EXERCISE 4.13 PAGE NO: 4.92

1. If $\cos^{-1}(x/2) + \cos^{-1}(y/3) = \alpha$, then prove that $9x^2 - 12xy\cos\alpha + 4y^2 = 36\sin^2\alpha$

Solution:

Given $\cos^{-1}(x/2) + \cos^{-1}(y/3) = \alpha$

We know that,

$$\cos^{-1}x + \cos^{-1}y = \cos^{-1}\left[xy - \sqrt{1-x^2}\sqrt{1-y^2}\right]$$

Now by substituting, we get

$$\Rightarrow \cos^{-1}\left[\frac{x}{2}\times\frac{y}{3} - \sqrt{1-\left(\frac{x}{2}\right)^2}\sqrt{1-\left(\frac{y}{3}\right)^2}\right] = \alpha$$

$$\Rightarrow \left[\frac{xy}{6} - \frac{\sqrt{4-x^2}}{2}\times\frac{\sqrt{9-y^2}}{3}\right] = \cos\alpha$$

$$\Rightarrow xy - \sqrt{4-x^2}\times\sqrt{9-y^2} = 6\cos\alpha$$

$$\Rightarrow xy - 6\cos\alpha = \sqrt{4-x^2}\sqrt{9-y^2}$$

On squaring both the sides we get

$$\Rightarrow (xy - 6\cos\alpha)^2 = (4-x^2)(9-y^2)$$

$$\Rightarrow x^2y^2 + 36\cos^2\alpha - 12xy\cos\alpha = 36 - 9x^2 - 4y^2 + x^2y^2$$

$$\Rightarrow 9x^2 + 4y^2 - 36 + 36\cos^2\alpha - 12xy\cos\alpha = 0$$

$\Rightarrow 9x^2 + 4y^2 - 12xy\cos\alpha - 36(1 - \cos^2\alpha) = 0$

$\Rightarrow 9x^2 + 4y^2 - 12xy\cos\alpha - 36\sin^2\alpha = 0$

$\Rightarrow 9x^2 + 4y^2 - 12xy\cos\alpha = 36\sin^2\alpha$

Hence, proved.

2. Solve the equation: $\cos^{-1}(a/x) - \cos^{-1}(b/x) = \cos^{-1}(1/b) - \cos^{-1}(1/a)$

Solution:

Given $\cos^{-1}(a/x) - \cos^{-1}(b/x) = \cos^{-1}(1/b) - \cos^{-1}(1/a)$

$\Rightarrow \cos^{-1}\frac{a}{x} + \cos^{-1}\frac{1}{a} = \cos^{-1}\frac{1}{b} + \cos^{-1}\frac{b}{x}$

We know that,

$$\cos^{-1}x + \cos^{-1}y = \cos^{-1}\left[xy - \sqrt{1-x^2}\sqrt{1-y^2}\right]$$

By substituting this formula we get,

$$\Rightarrow \cos^{-1}\left[\frac{1}{x} - \sqrt{1-\left(\frac{a}{x}\right)^2}\sqrt{1-\left(\frac{1}{a}\right)^2}\right] = \cos^{-1}\left[\frac{1}{x} - \sqrt{1-\left(\frac{b}{x}\right)^2}\sqrt{1-\left(\frac{1}{b}\right)^2}\right]$$

$$\Rightarrow \frac{1}{x} - \sqrt{1-\left(\frac{a}{x}\right)^2}\sqrt{1-\left(\frac{1}{a}\right)^2} = \frac{1}{x} - \sqrt{1-\left(\frac{b}{x}\right)^2}\sqrt{1-\left(\frac{1}{b}\right)^2}$$

$$\Rightarrow \sqrt{1-\left(\frac{a}{x}\right)^2}\sqrt{1-\left(\frac{1}{a}\right)^2} = \sqrt{1-\left(\frac{b}{x}\right)^2}\sqrt{1-\left(\frac{1}{b}\right)^2}$$

Squaring on both the sides, we get

$$\Rightarrow \left(1-\left(\frac{a}{x}\right)^2\right)\left(1-\left(\frac{1}{a}\right)^2\right) = \left(1-\left(\frac{b}{x}\right)^2\right)\left(1-\left(\frac{1}{b}\right)^2\right)$$

$\Rightarrow 1-\left(\frac{a}{x}\right)^2-\left(\frac{1}{a}\right)^2+\left(\frac{1}{x}\right)^2 = 1-\left(\frac{b}{x}\right)^2-\left(\frac{1}{b}\right)^2+\left(\frac{1}{x}\right)^2$

$\Rightarrow \left(\frac{b}{x}\right)^2-\left(\frac{a}{x}\right)^2=\left(\frac{1}{a}\right)^2-\left(\frac{1}{b}\right)^2$

On simplifying, we get

$\Rightarrow (b^2 - a^2)\, a^2b^2 = x^2(b^2 - a^2)$

$\Rightarrow x^2 = a^2b^2$

$\Rightarrow x = a\, b$