EXERCISE 13.1 PAGE NO: 13.3

1. Evaluate the following:
(i) i^{457}
(ii) i^{528}
(iii) $1/ i^{58}$
(iv) $i^{37} + 1/i^{67}$
(v) $[i^{41} + 1/i^{257}]$
(vi) $(i^{77} + i^{70} + i^{87} + i^{414})^3$
(vii) $i^{30} + i^{40} + i^{60}$
(viii) $i^{49} + i^{68} + i^{89} + i^{110}$
Solution:
(i) i^{457}
Let us simplify we get,
$i^{457} = i^{(456 + 1)}$
$= i^{4(114)} \times i$
$= (1)^{114} \times i$
$= i$ [since $i^4 = 1$]

(ii) i^{528}
Let us simplify we get,
$i^{528} = i^{4(132)}$
$= (1)^{132}$
$= 1$ [since $i^4 = 1$]

(iii) $1/ i^{58}$
Let us simplify we get,
$1/ i^{58} = 1/ i^{56+2}$
$= 1/ i^{56} \times i^2$
$= 1/ (i^4)^{14} \times i^2$
$= 1/ i^2$ [since, $i^4 = 1$]
$= 1/-1$ [since, $i^2 = -1$]
$= -1$

(iv) $i^{37} + 1/i^{67}$
Let us simplify we get,
$i^{37} + 1/i^{67} = i^{36+1} + 1/ i^{64+3}$
$= i + 1/i^3$ [since, $i^4 = 1$]
$= i + i/i^4$

$= i + i$
$= 2i$

(v) $[i^{41} + 1/i^{257}]$
Let us simplify we get,
$[i^{41} + 1/i^{257}] = [i^{40+1} + 1/ i^{256+1}]$
$= [i + 1/i]^9$ [since, $1/i = -1$]
$= [i - i]$
$= 0$

(vi) $(i^{77} + i^{70} + i^{87} + i^{414})^3$
Let us simplify we get,
$(i^{77} + i^{70} + i^{87} + i^{414})^3 = (i^{(76 + 1)} + i^{(68 + 2)} + i^{(84 + 3)} + i^{(412 + 2)})^3$
$= (i + i^2 + i^3 + i^2)^3$ [since $i^3 = – i$, $i^2 = – 1$]
$= (i + (– 1) + (– i) + (– 1))^3$
$= (– 2)^3$
$= – 8$

(vii) $i^{30} + i^{40} + i^{60}$
Let us simplify we get,
$i^{30} + i^{40} + i^{60} = i^{(28 + 2) +} i^{40} + i^{60}$
$= (i^4)^7 i^2 + (i^4)^{10} + (i^4)^{15}$
$= i^2 + 1^{10} + 1^{15}$
$= – 1 + 1 + 1$
$= 1$

(viii) $i^{49} + i^{68} + i^{89} + i^{110}$
Let us simplify we get,
$i^{49} + i^{68} + i^{89} + i^{110} = i^{(48 + 1)} + i^{68} + i^{(88 + 1)} + i^{(116 + 2)}$
$= (i^4)^{12}\times i + (i^4)^{17} + (i^4)^{22}\times i + (i^4)^{29}\times i^2$
$= i + 1 + i – 1$
$= 2i$

2. Show that $1 + i^{10} + i^{20} + i^{30}$ is a real number?
Solution:
Given:
$1 + i^{10} + i^{20} + i^{30} = 1 + i^{(8 + 2) +} i^{20} + i^{(28 + 2)}$
$= 1 + (i^4)^2 \times i^2 + (i^4)^5 + (i^4)^7 \times i^2$
$= 1 – 1 + 1 – 1$ [since, $i^4 = 1$, $i^2 = – 1$]
$= 0$

Hence , $1 + i^{10} + i^{20} + i^{30}$ is a real number.

3. Find the values of the following expressions:

(i) $i^{49} + i^{68} + i^{89} + i^{110}$

(ii) $i^{30} + i^{80} + i^{120}$

(iii) $i + i^2 + i^3 + i^4$

(iv) $i^5 + i^{10} + i^{15}$

(v) $[i^{592} + i^{590} + i^{588} + i^{586} + i^{584}] / [i^{582} + i^{580} + i^{578} + i^{576} + i^{574}]$

(vi) $1 + i^2 + i^4 + i^6 + i^8 + ... + i^{20}$

(vii) $(1 + i)^6 + (1 – i)^3$

Solution:

(i) $i^{49} + i^{68} + i^{89} + i^{110}$

Let us simplify we get,

$i^{49} + i^{68} + i^{89} + i^{110} = i^{(48 + 1)} + i^{68} + i^{(88 + 1)} + i^{(108 + 2)}$

$= (i^4)^{12} \times i + (i^4)^{17} + (i^4)^{22} \times i + (i^4)^{27} \times i^2$

$= i + 1 + i – 1$ [since $i^4 = 1, i^2 = – 1$]

$= 2i$

$\therefore i^{49} + i^{68} + i^{89} + i^{110} = 2i$

(ii) $i^{30} + i^{80} + i^{120}$

Let us simplify we get,

$i^{30} + i^{80} + i^{120} = i^{(28 + 2)} + i^{80} + i^{120}$

$= (i^4)^7 \times i^2 + (i^4)^{20} + (i^4)^{30}$

$= – 1 + 1 + 1$ [since $i^4 = 1, i^2 = – 1$]

$= 1$

$\therefore i^{30} + i^{80} + i^{120} = 1$

(iii) $i + i^2 + i^3 + i^4$

Let us simplify we get,

$i + i^2 + i^3 + i^4 = i + i^2 + i^2 \times i + i^4$

$= i – 1 + (– 1) \times i + 1$ [since $i^4 = 1, i^2 = – 1$]

$= i – 1 – i + 1$

$= 0$

$\therefore i + i^2 + i^3 + i^4 = 0$

(iv) $i^5 + i^{10} + i^{15}$

Let us simplify we get,

$i^5 + i^{10} + i^{15} = i^{(4 + 1)} + i^{(8 + 2)} + i^{(12 + 3)}$

$= (i^4)^1 \times i + (i^4)^2 \times i^2 + (i^4)^3 \times i^3$

$= (i^4)^1 \times i + (i^4)^2 \times i^2 + (i^4)^3 \times i^2 \times i$

$= 1 \times i + 1 \times (-1) + 1 \times (-1) \times i$

$= i - 1 - i$

$= -1$

$\therefore i^5 + i^{10} + i^{15} = -1$

(v) $[i^{592} + i^{590} + i^{588} + i^{586} + i^{584}] / [i^{582} + i^{580} + i^{578} + i^{576} + i^{574}]$

Let us simplify we get,

$[i^{592} + i^{590} + i^{588} + i^{586} + i^{584}] / [i^{582} + i^{580} + i^{578} + i^{576} + i^{574}]$

$= [i^{10} (i^{582} + i^{580} + i^{578} + i^{576} + i^{574}) / (i^{582} + i^{580} + i^{578} + i^{576} + i^{574})]$

$= i^{10}$

$= i^8 i^2$

$= (i^4)^2 i^2$

$= (1)^2 (-1)$ [since $i^4 = 1$, $i^2 = -1$]

$= -1$

$\therefore [i^{592} + i^{590} + i^{588} + i^{586} + i^{584}] / [i^{582} + i^{580} + i^{578} + i^{576} + i^{574}] = -1$

(vi) $1 + i^2 + i^4 + i^6 + i^8 + \ldots + i^{20}$

Let us simplify we get,

$1 + i^2 + i^4 + i^6 + i^8 + \ldots + i^{20} = 1 + (-1) + 1 + (-1) + 1 + \ldots + 1$

$= 1$

$\therefore 1 + i^2 + i^4 + i^6 + i^8 + \ldots + i^{20} = 1$

(vii) $(1 + i)^6 + (1 - i)^3$

Let us simplify we get,

$(1 + i)^6 + (1 - i)^3 = \{(1 + i)^2\}^3 + (1 - i)^2 (1 - i)$

$= \{1 + i^2 + 2i\}^3 + (1 + i^{2-} 2i)(1 - i)$

$= \{1 - 1 + 2i\}^3 + (1 - 1 - 2i)(1 - i)$

$= (2i)^3 + (-2i)(1 - i)$

$= 8i^3 + (-2i) + 2i^2$

$= -8i - 2i - 2$ [since $i^3 = -i$, $i^2 = -1$]

$= -10i - 2$

$= -2(1 + 5i)$

$= -2 - 10i$

$\therefore (1 + i)^6 + (1 - i)^3 = -2 - 10i$

EXERCISE 13.2 PAGE NO: 13.31

1. Express the following complex numbers in the standard form a + ib:
(i) (1 + i) (1 + 2i)
(ii) (3 + 2i) / (-2 + i)
(iii) $1/(2 + i)^2$
(iv) (1 - i) / (1 + i)
(v) $(2 + i)^3 / (2 + 3i)$
(vi) [(1 + i) (1 +√3i)] / (1 - i)
(vii) (2 + 3i) / (4 + 5i)
(viii) $(1 - i)^3 / (1 – i^3)$
(ix) $(1 + 2i)^{-3}$
(x) (3 – 4i) / [(4 – 2i) (1 + i)]
(xi)
$$\left(\frac{1}{1-4i}-\frac{2}{1+i}\right)\left(\frac{3-4i}{5+i}\right)$$
(xii) (5 +√2i) / (1-√2i)
Solution:
(i) (1 + i) (1 + 2i)
Let us simplify and express in the standard form of (a + ib),
(1 + i) (1 + 2i) = (1+i)(1+2i)
= 1(1+2i)+i(1+2i)
= $1+2i+i+2i^2$
= 1+3i+2(-1) [since, $i^2 = -1$]
= 1+3i-2
= -1+3i
∴ The values of a, b are -1, 3.

(ii) (3 + 2i) / (-2 + i)
Let us simplify and express in the standard form of (a + ib),
(3 + 2i) / (-2 + i) = [(3 + 2i) / (-2 + i)] × (-2-i) / (-2-i) [multiply and divide with (-2-i)]
= $[3(-2-i) + 2i (-2-i)] / [(-2)^2 – (i)^2]$
= $[-6 -3i – 4i -2i^2] / (4-i^2)$
= [-6 -7i -2(-1)] / (4 – (-1)) [since, $i^2 = -1$]
= [-4 -7i] / 5
∴ The values of a, b are -4/5, -7/5

(iii) $1/(2 + i)^2$
Let us simplify and express in the standard form of (a + ib),

$1/(2 + i)^2 = 1/(2^2 + i^2 + 2(2)(i))$
$= 1/(4 – 1 + 4i)$ [since, $i^2 = -1$]
$= 1/(3 + 4i)$ [multiply and divide with $(3 – 4i)$]
$= 1/(3 + 4i) \times (3 – 4i)/(3 – 4i)]$
$= (3-4i)/(3^2 – (4i)^2)$
$= (3-4i)/(9 – 16i^2)$
$= (3-4i)/(9 – 16(-1))$ [since, $i^2 = -1$]
$= (3-4i)/25$
∴ The values of a, b are 3/25, -4/25

(iv) $(1 - i) / (1 + i)$
Let us simplify and express in the standard form of $(a + ib)$,
$(1 - i) / (1 + i) = (1 - i) / (1 + i) \times (1-i)/(1-i)$ [multiply and divide with $(1-i)$]
$= (1^2 + i^2 – 2(1)(i)) / (1^2 – i^2)$
$= (1 + (-1) -2i) / (1 – (-1))$
$= -2i/2$
$= -i$
∴ The values of a, b are 0, -1

(v) $(2 + i)^3 / (2 + 3i)$
Let us simplify and express in the standard form of $(a + ib)$,
$(2 + i)^3 / (2 + 3i) = (2^3 + i^3 + 3(2)^2(i) + 3(i)^2(2)) / (2 + 3i)$
$= (8 + (i^2.i) + 3(4)(i) + 6i^2) / (2 + 3i)$
$= (8 + (-1)i + 12i + 6(-1)) / (2 + 3i)$
$= (2 + 11i) / (2 + 3i)$
[multiply and divide with $(2-3i)$]
$= (2 + 11i)/(2 + 3i) \times (2-3i)/(2-3i)$
$= [2(2-3i) + 11i(2-3i)] / (2^2 – (3i)^2)$
$= (4 – 6i + 22i – 33i^2) / (4 – 9i^2)$
$= (4 + 16i – 33(-1)) / (4 – 9(-1))$ [since, $i^2 = -1$]
$= (37 + 16i) / 13$
∴ The values of a, b are 37/13, 16/13

(vi) $[(1 + i) (1 +\sqrt{3}i)] / (1 - i)$
Let us simplify and express in the standard form of $(a + ib)$,
$[(1 + i) (1 +\sqrt{3}i)] / (1 - i) = [1(1+\sqrt{3}i) + i(1+\sqrt{3}i)] / (1-i)$
$= (1 + \sqrt{3}i + i + \sqrt{3}i^2) / (1 - i)$
$= (1 + (\sqrt{3}+1)i + \sqrt{3}(-1)) / (1-i)$ [since, $i^2 = -1$]
$= [(1-\sqrt{3}) + (1+\sqrt{3})i] / (1-i)$

[multiply and divide with (1+i)]
$= [(1-\sqrt{3}) + (1+\sqrt{3})i] / (1-i) \times (1+i)/(1+i)$
$= [(1-\sqrt{3})(1+i) + (1+\sqrt{3})i(1+i)] / (1^2 - i^2)$
$= [1-\sqrt{3} + (1-\sqrt{3})i + (1+\sqrt{3})i + (1+\sqrt{3})i^2] / (1-(-1))$ [since, $i^2 = -1$]
$= [(1-\sqrt{3})+(1-\sqrt{3}+1+\sqrt{3})i+(1+\sqrt{3})(-1)] / 2$
$= (-2\sqrt{3} + 2i) / 2$
$= -\sqrt{3} + i$
∴ The values of a, b are $-\sqrt{3}$, 1

(vii) $(2 + 3i) / (4 + 5i)$
Let us simplify and express in the standard form of (a + ib),
$(2 + 3i) / (4 + 5i)$ = [multiply and divide with (4-5i)]
$= (2 + 3i) / (4 + 5i) \times (4-5i)/(4-5i)$
$= [2(4-5i) + 3i(4-5i)] / (4^2 - (5i)^2)$
$= [8 - 10i + 12i - 15i^2] / (16 - 25i^2)$
$= [8+2i-15(-1)] / (16 - 25(-1))$ [since, $i^2 = -1$]
$= (23 + 2i) / 41$
∴ The values of a, b are 23/41, 2/41

(viii) $(1 - i)^3 / (1 - i^3)$
Let us simplify and express in the standard form of (a + ib),
$(1 - i)^3 / (1 - i^3) = [1^3 - 3(1)^2 i + 3(1)(i)^2 - i^3] / (1-i^2.i)$
$= [1 - 3i + 3(-1)-i^2.i] / (1 - (-1)i)$ [since, $i^2 = -1$]
$= [-2 - 3i - (-1)i] / (1+i)$
$= [-2-4i] / (1+i)$
[Multiply and divide with (1-i)]
$= [-2-4i] / (1+i) \times (1-i)/(1-i)$
$= [-2(1-i)-4i(1-i)] / (1^2 - i^2)$
$= [-2+2i-4i+4i^2] / (1 - (-1))$
$= [-2-2i+4(-1)] /2$
$= (-6-2i)/2$
$= -3 - i$
∴ The values of a, b are -3, -1

(ix) $(1 + 2i)^{-3}$
Let us simplify and express in the standard form of (a + ib),
$(1 + 2i)^{-3} = 1/(1 + 2i)^3$
$= 1/(1^3+3(1)^2 (2i)+2(1)(2i)^2 + (2i)^3)$
$= 1/(1+6i+4i^2+8i^3)$

= $1/(1+6i+4(-1)+8i^2.i)$ [since, $i^2 = -1$]
= $1/(-3+6i+8(-1)i)$ [since, $i^2 = -1$]
= $1/(-3-2i)$
= $-1/(3+2i)$
[Multiply and divide with (3-2i)]
= $-1/(3+2i) \times (3-2i)/(3-2i)$
= $(-3+2i)/(3^2 – (2i)^2)$
= $(-3+2i) / (9-4i^2)$
= $(-3+2i) / (9-4(-1))$
= $(-3+2i) /13$
∴ The values of a, b are -3/13, 2/13

(x) $(3 – 4i) / [(4 – 2i) (1 + i)]$
Let us simplify and express in the standard form of (a + ib),
$(3 – 4i) / [(4 – 2i) (1 + i)] = (3-4i)/ [4(1+i)-2i(1+i)]$
= $(3-4i)/ [4+4i-2i-2i^2]$
= $(3-4i)/ [4+2i-2(-1)]$ [since, $i^2 = -1$]
= $(3-4i)/ (6+2i)$
[Multiply and divide with (6-2i)]
= $(3-4i)/ (6+2i) \times (6-2i)/(6-2i)$
= $[3(6-2i)-4i(6-2i)] / (6^2 – (2i)^2)$
= $[18 – 6i – 24i + 8i^2] / (36 – 4i^2)$
= $[18 – 30i + 8 (-1)] / (36 – 4 (-1))$ [since, $i^2 = -1$]
= $[10-30i] / 40$
= $(1 – 3i) / 4$
∴ The values of a, b are 1/4, -3/4

(xi)

$$\left(\frac{1}{1-4i}-\frac{2}{1+i}\right)\left(\frac{3-4i}{5+i}\right)$$

Let us simplify and express in the standard form of (a + ib),

$$\left(\frac{1}{1-4i}-\frac{2}{1+i}\right)\left(\frac{3-4i}{5+i}\right) = \left(\frac{1+i-2(1-4i)}{(1-4i)(1+i)}\right)\left(\frac{3-4i}{5+i}\right)$$

$$= \left(\frac{1+i-2+8i}{1(1+i)-4i(1+i)}\right)\left(\frac{3-4i}{5+i}\right)$$

$$= \left(\frac{-1+9i}{1+i-4i-4i^2}\right)\left(\frac{3-4i}{5+i}\right)$$

$$= \left(\frac{-1+9i}{1-3i-4(-1)}\right)\left(\frac{3-4i}{5+i}\right)$$

$= \frac{(-1+9i)(3-4i)}{(5-3i)(5+i)}$

$= \frac{-1(3-4i)+9i(3-i)}{5(5+i)-3i(5+i)}$

$= \frac{-3+4i+27i-9i^2}{25+5i-15i-3i^2}$

$= \frac{-3+31i-9(-1)}{25-10i-3(-1)}$

$= \frac{6+31i}{28-10i}$

[Multiply and divide with (28+10i)]

$= \frac{6+31i}{28-10i} \times \frac{28+10i}{28+10i}$

$= \frac{6(28+10i)+31i(28+10i)}{28^2-(10i)^2}$

$= \frac{168+60i+868i+310i^2}{784-100i^2}$

$= \frac{168+928i+310(-1)}{784-100(-1)}$

$= \frac{478+928i}{884}$

Therefore value of a, b are 478/884, 928/884.

(xii) $(5 +\sqrt{2}i) / (1-\sqrt{2}i)$

Let us simplify and express in the standard form of (a + ib),

$(5 +\sqrt{2}i) / (1-\sqrt{2}i)$ = [Multiply and divide with $(1+\sqrt{2}i)$]

$= (5 +\sqrt{2}i) / (1-\sqrt{2}i) \times (1+\sqrt{2}i)/(1+\sqrt{2}i)$

$= [5(1+\sqrt{2}i) + \sqrt{2}i(1+\sqrt{2}i)] / (1^2 – (\sqrt{2})^2)$

$= [5+5\sqrt{2}i + \sqrt{2}i + 2i^2] / (1 – 2i^2)$

$= [5 + 6\sqrt{2}i + 2(-1)] / (1-2(-1))$ [since, $i^2 = -1$]

$= [3+6\sqrt{2}i]/3$

$= 1+ 2\sqrt{2}i$

∴ The values of a, b are 1, $2\sqrt{2}$

2. Find the real values of x and y, if

(i) $(x + iy) (2 – 3i) = 4 + i$

(ii) $(3x – 2i y) (2 + i)^2 = 10(1 + i)$

(iii) $\frac{(1+i)x-2i}{3+i} + \frac{(2-3i)y+i}{3-i} = i$

(iv) $(1 + i)(x + iy) = 2 – 5i$

Solution:

(i) $(x + iy)(2 – 3i) = 4 + i$

Given:

$(x + iy)(2 - 3i) = 4 + i$

Let us simplify the expression we get,

$x(2 - 3i) + iy(2 - 3i) = 4 + i$

$2x - 3xi + 2yi - 3yi^2 = 4 + i$

$2x + (-3x+2y)i - 3y(-1) = 4 + i$ [since, $i^2 = -1$]

$2x + (-3x+2y)i + 3y = 4 + i$ [since, $i^2 = -1$]

$(2x+3y) + i(-3x+2y) = 4 + i$

Equating Real and Imaginary parts on both sides, we get

$2x+3y = 4$… (i)

And $-3x+2y = 1$… (ii)

Multiply (i) by 3 and (ii) by 2 and add

On solving we get,

$6x – 6x – 9y + 4y = 12 + 2$

$13y = 14$

$y = 14/13$

Substitute the value of y in (i) we get,

$2x+3y = 4$

$2x + 3(14/13) = 4$

$2x = 4 – (42/13)$

$= (52-42)/13$

$2x = 10/13$

$x = 5/13$

$x = 5/13,\ y = 14/13$

∴ The real values of x and y are 5/13, 14/13

(ii) $(3x – 2i y)(2 + i)^2 = 10(1 + i)$

Given:

$(3x - 2iy)(2+i)^2 = 10(1+i)$

$(3x - 2yi)(2^2+i^2+2(2)(i)) = 10+10i$

$(3x - 2yi)(4 + (-1)+4i) = 10+10i$ [since, $i^2 = -1$]

$(3x - 2yi)(3+4i) = 10+10i$

Let us divide with 3+4i on both sides we get,

$(3x - 2yi) = (10+10i)/(3+4i)$

= Now multiply and divide with (3-4i)

$= [10(3-4i) + 10i(3-4i)] / (3^2 – (4i)^2)$

= [30-40i+30i-40i^2] / (9 – 16i^2)
= [30-10i-40(-1)] / (9-16(-1))
= [70-10i]/25
Now, equating Real and Imaginary parts on both sides we get
3x = 70/25 and -2y = -10/25
x = 70/75 and y = 1/5
x = 14/15 and y = 1/5
∴ The real values of x and y are 14/15, 1/5

(iii) $\frac{(1+i)x-2i}{3+i}+\frac{(2-3i)y+i}{3-i}=i$

Given:

$$\frac{(1+i)x-2i}{3+i}+\frac{(2-3i)y+i}{3-i}=i$$

$$\frac{\left(\left((1+i)x-2i\right)(3-i)\right)+\left(\left((2-3i)y+i\right)(3+i)\right)}{(3+i)(3-i)}=i$$

$$\frac{\left((1+i)(3-i)x\right)-(2i)(3-i)+\left((2-3i)(3+i)y\right)+(i)(3+i)}{3^2-i^2}=i$$

$$\frac{(3-i+3i-i^2)x-6i+2i^2+(6+2i-9i-3i^2)y+3i+i^2}{9-(-1)}=i$$

$$\frac{(3+2i-(-1))x-6i+2(-1)+(6-7i-3(-1))y+3i+(-1)}{10}=i \text{ [since, } i^2=-1]$$

(4+2i) x-3i-3 + (9-7i)y = 10i
(4x+9y-3) + i(2x-7y-3) = 10i
Now, equating Real and Imaginary parts on both sides we get,
4x+9y-3 = 0 … (i)
And 2x-7y-3 = 10
2x-7y = 13 … (ii)
Multiply (i) by 7 and (ii) by 9 and add
On solving these equations we get
28x + 18x + 63y – 63y = 117 + 21
46x = 117 + 21
46x = 138
x = 138/46
= 3
Substitute the value of x in (i) we get,
4x+9y-3 = 0

9y = -9
y = -9/9
  = -1
x = 3 and y = -1
∴ The real values of x and y are 3 and -1

(iv) (1 + i) (x + iy) = 2 – 5i
Given:
(1 + i) (x + iy) = 2 – 5i
Divide with (1+i) on both the sides we get,
(x + iy) = (2 – 5i)/(1+i)
Multiply and divide by (1-i)
= (2 – 5i)/(1+i) × (1-i)/(1-i)
= [2(1-i) – 5i (1-i)] / (1^2 – i^2)
= [2 – 7i + 5(-1)] / 2 [since, i^2 = -1]
= (-3-7i)/2
Now, equating Real and Imaginary parts on both sides we get
x = -3/2 and y = -7/2
∴ The real values of x and y are -3/2, -7/2

3. Find the conjugates of the following complex numbers:
(i) 4 – 5i
(ii) 1 / (3 + 5i)
(iii) 1 / (1 + i)
(iv) $(3 - i)^2$ / (2 + i)
(v) [(1 + i) (2 + i)] / (3 + i)
(vi) [(3 – 2i) (2 + 3i)] / [(1 + 2i) (2 - i)]
Solution:
(i) 4 – 5i
Given:
4 – 5i
We know the conjugate of a complex number (a + ib) is (a - ib)
So,
∴ The conjugate of (4 – 5i) is (4 + 5i)

(ii) 1 / (3 + 5i)
Given:
1 / (3 + 5i)
Since the given complex number is not in the standard form of (a + ib)

Let us convert to standard form by multiplying and dividing with (3 – 5i)
We get,

$$\frac{1}{3+5i} = \frac{1}{3+5i} \times \frac{3-5i}{3-5i}$$
$$= \frac{3-5i}{3^2-(5i)^2}$$
$$= \frac{3-5i}{9-25i^2}$$
$$= \frac{3-5i}{9-25(-1)} \text{ [Since, } i^2 = -1]$$
$$= \frac{3-5i}{34}$$

We know the conjugate of a complex number (a + ib) is (a - ib)
So,
∴ The conjugate of (3 – 5i)/34 is (3 + 5i)/34

(iii) 1 / (1 + i)
Given:
1 / (1 + i)
Since the given complex number is not in the standard form of (a + ib)
Let us convert to standard form by multiplying and dividing with (1 – i)
We get,

$$\frac{1}{1+i} = \frac{1}{1+i} \times \frac{1-i}{1-i}$$
$$= \frac{1-i}{1^2-i^2}$$
$$= \frac{1-i}{1-(-1)} \text{ [since, } i^2 = -1]$$
$$= \frac{1-i}{2}$$

We know the conjugate of a complex number (a + ib) is (a - ib)
So,
∴ The conjugate of (1-i)/2 is (1+i)/2

(iv) $(3 - i)^2 / (2 + i)$
Given:
$(3 - i)^2 / (2 + i)$
Since the given complex number is not in the standard form of (a + ib)
Let us convert to standard form,

$\frac{(3-i)^2}{2+i} = \frac{3^2+i^2-2(3)(i)}{2+i}$

$= \frac{9+(-1)-6i}{2+i}$ [Since, i^2 = -1]

$= \frac{8-6i}{2+i}$

Now, let us multiply and divide with (2 - i) we get,

$\frac{8-6i}{2+i} = \frac{8-6i}{2+i} \times \frac{2-i}{2-i}$

$= \frac{8(2-i)-6i(2-i)}{2^2-i^2}$

$= \frac{16-8i-12i+6i^2}{4-(-1)}$ [Since, i^2 = -1]

$= \frac{16-20i+6(-1)}{5}$

$= \frac{10-20i}{5}$

= 10/5 – 20i/5

= 2 – 4i

We know the conjugate of a complex number (a + ib) is (a - ib)

So,

∴ The conjugate of (2 – 4i) is (2 + 4i)

(v) [(1 + i) (2 + i)] / (3 + i)

Given:

[(1 + i) (2 + i)] / (3 + i)

Since the given complex number is not in the standard form of (a + ib)

Let us convert to standard form,

$\frac{(1+i)(2+i)}{3+i} = \frac{1(2+i)+i(2+i)}{3+i}$

$= \frac{2+i+2i+i^2}{3+i}$

$= \frac{2+3i+(-1)}{3+i}$ [Since, i^2 = -1]

$= \frac{1+3i}{3+i}$

Now, let us multiply and divide with (3 - i) we get,

$\frac{1+3i}{3+i} = \frac{1+3i}{3+i} \times \frac{3-i}{3-i}$

$$= \frac{1(3-i)+3i(3-i)}{3^2-i^2}$$

$$= \frac{3-i+9i-3i^2}{9-(-1)}$$ [Since, $i^2 = -1$]

$$= \frac{3+8i-3(-1)}{10}$$

$$= \frac{6+8i}{10}$$

$$= \frac{3}{5} + \frac{4i}{5}$$

We know the conjugate of a complex number (a + ib) is (a - ib)
So,
∴ The conjugate of (3 + 4i)/5 is (3 – 4i)/5

(vi) [(3 – 2i) (2 + 3i)] / [(1 + 2i) (2 - i)]
Given:
[(3 – 2i) (2 + 3i)] / [(1 + 2i) (2 - i)]
Since the given complex number is not in the standard form of (a + ib)
Let us convert to standard form,

$$\frac{(3-2i)(2+3i)}{(1+2i)(2-i)} = \frac{3(2+3i)-2i(2+3i)}{1(2-i)+2i(2-i)}$$

$$= \frac{6+9i-4i-6i^2}{2-i+4i-2i^2}$$

$$= \frac{6+5i-6(-1)}{2+3i-2(-1)}$$

$$= \frac{12+5i}{4+3i}$$

Now, let us multiply and divide with (4 - 3i) we get,

$$\frac{12+5i}{4+3i} = \frac{12+5i}{4+3i} \times \frac{4-3i}{4-3i}$$

$$= \frac{12(4-3i)+5i(4-3i)}{4^2-(3i)^2}$$

$$= \frac{48-36i+20i-15i^2}{16-9i^2}$$

$$= \frac{48-16i-15(-1)}{16-9(-1)}$$

$$= \frac{63-16i}{25}$$

We know the conjugate of a complex number (a + ib) is (a - ib)

So,
∴ The conjugate of (63 - 16i)/25 is (63 + 16i)/25

4. Find the multiplicative inverse of the following complex numbers:
(i) 1 – i
(ii) $(1 + i\sqrt{3})^2$
(iii) 4 – 3i
(iv) √5 + 3i
Solution:
(i) 1 – i
Given:
1 – i
We know the multiplicative inverse of a complex number (Z) is Z^{-1} or 1/Z
So,

Z = 1 - i

$$Z^{-1} = \frac{1}{1-i}$$

Let us multiply and divide by (1 + i) we get,

$$= \frac{1}{1-i} \times \frac{1+i}{1+i}$$

$$= \frac{1+i}{1^2-(i)^2}$$

$$= \frac{1+i}{1-(-1)} \text{ [Since, } i^2 = -1]$$

$$= \frac{1+i}{2}$$

∴ The multiplicative inverse of (1 - i) is (1 + i)/2

(ii) $(1 + i\sqrt{3})^2$
Given:
$(1 + i\sqrt{3})^2$
$Z = (1 + i\sqrt{3})^2$
$= 1^2 + (i\sqrt{3})^2 + 2(1)(i\sqrt{3})$
$= 1 + 3i^2 + 2i\sqrt{3}$
$= 1 + 3(-1) + 2i\sqrt{3}$ [since, $i^2 = -1$]
$= 1 – 3 + 2i\sqrt{3}$
$= -2 + 2i\sqrt{3}$
We know the multiplicative inverse of a complex number (Z) is Z^{-1} or 1/Z
So,
$Z = -2 + 2i\sqrt{3}$

$$Z^{-1} = \frac{1}{-2 + 2i\sqrt{3}}$$

Let us multiply and divide by -2 - 2 i√3, we get

$$= \frac{1}{-2+2\sqrt{3}i} \times \frac{-2-2\sqrt{3}i}{-2-2\sqrt{3}i}$$

$$= \frac{-2-2\sqrt{3}i}{(-2)^2-\left(2\sqrt{3}i\right)^2}$$

$$= \frac{-2-2\sqrt{3}i}{4-12i^2}$$

$$= \frac{-2-2\sqrt{3}i}{4-12(-1)}$$

$$= \frac{-2-2\sqrt{3}i}{16}$$

$$= \frac{-1}{8}\frac{-i\sqrt{3}}{8}$$

∴ The multiplicative inverse of $(1 + i\sqrt{3})^2$ is $(-1-i\sqrt{3})/8$

(iii) 4 – 3i

Given:

4 – 3i

We know the multiplicative inverse of a complex number (Z) is Z^{-1} or 1/Z

So,

Z = 4 – 3i

$$Z^{-1} = \frac{1}{4 - 3i}$$

Let us multiply and divide by (4 + 3i), we get

$$= \frac{1}{4-3i} \times \frac{4+3i}{4+3i}$$

$$= \frac{4+3i}{4^2-(3i)^2}$$

$$= \frac{4+3i}{16-9i^2}$$

$$= \frac{4+3i}{16-9(-1)}$$

$$= \frac{4+3i}{25}$$

∴ The multiplicative inverse of (4 – 3i) is (4 + 3i)/25

(iv) √5 + 3i
Given:
√5 + 3i
We know the multiplicative inverse of a complex number (Z) is Z^{-1} or 1/Z
So,
Z = √5 + 3i

$$Z^{-1} = \frac{1}{\sqrt{5}+3i}$$

Let us multiply and divide by (√5 – 3i)

$$= \frac{1}{\sqrt{5}+3i} \times \frac{\sqrt{5}-3i}{\sqrt{5}-3i}$$

$$= \frac{\sqrt{5}-3i}{(\sqrt{5})^2-(3i)^2}$$

$$= \frac{\sqrt{5}-3i}{5-9i^2}$$

$$= \frac{\sqrt{5}-3i}{5-9(-1)}$$

$$= \frac{\sqrt{5}-3i}{14}$$

∴ The multiplicative inverse of (√5 + 3i) is (√5 - 3i)/14

5. If $z_1 = 2 - i$, $z_2 = 1 + i$, find

$$\left|\frac{z_1 + z_2 + 1}{z_1 - z_2 + i}\right|$$

Solution:
Given:
$z_1 = (2 - i)$ and $z_2 = (1 + i)$
We know that, |a/b| = |a| / |b|
So,

$$\left|\frac{z_1+z_2+1}{z_1-z_2+i}\right| = \frac{|z_1+z_2+1|}{|z_1-z_2+i|}$$

$$= \frac{|2-i+1+i+1|}{|2-i-(1+i)+i|}$$

$$= \frac{|4|}{|1-i|}$$

We know, |a + ib| is $\sqrt{a^2+b^2}$
So now,

$$= \frac{\sqrt{4^2+0^2}}{\sqrt{1^2+(-1)^2}}$$

$$= \frac{4}{\sqrt{2}}$$

$$= 2\sqrt{2}$$

∴ The value of $\left|\frac{z_1+z_2+1}{z_1-z_2+i}\right|$ is $2\sqrt{2}$

6. If z_1 = (2 - i), z_2 = (-2 + i), find

$$\textbf{(i)Re}\left(\frac{\mathbf{z_1 z_2}}{\bar{\mathbf{z}}_1}\right)$$

$$\textbf{(ii)Im}\left(\frac{\mathbf{1}}{\mathbf{z_1 \bar{z}_1}}\right)$$

Solution:

Given:

z_1 = (2 - i) and z_2 = (-2 + i)

$$\textbf{(i)Re}\left(\frac{\mathbf{z_1 z_2}}{\bar{\mathbf{z}}_1}\right)$$

We shall rationalise the denominator, we get

$$\frac{z_1 z_2}{\bar{z}_1} = \frac{z_1 z_2}{z_1} \times \frac{z_1}{z_1}$$

$$= \frac{(z_1)^2 z_2}{z_1 z_1}$$

$$= \frac{(2-i)^2(-2+i)}{|z_1|^2} \quad [since, z\bar{z} = |z|^2]$$

$$= \frac{(2^2 + i^2 - 2 \times 2 \times i)(-2+i)}{|2-i|^2}$$

$$= \frac{(4-1-4i)(-2+i)}{2^2+(-1)^2}$$

$$= \frac{(3-4i)(-2+i)}{4+i}$$

$$= \frac{3(-2+i)-4i(-2+i)}{4+i}$$

$$= \frac{-6+3i+8i+4}{5}$$

$$= \frac{-2+11i}{5}$$

$\therefore$ *The real value of* $\left(\frac{z_1 z_2}{\bar{z}_1}\right)$ *is* $\frac{-2}{5}$

$$(\mathbf{ii})\,\mathbf{Im}\left(\frac{1}{z_1\bar{z}_1}\right)$$

$$\frac{1}{z_1\bar{z}_1} = \frac{1}{|z_1|^2}$$
$$= \frac{1}{|2-i|^2}$$
$$= \frac{1}{2^2+(-1)^2}$$
$$= \frac{1}{4+1}$$
$$= \frac{1}{5}$$

$\therefore$ *The imaginary value of* $\left(\frac{1}{z_1\bar{z}_1}\right)$ *is* 0

7. Find the modulus of [(1 + i)/(1 - i)] – [(1 - i)/(1 + i)]

Solution:

Given:

[(1 + i)/(1 - i)] – [(1 - i)/(1 + i)]

So,

Z = [(1 + i)/(1 - i)] – [(1 - i)/(1 + i)]

Let us simplify, we get

= [(1+i) (1+i) – (1-i) (1-i)] / (1^2 – i^2)

= [1^2 + i^2 + 2(1)(i) – (1^2 + i^2 – 2(1)(i))] / (1 – (-1)) [Since, i^2 = -1]

= 4i/2

= 2i

We know that for a complex number Z = (a+ib) it's magnitude is given by |z| = $\sqrt{(a^2 + b^2)}$

So,

|Z| = $\sqrt{(0^2 + 2^2)}$

= 2

∴ The modulus of [(1 + i)/(1 - i)] – [(1 - i)/(1 + i)] is 2.

8. If x + iy = (a+ib)/(a-ib), prove that $x^2 + y^2 = 1$

Solution:

Given:

x + iy = (a+ib)/(a-ib)
We know that for a complex number Z = (a+ib) it's magnitude is given by |z| = √(a^2 + b^2)
So,
|a/b| is |a| / |b|
Applying Modulus on both sides we get,

$$|x+iy| = \left|\frac{a+ib}{a-ib}\right|$$

$$\sqrt{x^2+y^2} = \frac{|a+ib|}{|a-ib|}$$

$$= \frac{\sqrt{a^2+b^2}}{\sqrt{a^2+(-b)^2}}$$

$$= \frac{\sqrt{a^2+b^2}}{\sqrt{a^2+b^2}}$$

= 1
Squaring on both sides we get,

$$\left(\sqrt{x^2+y^2}\right)^2 = 1^2$$

$x^2+y^2=1$
∴ Hence Proved.

9. Find the least positive integral value of n for which [(1+i)/(1-i)]^n is real.
Solution:
Given:
[(1+i)/(1-i)]^n
Z = [(1+i)/(1-i)]^n
Now let us multiply and divide by (1+i), we get

$$= \frac{1+i}{1-i} \times \frac{1+i}{1+i}$$

$$= \frac{(1+i)^2}{1^2-i^2}$$

$$= \frac{1^2+i^2+2(1)(i)}{1-(-1)}$$

$$= \frac{1-1+2i}{2}$$

$$= \frac{2i}{2}$$

= i [which is not real]
For n = 2, we have

$[(1+i)/(1-i)]^2 = i^2$

$= -1$ [which is real]

So, the smallest positive integral 'n' that can make $[(1+i)/(1-i)]^n$ real is 2.

∴ The smallest positive integral value of 'n' is 2.

10. Find the real values of θ for which the complex number (1 + i cos θ) / (1 – 2i cos θ) is purely real.

Solution:

Given:

(1 + i cos θ) / (1 – 2i cos θ)

Z = (1 + i cos θ) / (1 – 2i cos θ)

Let us multiply and divide by (1 + 2i cos θ)

$$= \frac{1+icos\theta}{1-2icos\theta} \times \frac{1+2icos\theta}{1+2icos\theta}$$

$$= \frac{1(1+2icos\theta)+icos\theta(1+2icos\theta)}{1^2-(2icos\theta)^2}$$

$$= \frac{1+2icos\theta+icos\theta+2i^2\cos^2\theta}{1-4i^2\cos^2\theta}$$

$$= \frac{1+3icos\theta+2(-1)\cos^2\theta}{1-4(-1)\cos^2\theta}$$

$$= \frac{1-2\cos^2\theta+3icos\theta}{1+4\cos^2\theta}$$

For a complex number to be purely real, the imaginary part should be equal to zero.

So,

$$\frac{3cos\theta}{1+4\cos^2\theta} = 0$$

3cos θ = 0 (since, $1 + 4\cos^2\theta \geq 1$)

cos θ = 0

cos θ = cos π/2

θ = [(2n+1)π] / 2, for n ∈ Z

= 2nπ ± π/2, for n ∈ Z

∴ The values of θ to get the complex number to be purely real is 2nπ ± π/2, for n ∈ Z

11. Find the smallest positive integer value of n for which $(1+i)^n / (1-i)^{n-2}$ is a real number.

Solution:

Given:

$(1+i)^n / (1-i)^{n-2}$

$Z = (1+i)^n / (1-i)^{n-2}$

Let us multiply and divide by $(1 - i)^2$

$= \frac{(1+i)^n}{(1-i)^{n-2}} \times \frac{(1-i)^2}{(1-i)^2}$

$= \left(\frac{1+i}{1-i}\right)^n \times (1-i)^2$

$= \left(\frac{1+i}{1-i} \times \frac{1+i}{1+i}\right)^n \times \left(1^2 + i^2 - 2(1)(i)\right)$

$= \left(\frac{(1+i)^2}{1^2-i^2}\right)^n \times (1 + i^2 - 2i)$

$= \left(\frac{1^2+i^2+2(1)(i)}{1-(-1)}\right)^n \times (1 + (-1) - 2i)$

$= \left(\frac{1-1+2i}{2}\right)^n \times (-2i)$

$= \left(\frac{2i}{2}\right)^n \times (-2i)$

$= i^n \times (-2i)$

$= -2i^{n+1}$

For n = 1,

$Z = -2i^{1+1}$

$= -2i^2$

= 2, which is a real number.

∴ The smallest positive integer value of n is 1.

12. If $[(1+i)/(1-i)]^3 – [(1-i)/(1+i)]^3 = x + iy$, find (x, y)

Solution:

Given:

$[(1+i)/(1-i)]^3 – [(1-i)/(1+i)]^3 = x + iy$

Let us rationalize the denominator, we get

$\left(\frac{1+i}{1-i} \times \frac{1+i}{1+i}\right)^3 - \left(\frac{1-i}{1+i} \times \frac{1-i}{1-i}\right)^3 = x + iy$

$\left(\frac{(1+i)^2}{1^2-i^2}\right)^3 - \left(\frac{(1-i)^2}{1^2-i^2}\right)^3 = x + iy$

$\left(\frac{1^2+i^2+2(1)(i)}{1-(-1)}\right)^3 - \left(\frac{1^2+i^2-2(1)(i)}{1-(-1)}\right)^3 = x + iy$

$\left(\frac{1-1+2i}{2}\right)^3 - \left(\frac{1-1-2i}{2}\right)^3 = x + iy$

$\left(\frac{2i}{2}\right)^3 - \left(\frac{-2i}{2}\right)^3 = x + iy$

$i^3 – (-i)^3 = x + iy$

$2i^3 = x + iy$

$2i^2.i = x + iy$

$2(-1)I = x + iy$

$-2i = x + iy$

Equating Real and Imaginary parts on both sides we get

x = 0 and y = -2

∴ The values of x and y are 0 and -2.

13. If $(1+i)^2$ / (2-i) = x + iy, find x + y

Solution:

Given:

$(1+i)^2 / (2-i) = x + iy$

Upon expansion we get,

$$\frac{1^2+i^2+2(1)(i)}{2-i} = x + iy$$

$$\frac{1+(-1)+2i}{2-i} = x + iy$$

$$\frac{2i}{2-i} = x + iy$$

Now, let us multiply and divide by (2+i), we get

$$\frac{2i}{2-i} \times \frac{2+i}{2+i} = x + iy$$

$$\frac{4i+2i^2}{2^2-i^2} = x + iy$$

$$\frac{2(-1)+4i}{4-(-1)} = x + iy$$

$$\frac{-2+4i}{5} = x + iy$$

Let us equate real and imaginary parts on both sides we get,

x = -2/5 and y = 4/5

so,

x + y = -2/5 + 4/5

= (-2+4)/5

= 2/5

∴ The value of (x + y) is 2/5

EXERCISE 13.3 PAGE NO: 13.39

1. Find the square root of the following complex numbers.
(i) – 5 + 12i
(ii) -7 - 24i
(iii) 1 - i
(iv) - 8 - 6i
(v) 8 – 15i
(vi) -11 - 60√-1
(vii) 1 + 4√-3
(viii) 4i
(ix) -i
Solution:

$$if\ b>0,\ \sqrt{a+ib}=\pm\left[\left(\frac{a+\sqrt{a^2+b^2}}{2}\right)^{\frac{1}{2}}+i\left(\frac{-a+\sqrt{a^2+b^2}}{2}\right)^{\frac{1}{2}}\right]$$

$$if\ b<0,\ \sqrt{a+ib}=\pm\left[\left(\frac{a+\sqrt{a^2+b^2}}{2}\right)^{\frac{1}{2}}-i\left(\frac{-a+\sqrt{a^2+b^2}}{2}\right)^{\frac{1}{2}}\right]$$

(i) – 5 + 12i
Given:
– 5 + 12i
We know, Z = a + ib
So, √(a + ib) = √(-5+12i)
Here, b > 0
Let us simplify now,

$$\sqrt{-5+12i}=\pm\left[\left(\frac{-5+\sqrt{(-5)^2+12^2}}{2}\right)^{\frac{1}{2}}+i\left(\frac{5+\sqrt{(-5)^2+12^2}}{2}\right)^{\frac{1}{2}}\right]\ \text{[Since, b > 0]}$$

$$=\pm\left[\left(\frac{-5+\sqrt{25+144}}{2}\right)^{\frac{1}{2}}+i\left(\frac{5+\sqrt{25+144}}{2}\right)^{\frac{1}{2}}\right]$$

$$=\pm\left[\left(\frac{-5+\sqrt{169}}{2}\right)^{\frac{1}{2}}+i\left(\frac{5+\sqrt{169}}{2}\right)^{\frac{1}{2}}\right]$$

$$=\pm\left[\left(\frac{-5+13}{2}\right)^{\frac{1}{2}}+i\left(\frac{5+13}{2}\right)^{\frac{1}{2}}\right]$$

$$= \pm\left[\left(\frac{8}{2}\right)^{\frac{1}{2}} + i\left(\frac{18}{2}\right)^{\frac{1}{2}}\right]$$

$$= \pm\left[4^{\frac{1}{2}} + i9^{\frac{1}{2}}\right]$$

$$= \pm[2 + 3i]$$

∴ Square root of (– 5 + 12i) is ±[2 + 3i]

(ii) -7 – 24i

Given:

-7 – 24i

We know, Z = -7 – 24i

So, √(a + ib) = √(-7 – 24i)

Here, b < 0

Let us simplify now,

$$\sqrt{-7-24i} = \pm\left[\left(\frac{-7+\sqrt{(-7)^2+(-24)^2}}{2}\right)^{\frac{1}{2}} - i\left(\frac{7+\sqrt{(-7)^2+(-24)^2}}{2}\right)^{\frac{1}{2}}\right] \text{[Since, b < 0]}$$

$$= \pm\left[\left(\frac{-7+\sqrt{49+576}}{2}\right)^{\frac{1}{2}} - i\left(\frac{7+\sqrt{49+576}}{2}\right)^{\frac{1}{2}}\right]$$

$$= \pm\left[\left(\frac{-7+\sqrt{625}}{2}\right)^{\frac{1}{2}} - i\left(\frac{7+\sqrt{625}}{2}\right)^{\frac{1}{2}}\right]$$

$$= \pm\left[\left(\frac{-7+25}{2}\right)^{\frac{1}{2}} - i\left(\frac{7+25}{2}\right)^{\frac{1}{2}}\right]$$

$$= \pm\left[\left(\frac{18}{2}\right)^{\frac{1}{2}} - i\left(\frac{32}{2}\right)^{\frac{1}{2}}\right]$$

$= \pm [9^{1/2} – i16^{1/2}]$

$= \pm [3 – 4i]$

∴ Square root of (-7 – 24i) is ± [3 - 4i]

(iii) 1 - i

Given:

1 – i

We know, Z = (1 - i)

So, √(a + ib) = √(1 - i)

Here, b < 0
Let us simplify now,

$$\sqrt{1-i} = \pm\left[\left(\frac{1+\sqrt{(1)^2+(-1)^2}}{2}\right)^{\frac{1}{2}} - i\left(\frac{-1+\sqrt{(1)^2+(-1)^2}}{2}\right)^{\frac{1}{2}}\right] \text{[Since, b < 0]}$$

$$= \pm\left[\left(\frac{1+\sqrt{1+1}}{2}\right)^{\frac{1}{2}} - i\left(\frac{-1+\sqrt{1+1}}{2}\right)^{\frac{1}{2}}\right]$$

$$= \pm\left[\left(\frac{1+\sqrt{2}}{2}\right)^{\frac{1}{2}} - i\left(\frac{-1+\sqrt{2}}{2}\right)^{\frac{1}{2}}\right]$$

$$= \pm\left[\left(\sqrt{\frac{\sqrt{2}+1}{2}}\right) - i\left(\sqrt{\frac{\sqrt{2}-1}{2}}\right)\right]$$

∴ Square root of (1 - i) is ± [(√(√2+1)/2) – i (√(√2-1)/2)]

(iv) -8 -6i
Given:
-8 -6i
We know, Z = -8 -6i
So, √(a + ib) = -8 -6i
Here, b < 0
Let us simplify now,

$$\sqrt{-8-6i} = \pm\left[\left(\frac{-8+\sqrt{(-8)^2+(-6)^2}}{2}\right)^{\frac{1}{2}} - i\left(\frac{8+\sqrt{(-8)^2+(-6)^2}}{2}\right)^{\frac{1}{2}}\right] \text{[Since, b < 0]}$$

$$= \pm\left[\left(\frac{-8+\sqrt{64+36}}{2}\right)^{\frac{1}{2}} - i\left(\frac{8+\sqrt{64+36}}{2}\right)^{\frac{1}{2}}\right]$$

$$= \pm\left[\left(\frac{-8+\sqrt{100}}{2}\right)^{\frac{1}{2}} - i\left(\frac{8+\sqrt{100}}{2}\right)^{\frac{1}{2}}\right]$$

$$= \pm\left[\left(\frac{-8+10}{2}\right)^{\frac{1}{2}} - i\left(\frac{8+10}{2}\right)^{\frac{1}{2}}\right]$$

$$= \pm\left[\left(\frac{2}{2}\right)^{\frac{1}{2}} - i\left(\frac{18}{2}\right)^{\frac{1}{2}}\right]$$

$= [1^{1/2} – i\ 9^{1/2}]$

= ± [1 – 3i]
∴ Square root of (-8 -6i) is ± [1 – 3i]

(v) 8 – 15i
Given:
8 – 15i
We know, Z = 8 – 15i
So, √(a + ib) = 8 – 15i
Here, b < 0
Let us simplify now,

$$\sqrt{8-15i} = \pm\left[\left(\frac{8+\sqrt{8^2+(-15)^2}}{2}\right)^{\frac{1}{2}} - i\left(\frac{-8+\sqrt{8^2+(-15)^2}}{2}\right)^{\frac{1}{2}}\right]$$ [Since, b < 0]

$$= \pm\left[\left(\frac{8+\sqrt{64+225}}{2}\right)^{\frac{1}{2}} - i\left(\frac{-8+\sqrt{64+225}}{2}\right)^{\frac{1}{2}}\right]$$

$$= \pm\left[\left(\frac{8+\sqrt{289}}{2}\right)^{\frac{1}{2}} - i\left(\frac{-8+\sqrt{289}}{2}\right)^{\frac{1}{2}}\right]$$

$$= \pm\left[\left(\frac{8+17}{2}\right)^{\frac{1}{2}} - i\left(\frac{-8+17}{2}\right)^{\frac{1}{2}}\right]$$

$$= \pm\left[\left(\frac{25}{2}\right)^{\frac{1}{2}} - i\left(\frac{9}{2}\right)^{\frac{1}{2}}\right]$$

$$= \pm\left[\frac{5}{\sqrt{2}} - \frac{i3}{\sqrt{2}}\right]$$

= ± 1/√2 (5 – 3i)
∴ Square root of (8 – 15i) is ± 1/√2 (5 – 3i)

(vi) -11 - 60√-1
Given:
-11 - 60√-1
We know, Z = -11 - 60√-1
So, √(a + ib) = -11 - 60√-1
= -11 – 60i
Here, b < 0
Let us simplify now,

$$\sqrt{-11-60i} = \pm\left[\left(\frac{-11+\sqrt{(-11)^2+(-60)^2}}{2}\right)^{\frac{1}{2}} - i\left(\frac{11+\sqrt{(-11)^2+(60)^2}}{2}\right)^{\frac{1}{2}}\right]$$ [Since, b < 0]

$$= \pm\left[\left(\frac{-11+\sqrt{121+3600}}{2}\right)^{\frac{1}{2}} - i\left(\frac{11+\sqrt{121+3600}}{2}\right)^{\frac{1}{2}}\right]$$

$$= \pm\left[\left(\frac{-11+\sqrt{3721}}{2}\right)^{\frac{1}{2}} - i\left(\frac{11+\sqrt{3721}}{2}\right)^{\frac{1}{2}}\right]$$

$$= \pm\left[\left(\frac{-11+61}{2}\right)^{\frac{1}{2}} - i\left(\frac{11+61}{2}\right)^{\frac{1}{2}}\right]$$

$$= \pm\left[\left(\frac{50}{2}\right)^{\frac{1}{2}} - i\left(\frac{72}{2}\right)^{\frac{1}{2}}\right]$$

$$= \pm\left[25^{\frac{1}{2}} - i36^{\frac{1}{2}}\right]$$

$$= \pm(5-6i)$$

∴ Square root of (-11 - 60√-1) is ± (5 – 6i)

(vii) 1 + 4√-3

Given:

1 + 4√-3

We know, Z = 1 + 4√-3

So, √(a + ib) = 1 + 4√-3

= 1 + 4(√3) (√-1)

= 1 + 4√3i

Here, b > 0

Let us simplify now,

$$\sqrt{1+4\sqrt{3}i} = \pm\left[\left(\frac{1+\sqrt{1^2+(4\sqrt{3})^2}}{2}\right)^{\frac{1}{2}} + i\left(\frac{-1+\sqrt{1^2+(4\sqrt{3})^2}}{2}\right)^{\frac{1}{2}}\right]$$ [Since, b > 0]

$$= \pm\left[\left(\frac{1+\sqrt{1+48}}{2}\right)^{\frac{1}{2}} + i\left(\frac{-1+\sqrt{1+48}}{2}\right)^{\frac{1}{2}}\right]$$

$$= \pm\left[\left(\frac{1+\sqrt{49}}{2}\right)^{\frac{1}{2}} + i\left(\frac{-1+\sqrt{49}}{2}\right)^{\frac{1}{2}}\right]$$

$$= \pm\left[\left(\frac{1+7}{2}\right)^{\frac{1}{2}} + i\left(\frac{-1+7}{2}\right)^{\frac{1}{2}}\right]$$

$$= \pm\left[\left(\frac{8}{2}\right)^{\frac{1}{2}} + i\left(\frac{6}{2}\right)^{\frac{1}{2}}\right]$$

$$= \pm\left[4^{\frac{1}{2}} + i3^{\frac{1}{2}}\right]$$

$$= \pm[2 + \sqrt{3}i]$$

∴ Square root of (1 + 4√-3) is ± (2 + √3i)

(viii) 4i

Given:

4i

We know, Z = 4i

So, √(a + ib) = 4i

Here, b > 0

Let us simplify now,

$$\sqrt{4i} = \pm\left[\left(\frac{0+\sqrt{0^2+4^2}}{2}\right)^{\frac{1}{2}} + i\left(\frac{0+\sqrt{0^2+4^2}}{2}\right)^{\frac{1}{2}}\right] \text{ [Since, b > 0]}$$

$$= \pm\left[\left(\frac{0+\sqrt{0+16}}{2}\right)^{\frac{1}{2}} + i\left(\frac{0+\sqrt{0+16}}{2}\right)^{\frac{1}{2}}\right]$$

$$= \pm\left[\left(\frac{0+\sqrt{16}}{2}\right)^{\frac{1}{2}} + i\left(\frac{0+\sqrt{16}}{2}\right)^{\frac{1}{2}}\right]$$

$$= \pm\left[\left(\frac{0+4}{2}\right)^{\frac{1}{2}} + i\left(\frac{0+4}{2}\right)^{\frac{1}{2}}\right]$$

$$= \pm\left[\left(\frac{4}{2}\right)^{\frac{1}{2}} + i\left(\frac{4}{2}\right)^{\frac{1}{2}}\right]$$

$$= \pm\left[2^{\frac{1}{2}} + i2^{\frac{1}{2}}\right]$$

$$= \pm[\sqrt{2} + \sqrt{2}i]$$

$$= \pm \sqrt{2}\,(1 + i)$$

∴ Square root of 4i is ± √2 (1 + i)

(ix) –i

Given:

-i
We know, Z = -i
So, √(a + ib) = -i
Here, b < 0
Let us simplify now,

$$\sqrt{-i} = \pm\left[\left(\frac{0+\sqrt{0^2+(-1)^2}}{2}\right)^{\frac{1}{2}} - i\left(\frac{0+\sqrt{0^2+(-1)^2}}{2}\right)^{\frac{1}{2}}\right] \text{[Since, b < 0]}$$

$$= \pm\left[\left(\frac{0+\sqrt{0+1}}{2}\right)^{\frac{1}{2}} - i\left(\frac{0+\sqrt{0+1}}{2}\right)^{\frac{1}{2}}\right]$$

$$= \pm\left[\left(\frac{0+\sqrt{1}}{2}\right)^{\frac{1}{2}} - i\left(\frac{0+\sqrt{1}}{2}\right)^{\frac{1}{2}}\right]$$

$$= \pm\left[\left(\frac{0+1}{2}\right)^{\frac{1}{2}} - i\left(\frac{0+1}{2}\right)^{\frac{1}{2}}\right]$$

$$= \pm\left[\left(\frac{1}{2}\right)^{\frac{1}{2}} - i\left(\frac{1}{2}\right)^{\frac{1}{2}}\right]$$

$$= \pm\left[\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right]$$

= ± 1/√2 (1 - i)

∴ Square root of –i is ± 1/√2 (1 - i)

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1. Find the modulus and argument of the following complex numbers and hence express each of them in the polar form:
(i) 1 + i
(ii) √3 + i
(iii) 1 - i
(iv) (1 - i) / (1 + i)
(v) 1/(1 + i)
(vi) (1 + 2i) / (1 – 3i)
(vii) sin 120° – i cos 120°
(viii) -16 / (1 + i√3)
Solution:
We know that the polar form of a complex number Z = x + iy is given by Z = |Z| (cos θ + i sin θ)
Where,
$|Z|$ = modulus of complex number = $\sqrt{(x^2 + y^2)}$
$\theta = \arg(z)$ = argument of complex number = $\tan^{-1}(|y| / |x|)$

(i) 1 + i
Given: Z = 1 + i
So now,
$|Z| = \sqrt{(x^2 + y^2)}$
$= \sqrt{(1^2 + 1^2)}$
$= \sqrt{(1 + 1)}$
$= \sqrt{2}$
$\theta = \tan^{-1}(|y| / |x|)$
$= \tan^{-1}(1 / 1)$
$= \tan^{-1} 1$
Since x > 0, y > 0 complex number lies in 1st quadrant and the value of θ is $0^0 \leq \theta \leq 90^0$.
$\theta = \pi/4$
$Z = \sqrt{2}(\cos(\pi/4) + i \sin(\pi/4))$
∴ Polar form of (1 + i) is $\sqrt{2}(\cos(\pi/4) + i \sin(\pi/4))$

(ii) √3 + i
Given: $Z = \sqrt{3} + i$
So now,
$|Z| = \sqrt{(x^2 + y^2)}$
$= \sqrt{((\sqrt{3})^2 + 1^2)}$

= √(3 + 1)
= √4
= 2
θ = $\tan^{-1}$ (|y| / |x|)
= $\tan^{-1}$ (1 / √3)
Since x > 0, y > 0 complex number lies in 1st quadrant and the value of θ is $0^0 \le \theta \le 90^0$.
θ = π/6
Z = 2 (cos (π/6) + i sin (π/6))
∴ Polar form of (√3 + i) is 2 (cos (π/6) + i sin (π/6))

(iii) 1 - i
Given: Z = 1 - i
So now,
|Z| = $\sqrt{(x^2 + y^2)}$
= $\sqrt{(1^2 + (-1)^2)}$
= √(1 + 1)
= √2
θ = $\tan^{-1}$ (|y| / |x|)
= $\tan^{-1}$ (1 / 1)
= $\tan^{-1}$ 1
Since x > 0, y < 0 complex number lies in 2nd quadrant and the value of θ is $-90^0 \le \theta \le 0^0$.
θ = -π/4
Z = √2 (cos (-π/4) + i sin (-π/4))
= √2 (cos (π/4) - i sin (π/4))
∴ Polar form of (1 - i) is √2 (cos (π/4) - i sin (π/4))

(iv) (1 - i) / (1 + i)
Given: Z = (1 - i) / (1 + i)
Let us multiply and divide by (1 - i), we get

$$z = \frac{1-i}{1+i} \times \frac{1-i}{1-i}$$
$$= \frac{(1-i)^2}{1^2-i^2}$$
$$= \frac{1^2+i^2-2(1)(i)}{1-(-1)}$$
$$= \frac{1+(-1)-2i}{2}$$
$$= \frac{-2i}{2}$$
= 0 – i

So now,
$|Z| = \sqrt{(x^2 + y^2)}$
$= \sqrt{(0^2 + (-1)^2)}$
$= \sqrt{(0 + 1)}$
$= \sqrt{1}$
$\theta = \tan^{-1} (|y| / |x|)$
$= \tan^{-1} (1 / 0)$
$= \tan^{-1} \infty$
Since $x \geq 0$, $y < 0$ complex number lies in 4th quadrant and the value of θ is $-90^0 \leq \theta \leq 0^0$.
$\theta = -\pi/2$
$Z = 1 (\cos (-\pi/2) + i \sin (-\pi/2))$
$= 1 (\cos (\pi/2) - i \sin (\pi/2))$
$\therefore$ Polar form of $(1 - i) / (1 + i)$ is $1 (\cos (\pi/2) - i \sin (\pi/2))$

(v) $1/(1 + i)$
Given: $Z = 1 / (1 + i)$
Let us multiply and divide by $(1 - i)$, we get
$$z = \frac{1}{1+i} \times \frac{1-i}{1-i}$$
$$= \frac{1-i}{1^2 - i^2}$$
$$= \frac{1-i}{1-(-1)}$$
$$= \frac{1-i}{2}$$
So now,
$|Z| = \sqrt{(x^2 + y^2)}$
$= \sqrt{((1/2)^2 + (-1/2)^2)}$
$= \sqrt{(1/4 + 1/4)}$
$= \sqrt{(2/4)}$
$= 1/\sqrt{2}$
$\theta = \tan^{-1} (|y| / |x|)$
$= \tan^{-1} ((1/2) / (1/2))$
$= \tan^{-1} 1$
Since $x > 0$, $y < 0$ complex number lies in 4th quadrant and the value of θ is $-90^0 \leq \theta \leq 0^0$.
$\theta = -\pi/4$
$Z = 1/\sqrt{2} (\cos (-\pi/4) + i \sin (-\pi/4))$
$= 1/\sqrt{2} (\cos (\pi/4) - i \sin (\pi/4))$
$\therefore$ Polar form of $1/(1 + i)$ is $1/\sqrt{2} (\cos (\pi/4) - i \sin (\pi/4))$

(vi) $(1 + 2i) / (1 – 3i)$
Given: $Z = (1 + 2i) / (1 – 3i)$
Let us multiply and divide by $(1 + 3i)$, we get

$$z = \frac{1+2i}{1-3i} \times \frac{1+3i}{1+3i}$$
$$= \frac{1(1+3i)+2i(1+3i)}{1^2-(3i)^2}$$
$$= \frac{1+3i+2i+6i^2}{1-9i^2}$$
$$= \frac{1+5i+6(-1)}{1-9(-1)}$$
$$= \frac{-5+5i}{10}$$
$$= \frac{-1+i}{2}$$

So now,
$|Z| = \sqrt{(x^2 + y^2)}$
$= \sqrt{((-1/2)^2 + (1/2)^2)}$
$= \sqrt{(1/4 + 1/4)}$
$= \sqrt{(2/4)}$
$= 1/\sqrt{2}$
$\theta = \tan^{-1} (|y| / |x|)$
$= \tan^{-1} ((1/2) / (1/2))$
$= \tan^{-1} 1$
Since $x < 0$, $y > 0$ complex number lies in 2nd quadrant and the value of θ is $90^0 \le \theta \le 180^0$.
$\theta = 3\pi/4$
$Z = 1/\sqrt{2} (\cos (3\pi/4) + i \sin (3\pi/4))$
∴ Polar form of $(1 + 2i) / (1 – 3i)$ is $1/\sqrt{2} (\cos (3\pi/4) + i \sin (3\pi/4))$

(vii) $\sin 120^o – i \cos 120^o$
Given: $Z = \sin 120^o – i \cos 120^o$
$= \sqrt{3}/2 – i (-1/2)$
$= \sqrt{3}/2 + i (1/2)$
So now,
$|Z| = \sqrt{(x^2 + y^2)}$
$= \sqrt{((\sqrt{3}/2)^2 + (1/2)^2)}$
$= \sqrt{(3/4 + 1/4)}$
$= \sqrt{(4/4)}$
$= \sqrt{1}$
$= 1$

$\theta = \tan^{-1} (|y| / |x|)$
$= \tan^{-1} ((1/2) / (\sqrt{3}/2))$
$= \tan^{-1} (1/\sqrt{3})$
Since $x > 0$, $y > 0$ complex number lies in 1st quadrant and the value of θ is $0^0 \leq \theta \leq 90^0$.
$\theta = \pi/6$
$Z = 1 (\cos (\pi/6) + i \sin (\pi/6))$
∴ Polar form of $\sqrt{3}/2 + i (1/2)$ is $1 (\cos (\pi/6) + i \sin (\pi/6))$

(viii) $-16 / (1 + i\sqrt{3})$
Given: $Z = -16 / (1 + i\sqrt{3})$
Let us multiply and divide by $(1 - i\sqrt{3})$, we get

$$z = \frac{-16}{1+i\sqrt{3}} \times \frac{1-i\sqrt{3}}{1-i\sqrt{3}}$$
$$= \frac{-16+i16\sqrt{3}}{1^2-\left(i\sqrt{3}\right)^2}$$
$$= \frac{-16+i16\sqrt{3}}{1-3i^2}$$
$$= \frac{-16+i16\sqrt{3}}{1-3(-1)}$$
$$= \frac{-16+i16\sqrt{3}}{4}$$
$= -4 + i\, 4\sqrt{3}$

So now,
$|Z| = \sqrt{(x^2 + y^2)}$
$= \sqrt{((-4)^2 + (4\sqrt{3})^2)}$
$= \sqrt{(16 + 48)}$
$= \sqrt{(64)}$
$= 8$
$\theta = \tan^{-1} (|y| / |x|)$
$= \tan^{-1} ((4\sqrt{3}) / 4)$
$= \tan^{-1} (\sqrt{3})$
Since $x < 0$, $y > 0$ complex number lies in 2nd quadrant and the value of θ is $90^0 \leq \theta \leq 180^0$.
$\theta = 2\pi/3$
$Z = 8 (\cos (2\pi/3) + i \sin (2\pi/3))$
∴ Polar form of $-16 / (1 + i\sqrt{3})$ is $8 (\cos (2\pi/3) + i \sin (2\pi/3))$

2. Write $(i^{25})^3$ in polar form.
Solution:
Given: $Z = (i^{25})^3$
$= i^{75}$

$= i^{74}. i$
$= (i^2)^{37}. i$
$= (-1)^{37}. i$
$= (-1). i$
$= - i$
$= 0 – i$

So now,
$|Z| = \sqrt{(x^2 + y^2)}$
$= \sqrt{(0^2 + (-1)^2)}$
$= \sqrt{(0 + 1)}$
$= \sqrt{1}$
$\theta = \tan^{-1} (|y| / |x|)$
$= \tan^{-1} (1 / 0)$
$= \tan^{-1} \infty$
Since $x \geq 0$, $y < 0$ complex number lies in 4^{th} quadrant and the value of θ is $-90^0 \leq \theta \leq 0^0$.
$\theta = -\pi/2$
$Z = 1 (\cos (-\pi/2) + i \sin (-\pi/2))$
$= 1 (\cos (\pi/2) - i \sin (\pi/2))$
∴ Polar form of $(i^{25})^3$ is $1 (\cos (\pi/2) - i \sin (\pi/2))$

3. Express the following complex numbers in the form r (cos θ + i sin θ):
(i) 1 + i tan α
(ii) tan α – i
(iii) 1 – sin α + i cos α
(iv) (1 - i) / (cos π/3 + i sin π/3)
Solution:
(i) $1 + i \tan \alpha$
Given: $Z = 1 + i \tan \alpha$
We know that the polar form of a complex number $Z = x + iy$ is given by $Z = |Z| (\cos \theta + i \sin \theta)$
Where,
$|Z|$ = modulus of complex number $= \sqrt{(x^2 + y^2)}$
$\theta = \arg (z)$ = argument of complex number $= \tan^{-1} (|y| / |x|)$
We also know that tan α is a periodic function with period π.
So α is lying in the interval $[0, \pi/2) \cup (\pi/2, \pi]$.

<u>Let us consider case 1:</u>
$\alpha \in [0, \pi/2)$
So now,

$|Z| = r = \sqrt{(x^2 + y^2)}$
$= \sqrt{(1^2 + \tan^2 \alpha)}$
$= \sqrt{(\sec^2 \alpha)}$
$= |\sec \alpha|$ since, $\sec \alpha$ is positive in the interval $[0, \pi/2)$
$\theta = \tan^{-1} (|y| / |x|)$
$= \tan^{-1} (\tan \alpha / 1)$
$= \tan^{-1} (\tan \alpha)$
$= \alpha$ since, $\tan \alpha$ is positive in the interval $[0, \pi/2)$
∴ Polar form is $Z = \sec \alpha (\cos \alpha + i \sin \alpha)$

<u>Let us consider case 2:</u>
$\alpha \in (\pi/2, \pi]$
So now,
$|Z| = r = \sqrt{(x^2 + y^2)}$
$= \sqrt{(1^2 + \tan^2 \alpha)}$
$= \sqrt{(\sec^2 \alpha)}$
$= |\sec \alpha|$
$= - \sec \alpha$ since, $\sec \alpha$ is negative in the interval $(\pi/2, \pi]$
$\theta = \tan^{-1} (|y| / |x|)$
$= \tan^{-1} (\tan \alpha / 1)$
$= \tan^{-1} (\tan \alpha)$
$= -\pi + \alpha$ since, $\tan \alpha$ is negative in the interval $(\pi/2, \pi]$
$\theta = -\pi + \alpha$ [since, θ lies in 4^{th} quadrant]
$Z = -\sec \alpha (\cos (\alpha - \pi) + i \sin (\alpha - \pi))$
∴ Polar form is $Z = -\sec \alpha (\cos (\alpha - \pi) + i \sin (\alpha - \pi))$

(ii) $\tan \alpha – i$
Given: $Z = \tan \alpha – i$
We know that the polar form of a complex number $Z = x + iy$ is given by $Z = |Z| (\cos \theta + i \sin \theta)$
Where,
$|Z|$ = modulus of complex number $= \sqrt{(x^2 + y^2)}$
$\theta = \arg (z)$ = argument of complex number $= \tan^{-1} (|y| / |x|)$
We also know that $\tan \alpha$ is a periodic function with period π.
So α is lying in the interval $[0, \pi/2) \cup (\pi/2, \pi]$.

<u>Let us consider case 1:</u>
$\alpha \in [0, \pi/2)$
So now,

$|Z| = r = \sqrt{(x^2 + y^2)}$
$= \sqrt{(\tan^2 \alpha + 1^2)}$
$= \sqrt{(\sec^2 \alpha)}$
$= |\sec \alpha|$ since, $\sec \alpha$ is positive in the interval $[0, \pi/2)$
$= \sec \alpha$
$\theta = \tan^{-1} (|y| / |x|)$
$= \tan^{-1} (1/\tan \alpha)$
$= \tan^{-1} (\cot \alpha)$ since, $\cot \alpha$ is positive in the interval $[0, \pi/2)$
$= \alpha - \pi/2$ [since, θ lies in 4th quadrant]
$Z = \sec \alpha (\cos (\alpha - \pi/2) + i \sin (\alpha - \pi/2))$
∴ Polar form is $Z = \sec \alpha (\cos (\alpha - \pi/2) + i \sin (\alpha - \pi/2))$

Let us consider case 2:
$\alpha \in (\pi/2, \pi]$
So now,
$|Z| = r = \sqrt{(x^2 + y^2)}$
$= \sqrt{(\tan^2 \alpha + 1^2)}$
$= \sqrt{(\sec^2 \alpha)}$
$= |\sec \alpha|$
$= - \sec \alpha$ since, $\sec \alpha$ is negative in the interval $(\pi/2, \pi]$
$\theta = \tan^{-1} (|y| / |x|)$
$= \tan^{-1} (1/\tan \alpha)$
$= \tan^{-1} (\cot \alpha)$
$= \pi/2 + \alpha$ since, $\cot \alpha$ is negative in the interval $(\pi/2, \pi]$
$\theta = \pi/2 + \alpha$ [since, θ lies in 3rd quadrant]
$Z = -\sec \alpha (\cos (\pi/2 + \alpha) + i \sin (\pi/2 + \alpha))$
∴ Polar form is $Z = -\sec \alpha (\cos (\pi/2 + \alpha) + i \sin (\pi/2 + \alpha))$

(iii) $1 – \sin \alpha + i \cos \alpha$
Given: $Z = 1 – \sin \alpha + i \cos \alpha$
By using the formulas,
$\sin^2 \theta + \cos^2 \theta = 1$
$\sin 2\theta = 2 \sin \theta \cos \theta$
$\cos 2\theta = \cos^2 \theta – \sin^2 \theta$
So,

$$z = \left(\sin^2\left(\frac{\alpha}{2}\right) + \cos^2\left(\frac{\alpha}{2}\right) - 2\sin\left(\frac{\alpha}{2}\right)\cos\left(\frac{\alpha}{2}\right)\right) + i\left(\cos^2\left(\frac{\alpha}{2}\right) - \sin^2\left(\frac{\alpha}{2}\right)\right)$$

$$= \left(\cos\left(\frac{\alpha}{2}\right) - \sin\left(\frac{\alpha}{2}\right)\right)^2 + i\left(\cos^2\left(\frac{\alpha}{2}\right) - \sin^2\left(\frac{\alpha}{2}\right)\right)$$

We know that the polar form of a complex number Z = x + iy is given by Z = |Z| (cos θ + i sin θ)
Where,
|Z| = modulus of complex number = $\sqrt{(x^2 + y^2)}$
θ = arg (z) = argument of complex number = $\tan^{-1}$ (|y| / |x|)
Now,

$$|z| = \sqrt{(1 - sin\alpha)^2 + \cos^2\alpha}$$
$$= \sqrt{1 + \sin^2\alpha - 2sin\alpha + \cos^2\alpha}$$
$$= \sqrt{1 + 1 - 2sin\alpha}$$
$$= \sqrt{(2)(1 - sin\alpha)}$$
$$= \sqrt{(2)\left(\sin^2\left(\frac{\alpha}{2}\right) + \cos^2\left(\frac{\alpha}{2}\right) - 2\sin\left(\frac{\alpha}{2}\right)\cos\left(\frac{\alpha}{2}\right)\right)}$$
$$= \sqrt{(2)\left(\cos\left(\frac{\alpha}{2}\right) - \sin\left(\frac{\alpha}{2}\right)\right)^2}$$
$$= \left|\sqrt{2}\left(\cos\left(\frac{\alpha}{2}\right) - \sin\left(\frac{\alpha}{2}\right)\right)\right|$$

$$\theta = \tan^{-1}\left(\frac{\cos^2\left(\frac{\alpha}{2}\right) - \sin^2\left(\frac{\alpha}{2}\right)}{\left(\cos\left(\frac{\alpha}{2}\right) - \sin\left(\frac{\alpha}{2}\right)\right)^2}\right)$$
$$= \tan^{-1}\left(\frac{\left(\cos\left(\frac{\alpha}{2}\right) - \sin\left(\frac{\alpha}{2}\right)\right)\left(\cos\left(\frac{\alpha}{2}\right) + \sin\left(\frac{\alpha}{2}\right)\right)}{\left(\cos\left(\frac{\alpha}{2}\right) - \sin\left(\frac{\alpha}{2}\right)\right)^2}\right)$$
$$= \tan^{-1}\left(\frac{\cos\left(\frac{\alpha}{2}\right) + \sin\left(\frac{\alpha}{2}\right)}{\cos\left(\frac{\alpha}{2}\right) - \sin\left(\frac{\alpha}{2}\right)}\right)$$
$$= \tan^{-1}\left(\frac{\cos\left(\frac{\alpha}{2}\right)\left(1 + \tan\left(\frac{\alpha}{2}\right)\right)}{\cos\left(\frac{\alpha}{2}\right)\left(1 - \tan\left(\frac{\alpha}{2}\right)\right)}\right)$$
$$= \tan^{-1}\left(\frac{\tan\left(\frac{\pi}{4}\right) + \tan\left(\frac{\alpha}{2}\right)}{1 - \tan\left(\frac{\pi}{4}\right)\tan\left(\frac{\alpha}{2}\right)}\right)$$
$$= \tan^{-1}\left(\tan\left(\frac{\pi}{4} + \frac{\alpha}{2}\right)\right)$$

We know that sine and cosine functions are periodic with period 2π
Here we have 3 intervals:
$0 \le \alpha \le \pi/2$
$\pi/2 \le \alpha \le 3\pi/2$
$3\pi/2 \le \alpha \le 2\pi$

Let us consider case 1:
In the interval $0 \le \alpha \le \pi/2$
Cos $(\alpha/2) > \sin(\alpha/2)$ and also $0 < \pi/4 + \alpha/2 < \pi/2$
So,

$$|z| = \left|\sqrt{2}\left(\cos\left(\frac{\alpha}{2}\right) - \sin\left(\frac{\alpha}{2}\right)\right)\right|$$
$$= \sqrt{2}\left(\cos\left(\frac{\alpha}{2}\right) - \sin\left(\frac{\alpha}{2}\right)\right)$$
$$\theta = \tan^{-1}\left(\tan\left(\frac{\pi}{4} + \frac{\alpha}{2}\right)\right)$$

$= \pi/4 + \alpha/2$ [since, θ lies in 1st quadrant]

∴ Polar form is $Z = \sqrt{2}(\cos(\alpha/2) - \sin(\alpha/2))(\cos(\pi/4 + \alpha/2) + i \sin(\pi/4 + \alpha/2))$

Let us consider case 2:
In the interval $\pi/2 \le \alpha \le 3\pi/2$
Cos $(\alpha/2) < \sin(\alpha/2)$ and also $\pi/2 < \pi/4 + \alpha/2 < \pi$
So,

$$|z| = \left|\sqrt{2}\left(\cos\left(\frac{\alpha}{2}\right) - \sin\left(\frac{\alpha}{2}\right)\right)\right|$$
$$= -\sqrt{2}\left(\cos\left(\frac{\alpha}{2}\right) - \sin\left(\frac{\alpha}{2}\right)\right)$$
$$= \sqrt{2}\left(\sin\left(\frac{\alpha}{2}\right) - \cos\left(\frac{\alpha}{2}\right)\right)$$
$$\theta = \tan^{-1}\left(\tan\left(\frac{\pi}{4} + \frac{\alpha}{2}\right)\right)$$

$= \pi - [\pi/4 + \alpha/2]$ [since, θ lies in 4th quadrant]
$= 3\pi/4 - \alpha/2$

Since, $(1 - \sin\alpha) > 0$ and $\cos\alpha < 0$ [Z lies in 4th quadrant]
$= \alpha/2 - 3\pi/4$

∴ Polar form is $Z = -\sqrt{2}(\cos(\alpha/2) - \sin(\alpha/2))(\cos(\alpha/2 - 3\pi/4) + i \sin(\alpha/2 - 3\pi/4))$

Let us consider case 3:
In the interval $3\pi/2 \le \alpha \le 2\pi$
Cos $(\alpha/2) < \sin(\alpha/2)$ and also $\pi < \pi/4 + \alpha/2 < 5\pi/4$
So,

$$|z| = \left|\sqrt{2}\left(\cos\left(\frac{\alpha}{2}\right) - \sin\left(\frac{\alpha}{2}\right)\right)\right|$$
$$= -\sqrt{2}\left(\cos\left(\frac{\alpha}{2}\right) - \sin\left(\frac{\alpha}{2}\right)\right)$$
$$= \sqrt{2}\left(\sin\left(\frac{\alpha}{2}\right) - \cos\left(\frac{\alpha}{2}\right)\right)$$

θ = $\tan^{-1}$ (tan (π/4 + α/2))

= π – (π/4 + α/2) [since, θ lies in 1st quadrant and tan's period is π]

= α/2 - 3π/4

∴ Polar form is Z = -√2 (cos (α/2) – sin (α/2)) (cos (α/2 - 3π/4) + i sin (α/2 - 3π/4))

(iv) (1 - i) / (cos π/3 + i sin π/3)

Given: Z = (1 - i) / (cos π/3 + i sin π/3)

Let us multiply and divide by (1 - i√3), we get

$$Z = \frac{1-i}{\frac{1}{2}+\frac{i\sqrt{3}}{2}}$$
$$= 2 \times \frac{1-i}{1+i\sqrt{3}} \times \frac{1-i\sqrt{3}}{1-i\sqrt{3}}$$
$$= 2 \times \frac{1+i^2\sqrt{3}-i(1+\sqrt{3})}{1-i^2 3}$$
$$= 2 \times \frac{(1+(-\sqrt{3})-i(1+\sqrt{3}))}{1-(-3)}$$
$$= 2 \times \frac{(1-\sqrt{3})-i(1+\sqrt{3})}{4}$$
$$= \frac{(1-\sqrt{3})-i(1+\sqrt{3})}{2}$$

We know that the polar form of a complex number Z = x + iy is given by Z = |Z| (cos θ + i sin θ)

Where,

|Z| = modulus of complex number = $\sqrt{(x^2 + y^2)}$

θ = arg (z) = argument of complex number = $\tan^{-1}$ (|y| / |x|)

Now,

$$|z| = \sqrt{\left(\frac{1-\sqrt{3}}{2}\right)^2 + \left(\frac{-1-\sqrt{3}}{2}\right)^2}$$
$$= \sqrt{\frac{1+3-2\sqrt{3}+1+2+2\sqrt{3}}{4}}$$
$$= \sqrt{\frac{8}{4}}$$
$$= \sqrt{2}$$

$$\theta = \tan^{-1}\left(\left|\frac{\frac{1+\sqrt{3}}{2}}{\frac{1-\sqrt{3}}{2}}\right|\right)$$

$$= \tan^{-1}\left(\left|\frac{1+\sqrt{3}}{1-\sqrt{3}}\right|\right)$$

$$= \tan^{-1}\left(\left|\frac{(1+\sqrt{3})(1+\sqrt{3})}{(1-\sqrt{3})(1+\sqrt{3})}\right|\right)$$

$$= \tan^{-1}\left(\left|\frac{1+3+2\sqrt{3}}{1-3}\right|\right)$$

$$= \tan^{-1}\left(\frac{4+2\sqrt{3}}{2}\right)$$

Since x < 0, y < 0 complex number lies in 3rd quadrant and the value of θ is $180^0 \le \theta \le -90^0$.

= $\tan^{-1}(2 + \sqrt{3})$

= -7π/12

Z = √2 (cos (-7π/12) + i sin (-7π/12))

= √2 (cos (7π/12) - i sin (7π/12))

∴ Polar form of (1 - i) / (cos π/3 + i sin π/3) is √2 (cos (7π/12) - i sin (7π/12))

4. If z_1 and z_2 are two complex number such that $|z_1| = |z_2|$ and arg (z_1) + arg (z_2) = π, then show that $z_1 = -\bar{z_2}$

Solution:

Given:

$|z_1| = |z_2|$ and arg (z_1) + arg (z_2) = π

Let us assume arg (z_1) = θ

arg (z_2) = π - θ

We know that in the polar form, z = |z| (cos θ + i sin θ)

$z_1 = |z_1|$ (cos θ + i sin θ) …………. (i)

$z_2 = |z_2|$ (cos (π - θ) + i sin (π - θ))

= $|z_2|$ (-cos θ + i sin θ)

= - $|z_2|$ (cos θ – i sin θ)

Now let us find the conjugate of

$\bar{z_2}$ = - $|z_2|$ (cos θ + i sin θ) …… (ii) (since, $|\bar{z_2}| = |z_2|$)

Now,

$z_1 / \bar{z_2}$ = [$|z_1|$ (cos θ + i sin θ)] / [-$|z_2|$ (cos θ + i sin θ)]

= - $|z_1|$ / $|z_2|$ [since, $|z_1| = |z_2|$]

= -1

When we cross multiply we get,

$z_1 = -\bar{z_2}$

Hence proved.

5. If z_1, z_2 and z_3, z_4 are two pairs of conjugate complex numbers, prove that arg (z_1/z_4) + arg (z_2/z_3) = 0

Solution:

Given:

$z_1 = \bar{z}_2$

$z_3 = \bar{z}_4$

We know that arg (z_1/z_2) = arg (z_1) – arg (z_2)

So,

arg (z_1/z_4) + arg (z_2/z_3) = arg (z_1) – arg (z_4) + arg (z_2) – arg (z_3)

= arg $(\bar{z}_2)$ – arg (z_4) + arg (z_2) – arg $(\bar{z}_4)$

= [arg (z_2) + arg $(\bar{z}_2)$] – [arg (z_4) + arg $(\bar{z}_4)$]

= 0 – 0 [since, arg (z) + arg $(\bar{z})$ = 0]

= 0

Hence proved.

6. Express sin π/5 + i (1 – cos π/5) in polar form.

Solution:

Given:

Z = sin π/5 + i (1 – cos π/5)

By using the formula,

sin 2θ = 2 sin θ cos θ

1- cos 2θ = 2 $\sin^2$ θ

So,

Z = 2 sin π/10 cos π/10 + i (2 $\sin^2$ π/10)

= 2 sin π/10 (cos π/10 + i sin π/10)

∴ The polar form of sin π/5 + i (1 – cos π/5) is 2 sin π/10 (cos π/10 + i sin π/10)