

### Exercise 1.3

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**Short Answer (S.A.)**

**1. Let  $A = \{a, b, c\}$  and the relation  $R$  be defined on  $A$  as follows:**

$$R = \{(a, a), (b, c), (a, b)\}.$$

**Then, write minimum number of ordered pairs to be added in  $R$  to make  $R$  reflexive and transitive.**

**Solution:**

Given relation,  $R = \{(a, a), (b, c), (a, b)\}$

To make  $R$  as reflexive we should add  $(b, b)$  and  $(c, c)$  to  $R$ . Also, to make  $R$  as transitive we should add  $(a, c)$  to  $R$ .

Hence, the minimum number of ordered pairs to be added are  $(b, b)$ ,  $(c, c)$  and  $(a, c)$  i.e. 3.

**2. Let  $D$  be the domain of the real valued function  $f$  defined by  $f(x) = \sqrt{25 - x^2}$ . Then, write  $D$ .**

**Solution:**

Given,  $f(x) = \sqrt{25 - x^2}$

The function is defined if  $25 - x^2 \geq 0$

$$\text{So, } x^2 \leq 25$$

$$-5 \leq x \leq 5$$

Therefore, the domain of the given function is  $[-5, 5]$

**3. Let  $f, g: \mathbb{R} \rightarrow \mathbb{R}$  be defined by  $f(x) = 2x + 1$  and  $g(x) = x^2 - 2, \forall x \in \mathbb{R}$ , respectively. Then, find  $g \circ f$ .**

**Solution:**

Given,

$$f(x) = 2x + 1 \text{ and } g(x) = x^2 - 2, \forall x \in \mathbb{R}$$

Thus,  $g \circ f = g(f(x))$

$$= g(2x + 1)$$

$$= (2x + 1)^2 - 2$$

$$= 4x^2 + 4x + 1 - 2$$

$$= 4x^2 + 4x - 1$$

**4. Let  $f: \mathbb{R} \rightarrow \mathbb{R}$  be the function defined by  $f(x) = 2x - 3, \forall x \in \mathbb{R}$ . write  $f^{-1}$ .**

**Solution:**

Given function,

$$f(x) = 2x - 3, \forall x \in \mathbb{R}$$

$$\text{Let } y = 2x - 3$$

$$x = (y + 3)/2$$

Thus,

$$f^{-1}(x) = (x + 3)/2$$

**5. If  $A = \{a, b, c, d\}$  and the function  $f = \{(a, b), (b, d), (c, a), (d, c)\}$ , write  $f^{-1}$ .**

**Solution:**

Given,

$$A = \{a, b, c, d\} \text{ and } f = \{(a, b), (b, d), (c, a), (d, c)\}$$

So,

$$f^{-1} = \{(b, a), (d, b), (a, c), (c, d)\}$$

**6. If  $f: \mathbb{R} \rightarrow \mathbb{R}$  is defined by  $f(x) = x^2 - 3x + 2$ , write  $f(f(x))$ .**

**Solution:**

Given,  $f(x) = x^2 - 3x + 2$

Then,

$$\begin{aligned} f(f(x)) &= f(x^2 - 3x + 2) \\ &= (x^2 - 3x + 2)^2 - 3(x^2 - 3x + 2) + 2, \\ &= x^4 + 9x^2 + 4 - 6x^3 + 4x^2 - 12x - 3x^2 + 9x - 6 + 2 \\ &= x^4 - 6x^3 + 10x^2 - 3x \end{aligned}$$

Thus,

$$f(f(x)) = x^4 - 6x^3 + 10x^2 - 3x$$

**7. Is  $g = \{(1, 1), (2, 3), (3, 5), (4, 7)\}$  a function? If  $g$  is described by  $g(x) = \alpha x + \beta$ , then what value should be assigned to  $\alpha$  and  $\beta$ .**

**Solution:**

Given,  $g = \{(1, 1), (2, 3), (3, 5), (4, 7)\}$

It's seen that every element of domain has a unique image. So,  $g$  is function.

Now, also given that  $g(x) = \alpha x + \beta$

So, we have

$$g(1) = \alpha(1) + \beta = 1$$

$$\alpha + \beta = 1 \dots\dots\dots (i)$$

And,  $g(2) = \alpha(2) + \beta = 3$

$$2\alpha + \beta = 3 \dots\dots\dots (ii)$$

Solving (i) and (ii), we have

$$\alpha = 2 \text{ and } \beta = -1$$

Therefore,  $g(x) = 2x - 1$

**8. Are the following set of ordered pairs functions? If so, examine whether the mapping is injective or surjective.**

(i)  $\{(x, y): x \text{ is a person, } y \text{ is the mother of } x\}$ .

(ii)  $\{(a, b): a \text{ is a person, } b \text{ is an ancestor of } a\}$ .

**Solution:**

(i) Given,  $\{(x, y): x \text{ is a person, } y \text{ is the mother of } x\}$

It's clearly seen that each person 'x' has only one biological mother.

Hence, the above set of ordered pairs make a function.

Now more than one person may have same mother. Thus, the function is many-many one and surjective.

(ii) Given,  $\{(a, b): a \text{ is a person, } b \text{ is an ancestor of } a\}$   
 It's clearly seen that any person 'a' has more than one ancestors.  
 Thus, it does not represent a function.

**9. If the mappings  $f$  and  $g$  are given by  $f = \{(1, 2), (3, 5), (4, 1)\}$  and  $g = \{(2, 3), (5, 1), (1, 3)\}$ , write  $f \circ g$ .**

**Solution:**

Given,  
 $f = \{(1, 2), (3, 5), (4, 1)\}$  and  $g = \{(2, 3), (5, 1), (1, 3)\}$   
 Now,  
 $f \circ g(2) = f(g(2)) = f(3) = 5$   
 $f \circ g(5) = f(g(5)) = f(1) = 2$   
 $f \circ g(1) = f(g(1)) = f(3) = 5$   
 Thus,  
 $f \circ g = \{(2, 5), (5, 2), (1, 5)\}$

**10. Let  $C$  be the set of complex numbers. Prove that the mapping  $f: C \rightarrow R$  given by  $f(z) = |z|, \forall z \in C$ , is neither one-one nor onto.**

**Solution:**

Given,  $f: C \rightarrow R$  such that  $f(z) = |z|, \forall z \in C$   
 Now, let take  $z = 6 + 8i$   
 Then,  
 $f(6 + 8i) = |6 + 8i| = \sqrt{6^2 + 8^2} = \sqrt{100} = 10$   
 And, for  $z = 6 - 8i$   
 $f(6 - 8i) = |6 - 8i| = \sqrt{6^2 + 8^2} = \sqrt{100} = 10$   
 Hence,  $f(z)$  is many-one.  
 Also,  $|z| \geq 0, \forall z \in C$   
 But the co-domain given is ' $R$ '  
 Therefore,  $f(z)$  is not onto.

**11. Let the function  $f: R \rightarrow R$  be defined by  $f(x) = \cos x, \forall x \in R$ . Show that  $f$  is neither one-one nor onto.**

**Solution:**

We have,  
 $f: R \rightarrow R, f(x) = \cos x$   
 Now,  
 $f(x_1) = f(x_2)$   
 $\cos x_1 = \cos x_2$   
 $x_1 = 2n\pi \pm x_2, n \in Z$   
 It's seen that the above equation has infinite solutions for  $x_1$  and  $x_2$   
 Hence,  $f(x)$  is many one function.  
 Also the range of  $\cos x$  is  $[-1, 1]$ , which is subset of given co-domain  $R$ .

Therefore, the given function is not onto.

**12. Let  $X = \{1, 2, 3\}$  and  $Y = \{4, 5\}$ . Find whether the following subsets of  $X \times Y$  are functions from  $X$  to  $Y$  or not.**

(i)  $f = \{(1, 4), (1, 5), (2, 4), (3, 5)\}$       (ii)  $g = \{(1, 4), (2, 4), (3, 4)\}$

(iii)  $h = \{(1, 4), (2, 5), (3, 5)\}$       (iv)  $k = \{(1, 4), (2, 5)\}$ .

**Solution:**

Given,  $X = \{1, 2, 3\}$  and  $Y = \{4, 5\}$

So,  $X \times Y = \{(1, 4), (1, 5), (2, 4), (2, 5), (3, 4), (3, 5)\}$

(i)  $f = \{(1, 4), (1, 5), (2, 4), (3, 5)\}$

$f$  is not a function as  $f(1) = 4$  and  $f(1) = 5$

Hence, pre-image '1' has not unique image.

(ii)  $g = \{(1, 4), (2, 4), (3, 4)\}$

It's seen clearly that  $g$  is a function in which each element of the domain has unique image.

(iii)  $h = \{(1, 4), (2, 5), (3, 5)\}$

It's seen clearly that  $h$  is a function as each pre-image with a unique image.

And, function  $h$  is many-one as  $h(2) = h(3) = 5$

(iv)  $k = \{(1, 4), (2, 5)\}$

Function  $k$  is not a function as '3' has not any image under the mapping.

**13. If functions  $f: A \rightarrow B$  and  $g: B \rightarrow A$  satisfy  $g \circ f = I_A$ , then show that  $f$  is one-one and  $g$  is onto.**

**Solution:**

Given,

$f: A \rightarrow B$  and  $g: B \rightarrow A$  satisfy  $g \circ f = I_A$

It's clearly seen that function 'g' is inverse of 'f'.

So, 'f' has to be one-one and onto.

Hence, 'g' is also one-one and onto.

**14. Let  $f: \mathbb{R} \rightarrow \mathbb{R}$  be the function defined by  $f(x) = 1/(2 - \cos x) \forall x \in \mathbb{R}$ . Then, find the range of  $f$ .**

**Solution:**

Given,

$f(x) = 1/(2 - \cos x) \forall x \in \mathbb{R}$

Let  $y = 1/(2 - \cos x)$

$2y - y \cos x = 1$

$\cos x = (2y - 1)/y$

$\cos x = 2 - 1/y$

Now, we know that  $-1 \leq \cos x \leq 1$

So,

$-1 \leq 2 - 1/y \leq 1$

$-3 \leq -1/y \leq -1$

$$1 \leq -1/y \leq 3$$

$$1/3 \leq y \leq 1$$

Thus, the range of the given function is  $[1/3, 1]$ .

**15. Let  $n$  be a fixed positive integer. Define a relation  $R$  in  $Z$  as follows:  $\forall a, b \in Z$ ,  $aRb$  if and only if  $a - b$  is divisible by  $n$ . Show that  $R$  is an equivalence relation.**

**Solution:**

Given  $\forall a, b \in Z$ ,  $aRb$  if and only if  $a - b$  is divisible by  $n$ .

Now, for

$aRa \Rightarrow (a - a)$  is divisible by  $n$ , which is true for any integer  $a$  as '0' is divisible by  $n$ .

Thus,  $R$  is reflexive.

Now,  $aRb$

So,  $(a - b)$  is divisible by  $n$ .

$\Rightarrow -(b - a)$  is divisible by  $n$ .

$\Rightarrow (b - a)$  is divisible by  $n$

$\Rightarrow bRa$

Thus,  $R$  is symmetric.

Let  $aRb$  and  $bRc$

Then,  $(a - b)$  is divisible by  $n$  and  $(b - c)$  is divisible by  $n$ .

So,  $(a - b) + (b - c)$  is divisible by  $n$ .

$\Rightarrow (a - c)$  is divisible by  $n$ .

$\Rightarrow aRc$

Thus,  $R$  is transitive.

So,  $R$  is an equivalence relation.

**Long Answer (L.A.)**

**16. If  $A = \{1, 2, 3, 4\}$ , define relations on  $A$  which have properties of being:**

**(a) reflexive, transitive but not symmetric**

**(b) symmetric but neither reflexive nor transitive**

**(c) reflexive, symmetric and transitive.**

**Solution:**

Given that,  $A = \{1, 2, 3\}$ .

(i) Let  $R_1 = \{(1, 1), (1, 2), (1, 3), (2, 3), (2, 2), (1, 3), (3, 3)\}$

$R_1$  is reflexive as  $(1, 1)$ ,  $(2, 2)$  and  $(3, 3)$  lie in  $R_1$ .

$R_1$  is transitive as  $(1, 2) \in R_1$ ,  $(2, 3) \in R_1 \Rightarrow (1, 3) \in R_1$

Now,  $(1, 2) \in R_1 \Rightarrow (2, 1) \notin R_1$ .

(ii) Let  $R_2 = \{(1, 2), (2, 1)\}$

Now,  $(1, 2) \in R_2$ ,  $(2, 1) \in R_2$

So, it is symmetric,

And, clearly  $R_2$  is not reflexive as  $(1, 1) \notin R_2$

Also,  $R_2$  is not transitive as  $(1, 2) \in R_2$ ,  $(2, 1) \in R_2$  but  $(1, 1) \notin R_2$

(iii) Let  $R_3 = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$

$R_3$  is reflexive as  $(1, 1)$   $(2, 2)$  and  $(3, 3) \in R_1$

$R_3$  is symmetric as  $(1, 2), (1, 3), (2, 3) \in R_1 \Rightarrow (2, 1), (3, 1), (3, 2) \in R_1$

Therefore,  $R_3$  is reflexive, symmetric and transitive.

**17. Let  $R$  be relation defined on the set of natural number  $N$  as follows:**

**$R = \{(x, y): x \in N, y \in N, 2x + y = 41\}$ . Find the domain and range of the relation  $R$ . Also verify whether  $R$  is reflexive, symmetric and transitive.**

**Solution:**

Given function:  $R = \{(x, y): x \in N, y \in N, 2x + y = 41\}$ .

So, the domain =  $\{1, 2, 3, \dots, 20\}$  [Since,  $y \in N$ ]

Finding the range, we have

$R = \{(1, 39), (2, 37), (3, 35), \dots, (19, 3), (20, 1)\}$

Thus, Range of the function =  $\{1, 3, 5, \dots, 39\}$

$R$  is not reflexive as  $(2, 2) \notin R$  as  $2 \times 2 + 2 \neq 41$

Also,  $R$  is not symmetric as  $(1, 39) \in R$  but  $(39, 1) \notin R$

Further  $R$  is not transitive as  $(11, 19) \notin R, (19, 3) \notin R$ ; but  $(11, 3) \notin R$ .

Thus,  $R$  is neither reflexive nor symmetric and nor transitive.

**18. Given  $A = \{2, 3, 4\}, B = \{2, 5, 6, 7\}$ . Construct an example of each of the following:**

**(a) an injective mapping from  $A$  to  $B$**

**(b) a mapping from  $A$  to  $B$  which is not injective**

**(c) a mapping from  $B$  to  $A$ .**

**Solution:**

Given,  $A = \{2, 3, 4\}, B = \{2, 5, 6, 7\}$

(i) Let  $f: A \rightarrow B$  denote a mapping

$f = \{(x, y): y = x + 3\}$  or

$f = \{(2, 5), (3, 6), (4, 7)\}$ , which is an injective mapping.

(ii) Let  $g: A \rightarrow B$  denote a mapping such that  $g = \{(2, 2), (3, 2), (4, 5)\}$ , which is not an injective mapping.

(iii) Let  $h: B \rightarrow A$  denote a mapping such that  $h = \{(2, 2), (5, 3), (6, 4), (7, 4)\}$ , which is one of the mapping from  $B$  to  $A$ .

**19. Give an example of a map**

**(i) which is one-one but not onto**

**(ii) which is not one-one but onto**

**(iii) which is neither one-one nor onto.**

**Solution:**

(i) Let  $f: N \rightarrow N$ , be a mapping defined by  $f(x) = x^2$

For  $f(x_1) = f(x_2)$

Then,  $x_1^2 = x_2^2$

$x_1 = x_2$  (Since  $x_1 + x_2 = 0$  is not possible)

Further ' $f$ ' is not onto, as for  $1 \in N$ , there does not exist any  $x$  in  $N$  such that  $f(x) = 2x + 1$ .

(ii) Let  $f: \mathbb{R} \rightarrow [0, \infty)$ , be a mapping defined by  $f(x) = |x|$   
 Then, it's clearly seen that  $f(x)$  is not one-one as  $f(2) = f(-2)$ .  
 But  $|x| \geq 0$ , so range is  $[0, \infty]$ .  
 Therefore,  $f(x)$  is onto.

(iii) Let  $f: \mathbb{R} \rightarrow \mathbb{R}$ , be a mapping defined by  $f(x) = x^2$   
 Then clearly  $f(x)$  is not one-one as  $f(1) = f(-1)$ . Also range of  $f(x)$  is  $[0, \infty)$ .  
 Therefore,  $f(x)$  is neither one-one nor onto.

**20. Let  $A = \mathbb{R} - \{3\}$ ,  $B = \mathbb{R} - \{1\}$ . Let  $f: A \rightarrow B$  be defined by  $f(x) = x - 2/x - 3 \forall x \in A$ . Then show that  $f$  is bijective.**

**Solution:**

Given,

$$A = \mathbb{R} - \{3\}, B = \mathbb{R} - \{1\}$$

And,

$$f: A \rightarrow B \text{ be defined by } f(x) = x - 2/x - 3 \forall x \in A$$

$$\text{Hence, } f(x) = (x - 3 + 1)/(x - 3) = 1 + 1/(x - 3)$$

$$\text{Let } f(x_1) = f(x_2)$$

$$1 + \frac{1}{x_1 - 3} = 1 + \frac{1}{x_2 - 3}$$

$$\frac{1}{x_1 - 3} = \frac{1}{x_2 - 3}$$

$$x_1 = x_2$$

So,  $f(x)$  is an injective function.

$$\text{Now let } y = (x - 2)/(x - 3)$$

$$x - 2 = xy - 3y$$

$$x(1 - y) = 2 - 3y$$

$$x = (3y - 2)/(y - 1)$$

$$y \in \mathbb{R} - \{1\} = B$$

Thus,  $f(x)$  is onto or surjective.

Therefore,  $f(x)$  is a bijective function.

**21. Let  $A = [-1, 1]$ . Then, discuss whether the following functions defined on  $A$  are one-one, onto or bijective:**

(i)  $f(x) = x/2$                       (ii)  $g(x) = |x|$

(iii)  $h(x) = x|x|$                       (iv)  $k(x) = x^2$

**Solution:**

$$\text{Given, } A = [-1, 1]$$

$$(i) f: [-1, 1] \rightarrow [-1, 1], f(x) = x/2$$

$$\text{Let } f(x_1) = f(x_2)$$



$$x_1/2 = x_2$$

So,  $f(x)$  is one-one.

Also  $x \in [-1, 1]$

$$x/2 = f(x) \in [-1/2, 1/2]$$

Hence, the range is a subset of co-domain 'A'

So,  $f(x)$  is not onto.

Therefore,  $f(x)$  is not bijective.

(ii)  $g(x) = |x|$

Let  $g(x_1) = g(x_2)$

$$|x_1| = |x_2|$$

$$x_1 = \pm x_2$$

So,  $g(x)$  is not one-one

Also  $g(x) = |x| \geq 0$ , for all real  $x$

Hence, the range is  $[0, 1]$ , which is subset of co-domain 'A'

So,  $f(x)$  is not onto.

Therefore,  $f(x)$  is not bijective.

(iii)  $h(x) = x|x|$

Let  $h(x_1) = h(x_2)$

$$x_1|x_1| = x_2|x_2|$$

If  $x_1, x_2 > 0$

$$x_1^2 = x_2^2$$

$$x_1^2 - x_2^2 = 0$$

$$(x_1 - x_2)(x_1 + x_2) = 0$$

$$x_1 = x_2 \text{ (as } x_1 + x_2 \neq 0 \text{)}$$

Similarly for  $x_1, x_2 < 0$ , we have  $x_1 = x_2$

It's clearly seen that for  $x_1$  and  $x_2$  of opposite sign,  $x_1 \neq x_2$ .

Hence,  $f(x)$  is one-one.

For  $x \in [0, 1]$ ,  $f(x) = x^2 \in [0, 1]$

For  $x < 0$ ,  $f(x) = -x^2 \in [-1, 0)$

Hence, the range is  $[-1, 1]$ .

So,  $h(x)$  is onto.

Therefore,  $h(x)$  is bijective.

(iv)  $k(x) = x^2$

Let  $k(x_1) = k(x_2)$

$$x_1^2 = x_2^2$$

$$x_1 = \pm x_2$$

Therefore,  $k(x)$  is not one-one.

**22. Each of the following defines a relation on  $\mathbb{N}$ :**

(i)  $x$  is greater than  $y$ ,  $x, y \in \mathbb{N}$

(ii)  $x + y = 10$ ,  $x, y \in \mathbb{N}$

(iii)  $x$  is square of an integer  $x, y \in \mathbb{N}$

(iv)  $x + 4y = 10$ ,  $x, y \in \mathbb{N}$ .



**Determine which of the above relations are reflexive, symmetric and transitive.**

**Solution:**

(i) Given,  $x$  is greater than  $y$ ;  $x, y \in \mathbb{N}$

If  $(x, x) \in R$ , then  $x > x$ , which is not true for any  $x \in \mathbb{N}$ .

Thus,  $R$  is not reflexive.

Let  $(x, y) \in R$

$\Rightarrow xRy$

$\Rightarrow x > y$

So,  $y > x$  is not true for any  $x, y \in \mathbb{N}$

Hence,  $R$  is not symmetric.

Let  $xRy$  and  $yRz$

$\Rightarrow x > y$  and  $y > z$

$\Rightarrow x > z$

$\Rightarrow xRz$

Hence,  $R$  is transitive.

(ii)  $x + y = 10$ ;  $x, y \in \mathbb{N}$

Thus,

$R = \{(x, y); x + y = 10, x, y \in \mathbb{N}\}$

$R = \{(1, 9), (2, 8), (3, 7), (4, 6), (5, 5), (6, 4), (7, 3), (8, 2), (9, 1)\}$

It's clear  $(1, 1) \notin R$

So,  $R$  is not reflexive.

$(x, y) \in R \Rightarrow (y, x) \in R$

Therefore,  $R$  is symmetric.

Now  $(1, 9) \in R$ ,  $(9, 1) \in R$ , but  $(1, 1) \notin R$

Therefore,  $R$  is not transitive.

(iii) Given,  $xy$  is square of an integer  $x, y \in \mathbb{N}$

$R = \{(x, y) : xy \text{ is a square of an integer } x, y \in \mathbb{N}\}$

It's clearly  $(x, x) \in R, \forall x \in \mathbb{N}$

As  $x^2$  is square of an integer for any  $x \in \mathbb{N}$

Thus,  $R$  is reflexive.

If  $(x, y) \in R \Rightarrow (y, x) \in R$

So,  $R$  is symmetric.

Now, if  $xy$  is square of an integer and  $yz$  is square of an integer.

Then, let  $xy = m^2$  and  $yz = n^2$  for some  $m, n \in \mathbb{Z}$

$x = m^2/y$  and  $z = x^2/y$

$xz = m^2n^2/y^2$ , which is square of an integer.

Thus,  $R$  is transitive.

(iv)  $x + 4y = 10$ ;  $x, y \in \mathbb{N}$

$R = \{(x, y); x + 4y = 10; x, y \in \mathbb{N}\}$

$R = \{(2, 2), (6, 1)\}$

It's clearly seen  $(1, 1) \notin R$

Hence,  $R$  is not symmetric.

$$(x, y) \in R \Rightarrow x + 4y = 10$$

$$\text{And } (y, z) \in R \Rightarrow y + 4z = 10$$

$$\Rightarrow x - 16z = -30$$

$$\Rightarrow (x, z) \notin R$$

Therefore,  $R$  is not transitive.

**23. Let  $A = \{1, 2, 3, \dots, 9\}$  and  $R$  be the relation in  $A \times A$  defined by  $(a, b) R (c, d)$  if  $a + d = b + c$  for  $(a, b), (c, d)$  in  $A \times A$ . Prove that  $R$  is an equivalence relation and also obtain the equivalent class  $[(2, 5)]$ .**

**Solution:**

Given,  $A = \{1, 2, 3, \dots, 9\}$  and  $(a, b) R (c, d)$  if  $a + d = b + c$  for  $(a, b), (c, d) \in A \times A$ .

Let  $(a, b) R (a, b)$

So,  $a + b = b + a$ ,  $\forall a, b \in A$  which is true for any  $a, b \in A$ .

Thus,  $R$  is reflexive.

Let  $(a, b) R (c, d)$

Then,

$$a + d = b + c$$

$$c + b = d + a$$

$$(c, d) R (a, b)$$

Thus,  $R$  is symmetric.

Let  $(a, b) R (c, d)$  and  $(c, d) R (e, f)$

$$a + d = b + c \text{ and } c + f = d + e$$

$$a + d = b + c \text{ and } d + e = c + f$$

$$(a + d) - (d + e) = (b + c) - (c + f)$$

$$a - e = b - f$$

$$a + f = b + e$$

$$(a, b) R (e, f)$$

So,  $R$  is transitive.

The equivalence class  $[(2, 5)] = \{(1, 4), (2, 5), (3, 6), (4, 7), (5, 8), (6, 9)\}$

Therefore,  $R$  is an equivalence relation.

**24. Using the definition, prove that the function  $f : A \rightarrow B$  is invertible if and only if  $f$  is both one-one and onto.**

**Solution:**

Let  $f : A \rightarrow B$  be many-one function.

Let  $f(a) = p$  and  $f(b) = p$

So, for inverse function we will have  $f^{-1}(p) = a$  and  $f^{-1}(p) = b$

Thus, in this case inverse function is not defined as we have two images 'a and b' for one pre-image 'p'. But for  $f$  to be invertible it must be one-one.

Now, let  $f : A \rightarrow B$  is not onto function.

Let  $B = \{p, q, r\}$  and range of  $f$  be  $\{p, q\}$ .

Here image 'r' has not any pre-image, which will have no image in set  $A$ .

And for  $f$  to be invertible it must be onto.

Thus, ' $f$ ' is invertible if and only if ' $f$ ' is both one-one and onto.

A function  $f = X \rightarrow Y$  is invertible iff  $f$  is a bijective function.

**25. Functions  $f, g : \mathbb{R} \rightarrow \mathbb{R}$  are defined, respectively, by  $f(x) = x^2 + 3x + 1$ ,  $g(x) = 2x - 3$ , find**

**(i)  $f \circ g$       (ii)  $g \circ f$       (iii)  $f \circ f$       (iv)  $g \circ g$**

**Solution:**

Given,  $f(x) = x^2 + 3x + 1$ ,  $g(x) = 2x - 3$

(i)  $f \circ g = f(g(x))$

$$= f(2x - 3)$$

$$= (2x - 3)^2 + 3(2x - 3) + 1$$

$$= 4x^2 + 9 - 12x + 6x - 9 + 1$$

$$= 4x^2 - 6x + 1$$

(ii)  $g \circ f = g(f(x))$

$$= g(x^2 + 3x + 1)$$

$$= 2(x^2 + 3x + 1) - 3$$

$$= 2x^2 + 6x - 1$$

(iii)  $f \circ f = f(f(x))$

$$= f(x^2 + 3x + 1)$$

$$= (x^2 + 3x + 1)^2 + 3(x^2 + 3x + 1) + 1$$

$$= x^4 + 9x^2 + 1 + 6x^3 + 6x + 2x^2 + 3x^2 + 9x + 3 + 1$$

$$= x^4 + 6x^3 + 14x^2 + 15x + 5$$

(iv)  $g \circ g = g(g(x))$

$$= g(2x - 3)$$

$$= 2(2x - 3) - 3$$

$$= 4x - 6 - 3$$

$$= 4x - 9$$

**26. Let  $*$  be the binary operation defined on  $\mathbb{Q}$ . Find which of the following binary operations are commutative**

**(i)  $a * b = a - b \forall a, b \in \mathbb{Q}$**

**(ii)  $a * b = a^2 + b^2 \forall a, b \in \mathbb{Q}$**

**(iii)  $a * b = a + ab \forall a, b \in \mathbb{Q}$**

**(iv)  $a * b = (a - b)^2 \forall a, b \in \mathbb{Q}$**

**Solution:**

Given that  $*$  is a binary operation defined on  $\mathbb{Q}$ .

(i)  $a * b = a - b, \forall a, b \in \mathbb{Q}$  and  $b * a = b - a$

So,  $a * b \neq b * a$

Thus,  $*$  is not commutative.

(ii)  $a * b = a^2 + b^2$

$$b * a = b^2 + a^2$$

Thus,  $*$  is commutative.

(iii)  $a * b = a + ab$

$$b * a = b + ab$$

So clearly,  $a + ab \neq b + ab$

Thus,  $*$  is not commutative.

(iv)  $a * b = (a - b)^2, \forall a, b \in \mathbb{Q}$

$$b * a = (b - a)^2$$

Since,  $(a - b)^2 = (b - a)^2$   
Thus,  $*$  is commutative.

**27. Let  $*$  be binary operation defined on  $R$  by  $a * b = 1 + ab, \forall a, b \in R$ . Then the operation  $*$  is**

- (i) commutative but not associative**
- (ii) associative but not commutative**
- (iii) neither commutative nor associative**
- (iv) both commutative and associative**

**Solution:**

(i) Given that  $*$  is a binary operation defined on  $R$  by  $a * b = 1 + ab, \forall a, b \in R$

So, we have  $a * b = ab + 1 = b * a$

So,  $*$  is a commutative binary operation.

Now,  $a * (b * c) = a * (1 + bc) = 1 + a(1 + bc) = 1 + a + abc$

Also,

$(a * b) * c = (1 + ab) * c = 1 + (1 + ab)c = 1 + c + abc$

Thus,  $a * (b * c) \neq (a * b) * c$

Hence,  $*$  is not associative.

Therefore,  $*$  is commutative but not associative.

### Objective Type Questions

Choose the correct answer out of the given four options in each of the Exercises from 28 to 47 (M.C.Q.)

**28. Let  $T$  be the set of all triangles in the Euclidean plane, and let a relation  $R$  on  $T$  be defined as  $aRb$  if  $a$  is congruent to  $b \forall a, b \in T$ . Then  $R$  is**

- (A) reflexive but not transitive**
- (B) transitive but not symmetric**
- (C) equivalence**
- (D) none of these**

**Solution:**

(C) equivalence

Given  $aRb$ , if  $a$  is congruent to  $b, \forall a, b \in T$ .

Then, we have  $aRa \Rightarrow a$  is congruent to  $a$ ; which is always true.

So,  $R$  is reflexive.

Let  $aRb \Rightarrow a \sim b$

$b \sim a$

$bRa$

So,  $R$  is symmetric.

Let  $aRb$  and  $bRc$

$a \sim b$  and  $b \sim c$

$a \sim c$

$aRc$

So,  $R$  is transitive.

Therefore,  $R$  is equivalence relation.

**29. Consider the non-empty set consisting of children in a family and a relation  $R$  defined as  $aRb$  if  $a$  is brother of  $b$ . Then  $R$  is**

- (A) symmetric but not transitive  
(C) neither symmetric nor transitive

- (B) transitive but not symmetric  
(D) both symmetric and transitive

**Solution:**

(B) transitive but not symmetric

$aRb \Rightarrow a$  is brother of  $b$ .

This does not mean  $b$  is also a brother of  $a$  as  $b$  can be a sister of  $a$ .

Thus,  $R$  is not symmetric.

$aRb \Rightarrow a$  is brother of  $b$ .

and  $bRc \Rightarrow b$  is brother of  $c$ .

So,  $a$  is brother of  $c$ .

Therefore,  $R$  is transitive.

**30. The maximum number of equivalence relations on the set  $A = \{1, 2, 3\}$  are**

- (A) 1                      (B) 2  
(C) 3                      (D) 5

**Solution:**

(D) 5

Given, set  $A = \{1, 2, 3\}$

Now, the number of equivalence relations as follows

$$R_1 = \{(1, 1), (2, 2), (3, 3)\}$$

$$R_2 = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1)\}$$

$$R_3 = \{(1, 1), (2, 2), (3, 3), (1, 3), (3, 1)\}$$

$$R_4 = \{(1, 1), (2, 2), (3, 3), (2, 3), (3, 2)\}$$

$$R_5 = \{(1, 2, 3) \Leftrightarrow A \times A = A^2\}$$

Thus, maximum number of equivalence relation is '5'.

**31. If a relation  $R$  on the set  $\{1, 2, 3\}$  be defined by  $R = \{(1, 2)\}$ , then  $R$  is**

- (A) reflexive                      (B) transitive  
(C) symmetric                      (D) none of these

**Solution:**

(D) none of these

$R$  on the set  $\{1, 2, 3\}$  be defined by  $R = \{(1, 2)\}$

Hence, its clear that  $R$  is not reflexive, transitive and symmetric.