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1. If 1/a, 1/b, 1/c are in A.P., prove that:
(i) (b+c)/a, (c+a)/b, (a+b)/c are in A.P.
(ii) a(b + c), b(c + a), c(a + b) are in A.P.
Solution:
(i) (b+c)/a, (c+a)/b, (a+b)/c are in A.P.
We know that, if a, b, c are in AP, then b - a = c – b
If, 1/a, 1/b, 1/c are in AP
Then, 1/b – 1/a = 1/c – 1/b
If (b+c)/a, (c+a)/b, (a+b)/c are in AP
Then, (c+a)/b – (b+c)/a = (a+b)/c – (c+a)/b
Let us take LCM

$$\frac{ca + a^2 - b^2 - cb}{ab} = \frac{ab + b^2 - c^2 - ac}{bc}$$

Now let us consider LHS:

$$\frac{ca + a^2 - b^2 - cb}{ab}$$

Multiply both numerator and denominator by 'c', we get,

$$\frac{ca + a^2 - b^2 - cb}{ab} = \frac{c^2a + ca^2 - cb^2 - c^2b}{abc}$$

$$= \frac{C(b-a)(a+b+c)}{abc}$$

Now let us consider RHS:

$$\frac{ab + b^2 - c^2 - ac}{bc}$$

Multiply both numerator and denominator by 'a', we get,

$$\frac{ab + b^2 - c^2 - ac}{bc} = \frac{a^2b + ab^2 - ac^2 - a^2c}{abc}$$

$$= \frac{a(b-c)(a+b+c)}{abc}$$

LHS = RHS

$$\frac{C(b-a)(a+b+c)}{abc} = \frac{a(b-c)(a+b+c)}{abc}$$

Since, 1/a, 1/b, 1/c are in AP
1/b – 1/a = 1/c – 1/b
C (b - a) = a (b-c)
Hence, the given terms are in AP.

(ii) a(b + c), b(c + a), c(a + b) are in A.P.
We know that if, b(c + a) – a(b+c) = c(a+b) – b(c+a)
Consider LHS:
b(c + a) – a(b+c)
Upon simplification we get,
b(c + a) – a(b+c) = bc + ba – ab – ac
= c (b-a)
Now, taking RHS
c(a+b) – b(c+a) = ca + cb – bc – ba
= a (c-b)
We know,
1/a, 1/b, 1/c are in AP
So, 1/a – 1/b = 1/b – 1/c
Or c(b-a) = a(c-b)
Hence, given terms are in AP.

2. If a^2, b^2, c^2 are in AP., prove that a/(b+c), b/(c+a), c/(a+b) are in AP.
Solution:
If a^2, b^2, c^2 are in AP then, $b^2 - a^2 = c^2 - b^2$
If a/(b+c), b/(c+a), c/(a+b) are in AP then,
b/(c+a) – a/(b+c) = c/(a+b) – b/(c+a)
Let us take LCM on both the sides we get,

$$\frac{b^2 + bc - a^2 - ac}{(a + c)(b + c)} = \frac{ca + c^2 - b^2 - ab}{(a + b)(b + c)}$$

$$\frac{(b - a)(a + b + c)}{(a + c)(b + c)} = \frac{(c - b)(a + b + c)}{(a + b)(b + c)}$$

Since, $b^2 - a^2 = c^2 - b^2$
Substituting $b^2 - a^2 = c^2 - b^2$ in above, we get
LHS = RHS
Hence, given terms are in AP

3. If a, b, c are in A.P., then show that:
(i) $a^2(b + c)$, $b^2(c + a)$, $c^2(a + b)$ are also in A.P.
(ii) b + c - a, c + a - b, a + b - c are in A.P.
(iii) bc – a^2, ca – b^2, ab – c^2 are in A.P.
Solution:

(i) $a^2(b + c)$, $b^2(c + a)$, $c^2(a + b)$ are also in A.P.
If $b^2(c + a) - a^2(b + c) = c^2(a + b) - b^2(c + a)$
$b^2c + b^2a – a^2b – a^2c = c^2a + c^2b – b^2a – b^2c$
Given, b - a = c - b
And since a, b, c are in AP,
$c(b^2 - a^2) + ab(b - a) = a(c^2 - b^2) + bc(c - b)$
(b - a) (ab + bc + ca) = (c - b) (ab + bc + ca)
Upon cancelling, ab + bc + ca from both sides
b - a = c – b
2b = c + a [which is true]
Hence, given terms are in AP

(ii) b + c - a, c + a - b, a + b - c are in A.P.
If (c + a - b) - (b + c - a) = (a + b - c) - (c + a - b)
Then, b + c - a, c + a - b, a + b - c are in A.P.
Let us consider LHS and RHS
(c + a - b) - (b + c - a) = (a + b - c) - (c + a - b)
2a - 2b = 2b - 2c
b - a = c - b
And since a, b, c are in AP,
b - a = c - b
Hence, given terms are in AP.

(iii) $bc – a^2$, $ca – b^2$, $ab – c^2$ are in A.P.
If $(ca - b^2) – (bc - a^2) = (ab - c^2) – (ca - b^2)$
Then, $bc – a^2$, $ca – b^2$, $ab – c^2$ are in A.P.
Let us consider LHS and RHS
$(ca - b^2) – (bc - a^2) = (ab - c^2) – (ca - b^2)$
$(a – b^2 – bc + a^2) = (ab – c^2 – ca + b^2)$
(a - b) (a + b + c) = (b - c) (a + b + c)
a - b = b - c
And since a, b, c are in AP,
b - c = a - b
Hence, given terms are in AP

4. If (b+c)/a, (c+a)/b, (a+b)/c are in AP., prove that:
(i) 1/a, 1/b, 1/c are in AP
(ii) bc, ca, ab are in AP
Solution:

(i) 1/a, 1/b, 1/c are in AP
If 1/a, 1/b, 1/c are in AP then,
1/b – 1/a = 1/c – 1/b
Let us consider LHS:
1/b – 1/a = (a-b)/ab
= c(a-b)/abc [by multiplying with 'c' on both the numerator and denominator]

Let us consider RHS:
1/c – 1/b = (b-c)/bc
= a(b-c)/bc [by multiplying with 'a' on both the numerator and denominator]
Since, (b+c)/a, (c+a)/b, (a+b)/c are in AP

$$\frac{c+a}{b}-\frac{b+c}{a}=\frac{a+b}{c}-\frac{c+a}{b}$$

$$\frac{b^2+bc-a^2-ac}{ab}=\frac{ca+c^2-b^2-ab}{bc}$$

$$\frac{(b-a)(a+b+c)}{ab}=\frac{(c-b)(a+b+c)}{bc}$$

$$\frac{a(b-c)}{abc}=\frac{c(a-b)}{abc}$$

a(b - c) = c(a - b)
LHS = RHS
Hence, the given terms are in AP

(ii) bc, ca, ab are in AP
If bc, ca, ab are in AP then,
ca – bc = ab – ca
c (a-b) = a (b-c)

If 1/a, 1/b, 1/c are in AP then,
1/b – 1/a = 1/c – 1/b
c (a-b) = a (b-c)
Hence, the given terms are in AP

5. If a, b, c are in A.P., prove that:
(i) $(a - c)^2 = 4 (a - b) (b - c)$
(ii) $a^2 + c^2 + 4ac = 2 (ab + bc + ca)$
(iii) $a^3 + c^3 + 6abc = 8b^3$

Solution:

(i) $(a - c)^2 = 4 (a - b) (b - c)$

Let us expand the above expression

$a^2 + c^2 - 2ac = 4(ab – ac – b^2 + bc)$

$a^2 + 4c^2b^2 + 2ac - 4ab - 4bc = 0$

$(a + c - 2b)^2 = 0$

$a + c - 2b = 0$

Since a, b, c are in AP

$b - a = c – b$

$a + c - 2b = 0$

$a + c = 2b$

Hence, $(a - c)^2 = 4 (a - b) (b - c)$

(ii) $a^2 + c^2 + 4ac = 2 (ab + bc + ca)$

Let us expand the above expression

$a^2 + c^2 + 4ac = 2 (ab + bc + ca)$

$a^2 + c^2 + 2ac - 2ab - 2bc = 0$

$(a + c - b)^2 – b^2 = 0$

$a + c - b = b$

$a + c - 2b = 0$

$2b = a+c$

$b = (a+c)/2$

Since a, b, c are in AP

$b - a = c - b$

$b = (a+c)/2$

Hence, $a^2 + c^2 + 4ac = 2 (ab + bc + ca)$

(iii) $a^3 + c^3 + 6abc = 8b^3$

Let us expand the above expression

$a^3 + c^3 + 6abc = 8b^3$

$a^3 + c^3 - (2b)^3 + 6abc = 0$

$a^3 + (-2b)^3 + c^3 + 3a(-2b)c = 0$

Since, if $a + b + c = 0$, $a^3 + b^3 + c^3 = 3abc$

$(a - 2b + c)^3 = 0$

$a - 2b + c = 0$

$a + c = 2b$

$b = (a+c)/2$

Since a, b, c are in AP

$a - b = c - b$

b = (a+c)/2
Hence, $a^3 + c^3 + 6abc = 8b^3$

6. If a(1/b + 1/c), b(1/c + 1/a), c(1/a + 1/b) are in AP., prove that a, b, c are in AP.
Solution:
Here, we know a(1/b + 1/c), b(1/c + 1/a), c(1/a + 1/b) are in AP
Also, a(1/b + 1/c) + 1, b(1/c + 1/a) + 1, c(1/a + 1/b) + 1 are in AP
Let us take LCM for each expression then we get,
(ac+ab+bc)/bc , (ab+bc+ac)/ac, (cb+ac+ab)/ab are in AP
1/bc, 1/ac, 1/ab are in AP
Let us multiply numerator with 'abc', we get
abc/bc, abc/ac, abc/ab are in AP
∴ a, b, c are in AP.
Hence proved.

7. Show that $x^2 + xy + y^2$, $z^2 + zx + x^2$ and $y^2 + yz + z^2$ are in consecutive terms of an A.P., if x, y and z are in A.P.
Solution:
x, y, z are in AP
Given, $x^2 + xy + y^2$, $z^2 + zx + x^2$ and $y^2 + yz + z^2$ are in AP
$(z^2 + zx + x^2) - (x^2 + xy + y^2) = (y^2 + yz + z^2) - (z^2 + zx + x^2)$
Let d = common difference,
So, Y = x + d and z = x + 2d
Let us consider the LHS:
$= (z^2 + zx + x^2) - (x^2 + xy + y^2)$
$= z^2 + zx – xy – y^2$
$= (x + 2d)^2 + (x + 2d)x - x(x + d) - (x + d)^2$
$= x^2 + 4xd + 4d^2 + x^2 + 2xd – x^2 – xd – x^2 - 2xd – d^2$
$= 3xd + 3d^2$
Now, let us consider RHS:
$= (y^2 + yz + z^2) - (z^2 + zx + x^2)$
$= y^2 + yz – zx – x^2$
$= (x + d)^2 + (x + d)(x + 2d) - (x + 2d)x – x^2$
$= x^2 + 2dx + d^2 + x^2 + 2dx + xd + 2d^2 – x^2 - 2dx – x^2$
$= 3xd + 3d^2$
LHS = RHS
∴ $x^2 + xy + y^2$, $z^2 + zx + x^2$ and $y^2 + yz + z^2$ are in consecutive terms of A.P
Hence proved.