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1. Verify Rolle's theorem for $f(x)=x^2+2x-8, x\in[-4,2]$.

Solution: Given function is $f(x)=x^2+2x-8, x\in[-4,2]$

(a) f(x) is a polynomial and polynomial function is always continuous.
So, function is continuous in [-4, 2].

(b) $f'(x)=2x+2,\ f'(x)$ exists in $[-4,2]$, so derivable.

(c) $f(-4)=0$ and $f(2)=0$

$f(-4)=f(2)$

All three conditions of Rolle's theorem are satisfied.

Therefore, there exists, at least one $c\in(-4,2)$ such that $f'(c)=0$

Which implies, 2c + 2 = 0 or c = -1.

2**. Examine if Rolles/ theorem is applicable to any of the following functions. Can you say something about the converse of Rolle's theorem from these examples:**

(i) $f(x)=[x]$ **for** $x\in[5,9]$
(ii) $f(x)=[x]$ **for** $x\in[-2,2]$
(iii) $f(x)=x^2-1$ **for** $x\in[1,2]$

Solution:
(i) Function is greatest integer function.
Given function is not differentiable and continuous
Hence Rolle's theorem is not applicable here.

(ii) Function is greatest integer function.
Given function is not differentiable and continuous.
Hence Rolle's theorem is not applicable.

(iii) $f(x)=x^2-1 \Rightarrow f(1)=(1)^2-1=1-1=0$

$f(2)=(2)^2-1=4-1=3 \therefore f(1)\neq f(2)$

Rolle's theorem is not applicable.

3. If $f:[-5,5]\to$ R is a differentiable function and if f' (x) does not vanish anywhere, then prove that $f(-5)\neq f(5)$.

Solution: As per Rolle's theorem, if

(a) f is continuous is $[a,b]$

(b) f is derivable in $[a,b]$

(c) $f(a)=f(b)$

Then, $f'(c)=0, c\in(a,b)$

It is given that f is continuous and derivable, but $f'(c)\neq 0$

$\Rightarrow f(a)\neq f(b)$

$\Rightarrow f(-5)\neq f(5)$

4. Verify Mean Value Theorem if

$f(x)=x^2-4x-3$

in the interval $[a,b]$ where a = 1 and b = 4

Solution:

(a) f(x) is a polynomial.

So, function is continuous in [1, 4] as polynomial function is always continuous.

(b) $f'(x)=2x-4,\ f'(x)$ exists in [1, 4], hence derivable.

Both the conditions of the theorem are satisfied, so there exists, at least one $c\in(1,4)$ such that

$$\frac{f(4)-f(1)}{4-1}=f'(c)$$

$$\frac{-3-(-6)}{3}=2c-4$$

$1=2c-4$

$c=\frac{5}{2}$

5. Verify Mean Value Theorem if $f(x)=x^3-5x^2-3x$ in the interval $[a,b]$ where $a=1$ and $b=3.$ Find all $c\in(1,3)$ for which $f'(c)=0.$

Solution:

(a) Function is a polynomial as polynomial function is always continuous.

So continuous in [1, 3]

(b) $f'(x)=3x^2-10x,\ f'(x)$ exists in [1, 3], hence derivable.

Conditions of MVT theorem are satisfied. So, there exists, at least one $c\in(1,3)$ such that

$\frac{f(3)-f(1)}{3-1}=f'(c)$

$\frac{-21-(-7)}{2}=3c^2-10c$

$-7=3c^2-10c$

$3c^2-7c-3c+7=0$

$c(3c-7)-1(3c-7)=0$

$(3c-7)(c-1)=0$

$(3c-7)=0$ or $(c-1)=0$

$3c=7$ or $c=1$

$c=\frac{7}{3}$ or $c=1$

Only $c=\frac{7}{3}\ \in(1,3)$

As, $f(1) \neq f(3)$, therefore the value of c does not exist such that $f(c)=0$.

6. Examine the applicability of Mean Value Theorem for all the three functions being given below: [Note for students: Check exercise 2]

(i) $f(x)=[x]$ **for** $x \in [5, 9]$

(ii) $f(x)=[x]$ **for** $x \in [-2, 2]$

(iii) $f(x)=x^2-1$ **for** $x \in [1, 2]$

Solution: According to Mean Value Theorem :

For a function $f:[a, b] \to$ R, if

(a) f is continuous on (a, b)

(b) f is differentiable on (a, b)

Then there exist some $c \in (a, b)$ such that $f'(c)=\frac{f(b)-f(a)}{b-a}$

(i) $f(x)=[x]$ for $x \in [5, 9]$

given function $f(x)$ is not continuous at $x=5$ and $x=9$.

Therefore,

$f(x)$ is not continuous at $[5,9]$.

Now let n be an integer such that $n \in [5, 9]$

$\therefore$ L.H.L. = $\lim_{h \to 0^-} \frac{f(n+h)-f(n)}{h} = \lim_{h \to 0^-} \frac{(n+h)-(n)}{h} = \lim_{h \to 0^-} \frac{n-1-n}{h} = \lim_{h \to 0^-} \frac{-1}{h} = \infty$

And R.H.L. = $\lim_{h \to 0^+} \frac{f(n+h)-f(n)}{h} = \lim_{h \to 0^+} \frac{(n+h)-(n)}{h} = \lim_{h \to 0^+} \frac{n-n}{h} = \lim_{h \to 0^+} 0 = 0$

Since, L.H.L. $\neq$ R.H.L.,

Therefore f is not differentiable at $[5,9]$.

Hence Mean Value Theorem is not applicable for this function.

(ii) $f(x)=[x]$ for $x\in[-2,2]$

Given function $f(x)$ is not continuous at $x=-2$ and $x=2$.

Therefore,

$f(x)$ is not continuous at $[-2,2]$.

Now let n be an integer such that $n\in[-2,2]$

$\therefore$ L.H.L. = $\lim_{h\to0^-}\frac{f(n+h)-f(n)}{h}=\lim_{h\to0^-}\frac{(n+h)-(n)}{h}=\lim_{h\to0^-}\frac{n-1-n}{h}=\lim_{h\to0^-}\frac{-1}{h}=\infty$

And R.H.L. = $\lim_{h\to0^+}\frac{f(n+h)-f(n)}{h}=\lim_{h\to0^+}\frac{(n+h)-(n)}{h}=\lim_{h\to0^+}\frac{n-n}{h}=\lim_{h\to0^+}0=0$

Since, L.H.L. $\neq$ R.H.L.,

Therefore f is not differentiable at $[-2,2]$.

Hence Mean Value Theorem is not applicable for this function.

(iii) $f(x)=x^2-1$ for $x\in[1,2]$ ……….(1)

Here, $f(x)$ is a polynomial function.

Therefore, $f(x)$ is continuous and derivable on the real line.

Hence $f(x)$ is continuous in the closed interval [1, 2] and derivable in open interval (1, 2).

Therefore, both conditions of Mean Value Theorem are satisfied.

Now, From equation (1), we have

$f'(x)=2x$

$f'(c)=2c$

Again, From equation (1):

$$f(a)=f(1)=(1)^2-1=1-1=0$$

And, $f(b)=f(2)=(2)^2-1=4-1=3$

Therefore,

$$f'c=\frac{f(b)-f(a)}{b-1}$$

$$2c=\frac{3-0}{2-1}$$

$$c=\frac{3}{2}\in(1,2)$$

Therefore, Mean Value Theorem is verified.