SECTION1 (Maximum Marks: 18)

- This section contains **SIX (06)** questions.
- The answer to each question is a **SINGLE DIGIT INTEGER** ranging from **0 TO 9, BOTH INCLUSIVE.**
- For Each question, enter the correct integer corresponding to the answer using the mouse and the on-screen virtual numeric keypad in the place designated to enter the answer.
- Answer to each question will be evaluated according to the following marking scheme:

Full Marks	:	+3	If ONLY the correct numerical value is entered;
Zero Marks	:	0	If the question is unanswered;
Negative Marks	:	–1	In all other cases.

1. For a complex number z, let Re(z) denote the real part of z. Let S be the set of all complex numbers z satisfying $z^4 - |z|^4 = 4iz^2$, where $i=\sqrt{-1}$. Then the minimum possible value of $|z_1 - z_2|^2$, where $z_1, z_2 \in S$ with Re(z_1) > 0 and Re(z_2) < 0, is ______

Answer: 1

Solution:

$z^4 - |z|^4 = 4iz^2$

$z^4 - 2|z|^2|\bar{z}|^2 = 4i\,z^2 \quad \because z\bar{z} = |z|^2$

$z^2\left(z^2 - (\bar{z})^2 - 4i\right) = 0$

$z^2 = 0$ or $z^2 - (\bar{z})^2 = 4i$

Let z = x + iy

We know

$z^2 - (\bar{z})^2 = 4i$

$(x+iy)^2 - (x-iy)^2 = 4i$

$x^2 - y^2 + 2ixy - \left(x^2 - y^2 - 2ixy\right) = 4i$

$4ixy = 4i$

$xy = 1$

Now $(z_1 - z_2)^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$

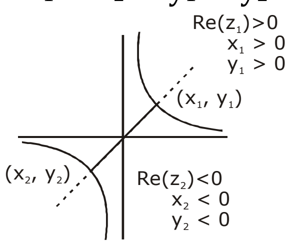
$= x_1^2 + x_2^2 + y_1^2 + y_2^2 - 2x_1x_2 - 2y_1y_2$

$= x_1^2 + x_2^2 + y_1^2 + y_2^2 + 2x_1(-x_2) + 2y_1(-y_2)$

Now A.M.≥ G.M.

$$\frac{\left(x_1^2 + x_2^2 + y_1^2 + y_2^2 + x_1(-x_2) + x_1(-x_2) + y_1(-y_2) + y_1(-y_2)\right)}{8} \geq (x_1^2\, x_2^2\, y_1^2 y_2^2 x_1^2\, x_2^2\, y_1^2 y_2^2)^{1/8}$$

$= x_1^2 + x_2^2 + y_1^2 + y_2^2 + 2x_1(-x_2) + 2y_1(-y_2) \geq 8 \quad \because x_1y_1 = 1;\ x_2y_2 = 1$



2. The probability that a missile hits a target successfully is 0.75.In order to destroy the target completely, at least three successful hits are required. Then the minimum number of missiles that have to be fired so that the probability of completely destroying the target is **NOT** less than 0.95, is_____.

Answer: 6

Solution:

Given

$P(\text{Hit}) = 0.75 = \frac{3}{4} = P(H)$

$\& \; P(\text{Hitnot}) = 0.25 = \frac{1}{4} \; P(\bar{H})$

$P(\text{target Hit}) \geq 0.95$

$1 - P$ (target not hit in n throws) ≥ 0.95

$$1 - {}^{n}C_0 \left(\bar{H}\right)^{n} - {}^{n}C_1 \left(\bar{H}\right)^{n-1} . \left(H\right) - {}^{n}C_2 \left(\bar{H}\right)^{n-2} \left(H\right)^{2} \geq 0.95$$

$$1 - \left(\frac{1}{4}\right)^{n} - n.\left(\frac{1}{4}\right)^{n-1} . \frac{3}{4} - \frac{n(n-1)}{2}\left(\frac{1}{4}\right)^{n-2}\left(\frac{3}{4}\right)^{2} \geq 0.95$$

$$1 - 0.95 \geq \left(\frac{1}{4}\right)^{n}\left[\frac{9n^2 - 3n + 2}{2}\right]$$

$$\left[9n^2 - 3n + 2\right] \leq \frac{4^n}{10}$$

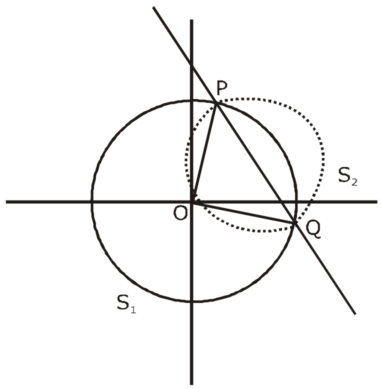
Now check n = 6

3. Let O be the centre of the circle $x^2 + y^2 = r^2$, where $r > \frac{\sqrt{5}}{2}$. Suppose PQ is a chord of this circle and the equation of the line passing through P and Q is $2x + 4y = 5$. If the centre of the circumcircle of the triangle OPQ lies on the line $x + 2y = 4$, then the value of ris_____

Answer: 2

Solution:

Given,



$S_1: x^2 + y^2 = r^2$ where $r > \frac{\sqrt{5}}{2} \Rightarrow C_1 = (0,0)$

Now let $S_2: x^2 + y^2 + ax + by = 0 \Rightarrow C_2 = \left(\frac{-a}{2}, \frac{-b}{2}\right)$

RA of $S_1 = 0$ & $S_2 = 0$ is PQ

PQ: $S_1 - S_2 = 0$

PQ: $ax + by + r^2 = 0$ ………………… (1)

Given PQ: $2x + 4y - 5 = 0$ ………………… (2)

On comparing equation (1) & (2)

$\frac{a}{2} = \frac{b}{4} = \frac{r^2}{-5}$ ………………(3)

Also centre of S_2 lies on $x + 2y = 4$

$\Rightarrow \frac{-a}{2} - b = 4$ ………………(4)

From equation (3) & (4)

$\frac{-r^2}{-5} - \frac{4r^2}{-5} = 4$

$-5r^2 = -20$

$r^2 = 4$

$r = 2$

4. The trace of a square matrix is defined to be the sum of its diagonal entries.If A is a 2 × 2 matrix such that the trace of A is 3 and the trace of A^3 is −18, then the value of the determinant of A is_____

Answer: 5

Solution:

$$\text{Let } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \ldots\ldots\ldots\ldots\ldots (1)$$

$$\Rightarrow T_r(A) = 3 \text{ (Given)}$$

a + d = 3

d = 3 – a put in equation (1)

$$\Rightarrow A = \begin{bmatrix} a & b \\ c & 3-a \end{bmatrix}$$

$T_r(A^3)$ = – 18 (Given)

$$A^3 = \begin{bmatrix} a & b \\ c & 3-a \end{bmatrix}\begin{bmatrix} a & b \\ c & 3-a \end{bmatrix}\begin{bmatrix} a & b \\ c & 3-a \end{bmatrix}$$

$$A^3 = \begin{bmatrix} a^2+bc & 3b \\ 3c & cb+(3-a)^2 \end{bmatrix}\begin{bmatrix} a & b \\ c & 3-a \end{bmatrix}$$

$$A^3 = \begin{bmatrix} a^3+abc+3bc & a^2b+b^2c+9b-3ab \\ 3ac+c^2b+c(3-a)^2 & 3bc+cb(3-a)+(3-a)^3 \end{bmatrix}$$

$T_r(A^3) = a^3 + abc + 3bc + 3bc + 3bc - abc + (3-a)^3 = -18$

$\Rightarrow a^3 + 9bc + (3-a)^3 = -18$

$\Rightarrow a^3 + 9bc + 27 - a^3 - 3.3a(3 - a) = -18$

$\Rightarrow a^2 - 3a + bc = -5$ ……………… (2)

Now $|A| = a(3 - a) - bc$

$= 3a - a^2 - bc = 5$ (From equation (2))

$|A| = 5$

5. Let the functions f: $(-1,1) \to$ R and $g: (-1, 1) \to (-1, 1)$ be defined by

$$f(x) = |2x - 1| + |2x + 1| \text{ and } g(x) = x - [x],$$

where $[x]$ denotes the greatest integer less than or equal to x. Let $f\circ g: (-1,1) \to R$ be the composite function defined by $(f\circ g)(x) = f(g(x))$. Suppose c is the number of points in the interval $(-1,1)$ at which $f\circ g$ is **NOT** continuous, and suppose d is the number of points in the interval $(-1,1)$ at which $f\circ g$ is **NOT** differentiable.Then the value of c $+d$ is ______

Answer: 4

Solution:

f(x) = |2x – 1| + |2x +1| (Given)

$$f(x) = \begin{cases} -4x & x \le \frac{-1}{2} \\ 2 & \frac{-1}{2} < x < \frac{1}{2} \\ 4x & x \ge \frac{1}{2} \end{cases}$$

g(x) = x – [x] = {x}

$$\text{Now } fog(x) = \begin{cases} -4g(x) & g(x) \le \frac{-1}{2} \\ 2 & \frac{-1}{2} < g(x) < \frac{1}{2} \\ 4g(x) & g(x) \ge \frac{1}{2} \end{cases}$$

$$fog\ (x) = \begin{cases} 2 & -1 < x < \frac{-1}{2} \\ 4\{x\} & \frac{-1}{2} \le x < 0 \\ 2 & 0 \le x < \frac{1}{2} \\ 4\{x\} & \frac{1}{2} \le x < 1 \end{cases}$$

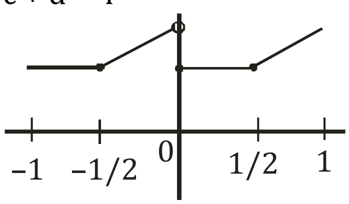
$$\text{fog}(x) = \begin{cases} 2 & -1 < x < \frac{-1}{2} \\ 4(x+1) & \frac{-1}{2} \le x < 0 \\ 2 & 0 \le x < \frac{1}{2} \\ 4x & \frac{1}{2} \le x < 1 \end{cases}$$

Now check

fog is not continuous at x = 0 only $\Rightarrow$c = 1

fog is not differentiable at x = $\frac{-1}{2}, 0, \frac{1}{2} \Rightarrow$d = 3

c + d = 4



6. The value of the limit

$$\lim_{x\to\frac{\pi}{2}} \frac{4\sqrt{2}(\sin 3x + \sin x)}{\left(2\sin 2x \sin\frac{3x}{2} + \cos\frac{5x}{2}\right) - \left(\sqrt{2} + \sqrt{2}\cos 2x + \cos\frac{3x}{2}\right)} \text{ is } \_\_\_\_$$

Answer: 8

Solution:

Using transformation and submultiple angle formula in denominator

$$\lim_{x\to\frac{\pi}{2}} \frac{4\sqrt{2}(\sin 3x + \sin x)}{\cos\frac{x}{2} - \cos\frac{7x}{2} + \cos\frac{5x}{2} - \sqrt{2}.2\cos^2 x - \cos\frac{3x}{2}}$$

$$\lim_{x\to\frac{\pi}{2}} \frac{4\sqrt{2}(\sin 3x + \sin x)}{\left(\cos\frac{x}{2} - \cos\frac{3x}{2}\right) + \left(\cos\frac{5x}{2} - \cos\frac{7x}{2}\right) - \sqrt{2}.2\cos^2 x}$$

$$\lim_{x\to\frac{\pi}{2}} \frac{8\sqrt{2}\sin 2x \cos x}{2\sin x \sin\frac{x}{2} + 2\sin 3x.\sin\frac{x}{2} - 2\sqrt{2}\cos^2 x}$$

$$\lim_{x\to\frac{\pi}{2}} \frac{8\sqrt{2}\sin 2x\cos x}{2\sin\frac{x}{2}\left(\sin x+\sin 3x\right)-2\sqrt{2}\cos^2 x}$$

$$\lim_{x\to\frac{\pi}{2}} \frac{16\sqrt{2}\sin x\cos^2 x}{2\sin\frac{x}{2}\left\{2\sin 2x.\cos x\right\}-2\sqrt{2}\cos^2 x} \because \ \sin 2x = 2\sin x \cos x$$

$$\lim_{x\to\frac{\pi}{2}} \frac{16\sqrt{2}\sin x}{2.4\sin\frac{x}{2}\sin x-2\sqrt{2}}$$

$$\frac{16\sqrt{2}}{8.\frac{1}{\sqrt{2}}-2\sqrt{2}}$$

$$\frac{32}{8-4}=\frac{32}{4}=8$$

SECTION 2 (Maximum Marks: 24)

- This section contains **SIX (06)** questions.
- Each question has **FOUR** options. **ONE OR MORE THAN ONE** of these four option(s) is(are) the correct answer(s).
- For each question, choose the option(s) corresponding to (all) the correct answer(s).
- Answer to each question will be evaluated according to the following marking scheme:

Full Marks : +4 If only (all) the correct option(s) is(are) chosen;

Partial Marks : +3 If all four options is correct but ONLY three options are chosen;

Partial Marks : +2 If three or more options are correct but ONLY two options are chosen; both of which are correct;

Partial Marks : +1 If two or more options are correct but ONLY one option is chosen and it is a correct option;

Zero Marks : 0 If none of the options is chosen (i.e. the questions is unanswered);

Negative Marks : –2 In all other cases.

7. Let b be a nonzero real number. Suppose f = R $\to$ R is a differentiable function such that f(0) = 1. If the derivative f′ of f satisfies the equation.

$$f'(x) = \frac{f(x)}{b^2 + x^2}$$

For all $x \in$ R, then which of the following statements is/are TRUE?

(A) If b>0, then f is an increasing function

(B) If b<0, then f is a decreasing function

(C) f(x)f(−x) = 1for all$x \in$ R

(D) f(x) − f(−x) = 0for all$x \in$ R

Answer: A,C

Solution:

Given

$$f'(x) = \frac{f(x)}{b^2 + x^2}$$

$$\frac{f'(x)}{f(x)} = \frac{1}{b^2 + x^2}$$

On integrating both sides

$$\int \frac{f'(x)}{f(x)} dx = \int \frac{1}{b^2 + x^2} dx$$

$$\ell n\,(f(x)) = \frac{1}{b}\tan^{-1}\left(\frac{x}{b}\right) + c$$

Put x = 0 $\Rightarrow$ c = 0 $\quad \because f(0) = 1$ (Given)

(A) $f(x) = e^{\frac{1}{b}\tan^{-1}\left(\frac{x}{b}\right)}$

$\because f'(x) = \frac{f(x)}{b^2 + x^2} > 0 \Rightarrow f(x)$ is increasing function

$f(x) > 0 \ \forall\ x \in R$

(C) $f(x)\,f(-x) = e^{\left(\frac{1}{b}\tan^{-1}\left(\frac{x}{b}\right)\right) - \left(\frac{1}{b}\tan^{-1}\left(\frac{x}{b}\right)\right)} = e^0 = 1$

(D) $f(x) - f(-x) = e^{\frac{1}{b}\tan^{-1}\left(\frac{x}{b}\right)} - e^{-\frac{1}{b}\tan^{-1}\left(\frac{x}{b}\right)} \neq 0 \ \forall\ x \in R$

8. Let a and b be positive real numbers such that $a > 1$ and b $< a$. Let P be a point in the first quadrant that lies on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. Suppose the tangent to the hyperbola at P passes through the point (1, 0), and suppose the normal to the hyperbola at P cuts off equal intercepts on the coordinate axes.Let Δdenote the area of the triangle formed by the tangent at P, the normal at Pand the x-axis. If e denotes the eccentricity of the hyperbola, then which of the following statements is/are TRUE?

(A) $1 < e < \sqrt{2}$

(B) $\sqrt{2} < e < 2$

(C) $\Delta = a^4$

(D) $\Delta = b^4$

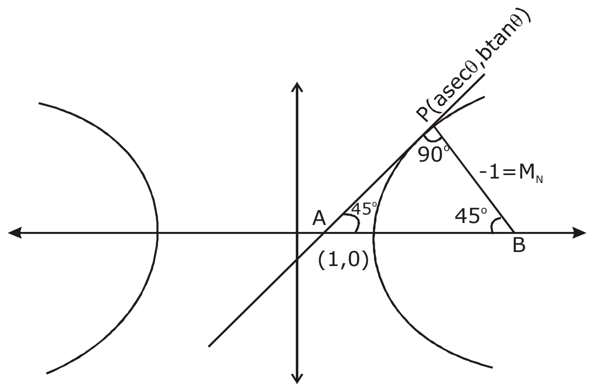
Answer: A,D

Solution:

Normal cuts equal intercepts

$\therefore$ $M_N = -1$

$M_T = 1$



Equation of tangent at P $\Rightarrow \dfrac{x\sec\theta}{a} - \dfrac{y\tan\theta}{b} = 1$

$\downarrow$ Passes through (1, 0)

$\sec\theta = a$

$M_T = 1 \Rightarrow +\left(\dfrac{b\sec\theta}{a\tan\theta}\right) = 1 \Rightarrow b = \tan\theta \qquad \because a = \sec\theta$

$b^2 = a^2\,(e^2 - 1) \Rightarrow e^2 - 1 = \sin^2\theta \Rightarrow e^2 = 1 + \sin^2\theta \;\; (\because\; 0 < \theta < \pi/2)$

$\Rightarrow 1 < e^2 < 2 \Rightarrow 1 < e < \sqrt{2}$

Area $\Delta = \dfrac{1}{2}$ (AP) (AP) $\qquad\qquad \because$ AP = BP

$= \dfrac{1}{2}\left[\left(1 - \sec^2\theta\right)^2 + \left(\tan^2\theta\right)^2\right] = \tan^4\theta = b^4 \because b = \tan\theta$

9. Let f: R → Rand g: R → R be functions satisfying

$$f(x + y) = f(x) + f(y) + f(x)f(y) \text{ and } f(x) = xg(x)$$

For all x, y $\in$ R. If $\lim\limits_{x\to 0} g(x) = 1$, then which of the following statements is/are TRUE?

(A) f is differentiable at every $x \in$ R

(B) If g(0) = 1,then gis differentiable at every $x \in$ R

(C) The derivative $f'(1)$ is equal to 1

(D) The derivative $f'(0)$ is equal to 1

Answer: A,B,D

Solution:

Given

$f(x+y) = f(x) + f(y) + f(x)f(y)$ ………………. (1)

$f(x) = xg(x)$ ………………. (2)

Differentiating equation (1) with respect to y

$f'(x+y).1 = f'(y) + f(x)\, f'(y)$ ……………….(3)

Put y = 0 in equation (3)

$f'(x) = f'(0) + f(x)f'(0)$ ……………….(4)

$\because$ From equation (2)

$f(x) = xg(x)$

differentiating with respect to x

$f'(x) = xg'(x) + g(x)$ ………. (5)

put x = 0 in equation (5)

we get $f'(0) = g(0) = 1$ (given)

$f'(0) = 1$ put in equation (4)

$f'(x) = f(x) + 1$

$$\frac{f'(x)}{1+f(x)} = 1$$

On integrating both sides

We get $\int \frac{f(x)}{f(x)+1}dx = \int dx$

$\Rightarrow \ln(f(x)+1) = x + c$ ………………. (6)

Put x = 0 in equation (6)

$\ell n\ (1+f(0)) = c \Rightarrow c = 0$ put in equation (6) $\because$ put $x = 0$ in equation(2) $\Rightarrow f(0) = 0$

$1 + f(x) = e^x$

$f(x) = e^x - 1$

$f(1) = e - 1$

$f'(1) = f(1) + 1 = e - 1 + 1 = e \because f'(x) = f(x) + 1$

$f'(0) = f(0) + 1 = 0 + 1 = 1$

$$g(x) = \frac{f(x)}{x} = \frac{e^x - 1}{x}$$

We have check differentiability at x = 0

$$g'(0^+) = \lim_{h\to 0}\left(\frac{\frac{e^h-1}{h} - 1}{h}\right)$$

$$\Rightarrow \lim_{h\to 0}\left(\frac{e^h - 1 - h}{h^2}\right) = \frac{1}{2}$$

$$g'(0^-) = \lim_{h\to 0}\frac{\frac{e^{-h}-1}{-h} - 1}{-h}$$

$$\lim_{h\to 0}\frac{e^{-h} - 1 + h}{h^2} = \frac{1}{2}$$

10. Let α, β, γ, δ be real numbers such that $\alpha^2 + \beta^2 + \gamma^2 \neq 0$ and $\alpha + \gamma = 1$.Suppose the point $(3, 2, -1)$ is the mirror image of the point $(1, 0, -1)$ with respect to the plane $\alpha x + \beta y + \gamma z = \delta$.Then which of the following statements is/are TRUE?
(A)$\alpha + \beta = 2$
(B)$\delta - \gamma = 3$
(C)$\delta + \beta = 4$
(D) $\alpha + \beta + \gamma = \delta$

Answer: A,B,C

Solution:

Given equation of plane is $\alpha x + \beta y + \gamma z = \delta$

Q point is mid point of pp' $= \left(\frac{1+3}{2}, \frac{0+2}{2}, \frac{-1-1}{2}\right) = (2, 1, -1)$

Since point Q lie on the plane $\alpha x + \beta y + \gamma z = \delta$

$\therefore \alpha(2) + \beta(1) + \gamma(-1) = \delta$

$\Rightarrow 2\alpha + \beta - \gamma = \delta$ ....(1)

pp' is normal to given plane

$\frac{\alpha}{2} = \frac{\beta}{2} = \frac{\gamma}{0} = \lambda(\text{let})$

$\Rightarrow \alpha = 2\lambda, \beta = 2\lambda, \gamma = 0$

$\because \alpha + \gamma = 1$ (given)

$\therefore 2\lambda + 0 = 1$

$\Rightarrow \lambda = \frac{1}{2}$

$\therefore \alpha = 2\lambda = 2 \times \frac{1}{2} = 1, \beta = 2\lambda = 2 \times \frac{1}{2} = 1, \gamma = 0$

Now putting value of α, β, γ in Eq. (1) we get.

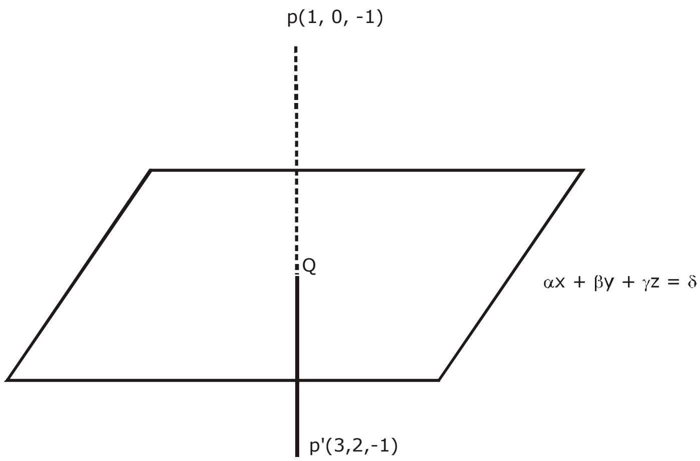
$2(1) + 1 - 0 = \delta \Rightarrow \delta = 3$

$\therefore \delta - \gamma = 3 - 0 = 3$

$\delta + \beta = 3 + 1 = 4$

$\alpha + \beta + \gamma = 1 + 1 + 0 = 2 (\neq \delta)$

$\alpha + \beta = 1 + 1 = 2$



11. Let a and b be positive real numbers. Suppose $\overrightarrow{PQ} = a\hat{i} + b\hat{j}$ and $\overrightarrow{PS} = a\hat{i} - b\hat{j}$ are adjacent sides of a parallelogram $PQRS$. Let, $\vec{u}$ and $\vec{v}$ be the projection vectors of $\vec{w} = \hat{i} + \hat{j}$ along $\overrightarrow{PQ}$ and $\overrightarrow{PS}$, respectively. If $|\vec{u}| + |\vec{v}| = |\vec{w}|$ and if the area of the parallelogram $PQRS$ is 8, then which of the following statements is/are TRUE?

(A) $a + b = 4$

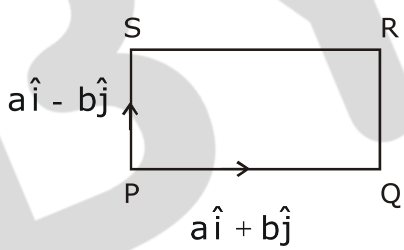
(B) $a - b = 2$

(C) The length of the diagonal PR of the parallelogram $PQRS$ is 4

(D) $\vec{w}$ is an angle bisector of the vectors $\overrightarrow{PQ}$ and $\overrightarrow{PS}$

Answer: A,C

Solution:



Given

$\overrightarrow{PQ} = a\hat{i} + b\hat{j}$

$\overrightarrow{PS} = a\hat{i} - b\hat{j}$

And $\vec{w} = \hat{i} + \hat{j}$

$$\therefore\ |\vec{u}| = \left|\frac{\vec{w}.\ \overrightarrow{PQ}}{|PQ|}\right| = \left|\frac{(\hat{i}+\hat{j}).(a\hat{i}+b\hat{j})}{|a\hat{i}+b\hat{j}|}\right|$$

$$\therefore |\vec{u}| = \frac{(a+b)}{\sqrt{a^2+b^2}}$$

$$|\vec{v}| = \left|\frac{(\vec{w}).\overrightarrow{ps}}{|\overrightarrow{ps}|}\right| = \left|\frac{(\hat{i}+\hat{j}).(a\hat{i}-b\hat{j})}{|a\hat{i}-b\hat{j}|}\right| = \frac{a-b}{\sqrt{a^2+b^2}}$$

$$\because |\vec{u}| + |\vec{v}| = |\vec{w}|$$

$$\frac{(a+b)+(a-b)}{\sqrt{a^2+b^2}} = \sqrt{1^2+1^2}$$

$$2a = \sqrt{2}\sqrt{a^2+b^2}$$

Squaring both side,

$4a^2 = 2(a^2 + b^2)$

$2a^2 = 2b^2$

$a = b$

Now, area of parallelogram = $\left\|\begin{matrix} \hat{i} & \hat{j} & \hat{k} \\ a & b & 0 \\ a & -b & 0 \end{matrix}\right\|$

$\Rightarrow 8 = \left|\hat{i}(0-0)-\hat{j}(0-0)+\hat{k}(-ab-ab)\right|$

$$\Rightarrow \left|-2ab\hat{k}\right| = 8$$

$ab = 4 \Rightarrow a^2 = 4 \quad (\because\ a = b)$

$a = 2 = b$

$a + b = 2 + 2 = 4$

$a - b = 2 - 2 = 0$

Length of diagonal of parallelogram $= \left|\left(a\hat{i} + bj\right) + \left(a\hat{i} - b\hat{j}\right)\right|$

$= \left|2a\hat{i}\right|$

$= 2a = 4$

$\overrightarrow{PQ} + \overrightarrow{PS} = 2a\hat{i}, 2b\hat{j} \neq \lambda\overrightarrow{w}$

12. For nonnegative integers s and r, let

$$\binom{s}{r} = \begin{cases} \dfrac{s!}{r!(s-r)!} & \text{if } r \leq s \\ & \text{if } r > s \end{cases}.$$

For positive integers m and n, let

$$g(m,n) = \sum_{p=0}^{m+n} \frac{f(m,n,p)}{\binom{n+p}{p}}$$

Where for any nonnegative integer p,

$$f(m,n,p) = \sum_{i=0}^{p} \binom{m}{i}\binom{n+i}{p}\binom{p+n}{p-i}.$$

Then which of the following statements is/are TRUE?

(A) $(m, n) = (n, m)$for all positive integers m, n

(B) $(m, n + 1) = (m + 1, n)$for all positive integers m, n

(C) $(2m, 2n) = 2\ (m, n)$for all positive integers m, n

(D)$(2m, 2n) = ((m, n))^2$for all positive integers m, n

Answer: A,B,D

Solution:

Given

$$f(m,n,p)=\sum_{i=0}^{p}\binom{m}{i}\binom{n+i}{p}\binom{p+n}{p-i}$$

$$=\sum_{i=0}^{p}{}^{m}C_i\cdot{}^{n+i}C_p\cdot{}^{n+p}C_{p-i}\qquad\left[\because\binom{n}{r}={}^{n}C_r\right]$$

$$=\sum_{i=0}^{p}{}^{m}C_i\frac{\lfloor\underline{n+i}}{\lfloor\underline{p}\;\lfloor\underline{n+i-p}}\cdot\frac{\lfloor\underline{n+p}}{\lfloor\underline{p-i}\;\lfloor\underline{n+p-p+i}}$$

$$=\sum_{i=0}^{p}{}^{m}C_i\left(\frac{\lfloor\underline{n+i}}{\lfloor\underline{p}\;\lfloor\underline{n+i-p}}\right)\cdot\left(\frac{\lfloor\underline{n+p}}{\lfloor\underline{p-i}\;\lfloor\underline{n+i}}\right)$$

$$=\sum_{i=0}^{p}{}^{m}C_i\left(\frac{\lfloor\underline{n+p}}{\lfloor\underline{p}\lfloor\underline{n+i-p}\;\lfloor\underline{p-i}}\right)$$

$$=\sum_{i=0}^{p}{}^{m}C_i\cdot\left(\frac{\lfloor\underline{n+p}}{\lfloor\underline{p}.\lfloor\underline{n}}\right)\left(\frac{\lfloor\underline{n}}{\lfloor\underline{n+i-p}\;\;\lfloor\underline{p-i}}\right)$$

$$={}^{n+p}C_p\left[\sum_{i=0}^{p}{}^{m}C_i\cdot{}^{n}C_{p-i}\right]$$

$$={}^{n+p}C_p\left[{}^{m}C_0\cdot{}^{n}C_p+{}^{m}C_1\,{}^{n}C_{p-1}+\ldots+{}^{m}C_m\,{}^{n}C_{p-m}\right]$$

Coefficient x^p in $(1+x)^n\,(x+1)^m$

$f(m,n,p)=({}^{n+p}C_p)\,({}^{m+n}C_p)$

$$g(m,n)=\sum_{p=0}^{m+n}\frac{f(m,n,p)}{\binom{n+p}{p}}=\sum_{p=0}^{m+n}\frac{\left({}^{n+p}C_p\right)\left({}^{m+n}C_p\right)}{\left({}^{n+p}C_p\right)}$$

$$g(m,n)=\sum_{p=0}^{m+n}{}^{m+n}C_p=2^{m+n}=2^{n+m}$$

$g(m,n)=g(n,m)$

$g(2m,2n)=2^{2(m+n)}=(2^{m+n})^2=(g(m,n))^2$

$g(m,n+1)=2^{m+n+1}=2^{(m+1)+n}=g(m+1,n)$

SECTION 3 (Maximum Marks: 24)

- This section contains **SIX (06)** questions. The answer to each question is a **NUMERICAL VALUE.**
- For each question, enter the correct numerical value of the answer using the mouse and the on-screen virtual numeric keypad in the place designated to enter the answer. If the numerical value has more than two decimal places, **truncate/round-off** the value to **TWO** decimal places.
- Answer to each question will be evaluated according to the following marking scheme:
  Full Marks : +4 If ONLY the correct numerical value is entered;
  Zero Marks : 0 In all other cases.

13. An engineer is required to visit a factory for exactly four days during the first 15 days of every month and it is mandatory that **no** two visits take place on consecutive days. Then the number of all possible ways in which such visits to the factory can be made by the engineer during 1-15 June 2021 is_____

Answer: 495.00
Solution:

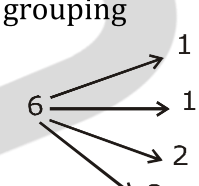
To select = 4 days
not selected days = 11 days
gaps = 12

$$\therefore {}^{12}C_4 = \frac{12\times 11\times 5\times 9}{24} = 495$$

14. In a hotel, four rooms are available. Six persons are to be accommodated in these four rooms in such a way that each of these rooms contains at least one person and at most two persons. Then the number of all possible ways in which this can be done is_____

Answer: 1080
Solution:

by grouping



$$\therefore \frac{6!}{1!1!2!2!2!2!} \times 4!$$

$$= \frac{720}{2\times 2\times 2\times 2} \times 24$$

$= 1080$

15. Two fair dice, each with faces numbered 1, 2, 3, 4, 5 and 6, are rolled together and the sum of the numbers on the faces is observed. This process is repeated till the sum is either a prime number or a perfect square. Suppose the sum turns out to be a perfect square before it turns out to be a prime number. If $p$is the probability that this perfect square is an odd number, then the value of 14 $p$is _____

Answer: 8.00

Solution:

Sum is prime =

2 (1,1)

3 (1,2)(2,1)

5 (2,3)(3,2) (1,4) (4,1)

7 (1,6) (2,5) (3,4) (4,3) (5,2) (6,1)

11 (5,6) (6,5)

P(prime) = $\frac{15}{36} = \frac{5}{12}$

Perfect square = $\underset{3}{4}, \underset{4}{9}$

P(perfect square) = $\frac{7}{36}$

Required probability(p) = $\dfrac{\frac{4}{36}+\left(\frac{14}{36}\right)\left(\frac{4}{36}\right)+\left(\frac{14}{36}\right)^2\left(\frac{4}{36}\right)+\dots\dots\dots}{\frac{7}{36}+\frac{14}{36}\times\frac{7}{36}+\left(\frac{14}{36}\right)^2\left(\frac{7}{36}\right)+\dots\dots\dots}$

$$=\frac{4\left(\frac{1}{36}+\left(\frac{14}{36}\right)\left(\frac{1}{36}\right)+\left(\frac{14}{36}\right)^2\left(\frac{1}{36}\right)+\dots\dots\dots\right)}{7\left(\frac{1}{36}+\frac{14}{36}\times\frac{1}{36}+\left(\frac{14}{36}\right)^2\left(\frac{1}{36}\right)+\dots\dots\dots\right)}$$

$p=\frac{4}{7}$

$\therefore$ 14p = 14 × $\frac{4}{7}$ = 8

16. Let the function f: [0, 1] →R be defined by $f\left(x\right)=\dfrac{4^x}{4^x+2}$ Then the value of

$$f\left(\frac{1}{40}\right)+f\left(\frac{2}{40}\right)+f\left(\frac{3}{40}\right)+\ldots+f\left(\frac{39}{40}\right)-f\left(\frac{1}{2}\right) \text{ is } \_\_\_\_$$

Answer: 19.00

Solution:

Given

$f:[0,1]\rightarrow R$

$$f(x)=\frac{4^x}{4^x+2} \quad \ldots\ldots\ldots\ldots(1)$$

Replace $x \rightarrow 1-x$

$$f(1-x)=\frac{4^{1-x}}{4^{1-x}+2}=\frac{\frac{4}{4^x}}{\frac{4}{4^x}+2}$$

$$=\frac{4}{4+2.4^x}$$

$$\Rightarrow f(1-x)=\frac{2}{2+4^x} \quad \ldots\ldots\ldots\ldots(2)$$

Adding (1) and (2) we get

$$\therefore f(x)+f(1-x)=\frac{4^x}{4^x+2}+\frac{2}{2+4^x}$$

$$\therefore f(x)+f(1-x)=1 \quad \ldots\ldots\ldots\ldots(3)$$

So, $f\left(\frac{1}{40}\right)+f\left(\frac{2}{40}\right)+\ldots\ldots f\left(\frac{37}{40}\right)+f\left(\frac{38}{40}\right)+f\left(\frac{39}{40}\right)-f\left(\frac{1}{2}\right)$

$$=f\left(\frac{1}{40}\right)+f\left(\frac{2}{40}\right)+\ldots\ldots+f\left(\frac{20}{40}\right)+\ldots\ldots+f\left(1-\frac{3}{40}\right)+f\left(1-\frac{2}{40}\right)+f\left(1-\frac{1}{40}\right)-f\left(\frac{1}{2}\right)$$

$$=\left\{f\left(\frac{1}{40}\right)+f\left(1-\frac{1}{40}\right)\right\}+\left\{f\left(\frac{2}{40}\right)+f\left(1-\frac{2}{40}\right)\right\}+\left\{f\left(\frac{3}{40}\right)+f\left(1-\frac{3}{40}\right)\right\}+\ldots\ldots+f\left(\frac{20}{40}\right)-f\left(\frac{1}{2}\right)$$

(From equation (3))

$$=\underbrace{1+1+1+\ldots\ldots\ldots1}_{19\ \text{times}}+f\left(\frac{20}{40}\right)-\left(\frac{1}{2}\right)$$

$$=19+f\left(\frac{1}{2}\right)-f\left(\frac{1}{2}\right)$$

$$=19$$

17. Let f: R → R be a differentiable function such that its derivative f' is continuous and $f(\pi) = -6$.

If F: [0, π] → R is defined by $F(x) = \int_0^x f(t)dt$, and if

$$\int_0^{\pi} (f'(x) + F(x))\cos x\, dx = 2,$$

Then the value of $f(0)$ is ______

Answer: 4.00

Solution:

Given, F: [0, π] → R

$F(x) = \int\limits_0^x f(t)dt$

Using Newton Leibniz

$F'(x) = f(x)\frac{d}{dx}x - f(0)\frac{d}{dx}(0)$

$\Rightarrow F'(x) = f(x)$ ………………….. (1)

$\because \int\limits_0^{\pi} (f'(x) + F(x))\cos x\, dx = 2$

$\therefore \int\limits_0^{\pi} f'(x)\cos x\, dx + \int\limits_0^{\pi} F(x)\cos x\, dx = 2$ ………………. (2)

$\therefore \int\limits_0^{\pi} f'(x)\cos x\, dx + \int\limits_0^{\pi} \underset{I}{F(x)}\underset{II}{\cos x}\, dx = 2$

Using by parts, we get

$\Rightarrow \int\limits_0^{\pi} f'(x)\cos x\, dx + (F(x).\sin x)_0^{\pi} - \int\limits_0^{\pi} F'(x)\sin x\, dx = 2$

$\Rightarrow \int\limits_0^{\pi} f'(x)\cos x\, dx + (F(\pi).\sin(\pi) - F(0)\sin(0)) - \int\limits_0^{\pi} F'(x)\sin x\, dx = 2$

$\Rightarrow \int\limits_0^{\pi} f'(x)\cos x\, dx - \int\limits_0^{\pi} f(x)\sin x\, dx = 2 \quad \left(\because \underset{\text{From }(1)}{F'(x) = f(x)}\right)$

$\Rightarrow \int\limits_0^{\pi} (f'(x)\cos x - f(x)\sin x)dx = 2$

$\Rightarrow \int\limits_0^{\pi} \frac{d}{dx}(f(x)\cos x)dx = 2$

$\Rightarrow f(x)\cos x\Big|_0^{\pi} = 2$

$\Rightarrow [f(\pi).\cos(\pi) - f(0).\cos(0)] = 2$

$\Rightarrow [(-6)(-1) - f(0)] = 2 \quad (\because f(\pi) = -6)$

$\Rightarrow [6 - f(0)] = 2$

$\Rightarrow f(0) = 4$

18. Let the function $f: (0, \pi) \to R$ be defined by $f(\theta) = (\sin\theta + \cos\theta)^2 + (\sin\theta - \cos\theta)^4$.

Suppose, the function f has a local minimum at θ precisely when $\theta \in \{\lambda_1\pi, \ldots, \lambda_r\pi\}$, where $0 < \lambda_1 < \cdots < \lambda_r < 1$. Then the value of $\lambda_1 + \cdots + \lambda_r$ is ______

Answer: 0.50

Solution:

Given, $f(\theta) = (\sin\theta + \cos\theta)^2 + (\sin\theta - \cos\theta)^4$

$= \sin^2\theta + \cos^2\theta + 2\sin\theta\cos\theta + ((\sin\theta - \cos\theta)^2)^2$

$= 1 + \sin 2\theta + (\sin^2\theta + \cos^2\theta - 2\sin\theta\cos\theta)^2$

$= 1 + \sin 2\theta + (1 - \sin 2\theta)^2$

$= 1 + \sin 2\theta + 1 + \sin^2 2\theta - 2\sin 2\theta$

$= \sin^2 2\theta - \sin 2\theta + 2$

$$\Rightarrow f(\theta) = \left(\sin 2\theta - \frac{1}{2}\right)^2 + \frac{7}{4}$$

$\because \theta \in [0, \pi]$

$\therefore 2\theta \in [0, 2\pi]$

$f(\theta)$ min. when $\sin 2\theta = \frac{1}{2}$

$$\therefore 2\theta = \frac{\pi}{6}, \frac{5\pi}{6}$$

$$\theta = \frac{\pi}{12}, \frac{5\pi}{12}$$

$$\lambda_1 = \frac{1}{12}, \qquad \lambda_2 = \frac{5}{12}$$

$$\lambda_1 + \lambda_2 = \frac{1}{12} + \frac{5}{12}$$

$$\lambda_1 + \lambda_2 = \frac{1}{2} = 0.50$$