqrt[3]{(x + 3)}$
Now, $f^{-1}(24) = \sqrt[3]{(24 + 3)}$
$= \sqrt[3]{27}$
$= \sqrt[3]{3^3}$
$= 3$
And $f^{-1}(5) = \sqrt[3]{(5 + 3)}$
$= \sqrt[3]{8}$

$= \sqrt[3]{2^3}$
$= 2$

11. A function f: R → R is defined as f(x) = x^3 + 4. Is it a bijection or not? In case it is a bijection, find f^{-1} (3).

Solution:
Given that f: R → R is defined as f(x) = $x^3 + 4$
Injectivity of f:
Let x and y be two elements of domain (R),
Such that f (x) = f (y)
$\Rightarrow x^3 + 4 = y^3 + 4$
$\Rightarrow x^3 = y^3$
$\Rightarrow x = y$
So, f is one-one.
Surjectivity of f:
Let y be in the co-domain (R),
Such that f(x) = y.
$\Rightarrow x^3 + 4 = y$
$\Rightarrow x^3 = y - 4$
$\Rightarrow x = \sqrt[3]{(y - 4)}$ in R (domain)
⇒ f is onto.
So, f is a bijection and, hence, it is invertible.
Finding f^{-1}:
Let $f^{-1}(x) = y$…… (1)
$\Rightarrow x = f(y)$
$\Rightarrow x = y^3 + 4$
$\Rightarrow x - 4 = y^3$
$\Rightarrow y = \sqrt[3]{(x-4)}$
So, $f^{-1}(x) = \sqrt[3]{(x-4)}$ [from (1)]
$f^{-1}(3) = \sqrt[3]{(3 - 4)}$
$= \sqrt[3]{-1}$
$= -1$