

EXERCISE

SHORT ANSWER TYPE

1. Give an example of a statement $P(n)$ which is true for all $n \geq 4$ but $P(1)$, $P(2)$ and $P(3)$ are not true. Justify your answer.

Solution:

According to the question,

$P(n)$ which is true for all $n \geq 4$ but $P(1)$, $P(2)$ and $P(3)$ are not true

Let $P(n)$ be $2^n < n!$

So, the examples of the given statements are,

$$P(0) \Rightarrow 2^0 < 0!$$

$$\text{i.e } 1 < 1 \Rightarrow \text{not true}$$

$$P(1) \Rightarrow 2^1 < 1!$$

$$\text{i.e } 2 < 1 \Rightarrow \text{not true}$$

$$P(2) \Rightarrow 2^2 < 2!$$

$$\text{i.e } 4 < 2 \Rightarrow \text{not true}$$

$$P(3) \Rightarrow 2^3 < 3!$$

$$\text{i.e } 8 < 6 \Rightarrow \text{not true}$$

$$P(4) \Rightarrow 2^4 < 4!$$

$$\text{i.e } 16 < 24 \Rightarrow \text{true}$$

$$P(5) \Rightarrow 2^5 < 5!$$

$$\text{i.e } 32 < 60 \Rightarrow \text{true, etc.}$$

2. Give an example of a statement $P(n)$ which is true for all n . Justify your answer.

Solution:

According to the question,

$P(n)$ which is true for all n .

Let $P(n)$ be,

$$1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$$

$$P(0) \text{ is } 0 = \frac{0(0+1)}{2} = 0 ; \text{ it's true}$$

$$P(1) \text{ is } 1 = \frac{1(1+1)}{2} = 1 ; \text{ it's true}$$

$$P(2) \text{ is } 1 + 2 = \frac{2(2+1)}{2} ; \text{ it's true}$$

$$P(k) \text{ is } 1 + 2 + 3 + \dots + k = \frac{k(k+1)}{2}$$

$$P(k) \text{ is } 1 + 2 + 3 + \dots + k + 1 = \frac{k(k+1)}{2} + k + 1 = \frac{(k+1)(k+2)}{2}$$

$\Rightarrow P(k)$ is true for all k .

Therefore, $P(n)$ is true for all n .

Prove each of the statements in Exercises 3 to 16 by the Principle of Mathematical Induction:

3. $4^n - 1$ is divisible by 3, for each natural number n .

Solution:

According to the question,

$P(n) = 4^n - 1$ is divisible by 3.

So, substituting different values for n , we get,

$P(0) = 4^0 - 1 = 0$ which is divisible by 3.

$P(1) = 4^1 - 1 = 3$ which is divisible by 3.

$P(2) = 4^2 - 1 = 15$ which is divisible by 3.

$P(3) = 4^3 - 1 = 63$ which is divisible by 3.

Let $P(k) = 4^k - 1$ be divisible by 3,

So, we get,

$\Rightarrow 4^k - 1 = 3x$.

Now, we also get that,

$\Rightarrow P(k+1) = 4^{k+1} - 1$

$= 4(3x + 1) - 1$

$= 12x + 3$ is divisible by 3.

$\Rightarrow P(k+1)$ is true when $P(k)$ is true

Therefore, by Mathematical Induction,

$P(n) = 4^n - 1$ is divisible by 3 is true for each natural number n .

4. $2^{3n} - 1$ is divisible by 7, for all natural numbers n .

Solution:

According to the question,

$P(n) = 2^{3n} - 1$ is divisible by 7.

So, substituting different values for n , we get,

$P(0) = 2^0 - 1 = 0$ which is divisible by 7.

$P(1) = 2^3 - 1 = 7$ which is divisible by 7.

$P(2) = 2^6 - 1 = 63$ which is divisible by 7.

$P(3) = 2^9 - 1 = 512$ which is divisible by 7.

Let $P(k) = 2^{3k} - 1$ be divisible by 7

So, we get,

$\Rightarrow 2^{3k} - 1 = 7x$.

Now, we also get that,

$\Rightarrow P(k+1) = 2^{3(k+1)} - 1$

$= 2^3(7x + 1) - 1$

$= 56x + 7$

$= 7(8x + 1)$ is divisible by 7.

$\Rightarrow P(k+1)$ is true when $P(k)$ is true.

Therefore, by Mathematical Induction,

$P(n) = 2^{3n} - 1$ is divisible by 7, for all natural numbers n .

5. $n^3 - 7n + 3$ is divisible by 3, for all natural numbers n .

Solution:

According to the question,

$P(n) = n^3 - 7n + 3$ is divisible by 3.

So, substituting different values for n , we get,

$$P(0) = 0^3 - 7 \times 0 + 3 = 3 \text{ which is divisible by } 3.$$

$$P(1) = 1^3 - 7 \times 1 + 3 = -3 \text{ which is divisible by } 3.$$

$$P(2) = 2^3 - 7 \times 2 + 3 = -3 \text{ which is divisible by } 3.$$

$$P(3) = 3^3 - 7 \times 3 + 3 = 9 \text{ which is divisible by } 3.$$

Let $P(k) = k^3 - 7k + 3$ be divisible by 3

So, we get,

$$\Rightarrow k^3 - 7k + 3 = 3x.$$

Now, we also get that,

$$\begin{aligned} \Rightarrow P(k+1) &= (k+1)^3 - 7(k+1) + 3 \\ &= k^3 + 3k^2 + 3k + 1 - 7k - 7 + 3 \\ &= 3x + 3(k^2 + k - 2) \text{ is divisible by } 3. \end{aligned}$$

$\Rightarrow P(k+1)$ is true when $P(k)$ is true.

Therefore, by Mathematical Induction,

$P(n) = n^3 - 7n + 3$ is divisible by 3, for all natural numbers n .

6. $3^{2n} - 1$ is divisible by 8, for all natural numbers n .

Solution:

According to the question,

$$P(n) = 3^{2n} - 1 \text{ is divisible by } 8.$$

So, substituting different values for n , we get,

$$P(0) = 3^0 - 1 = 0 \text{ which is divisible by } 8.$$

$$P(1) = 3^2 - 1 = 8 \text{ which is divisible by } 8.$$

$$P(2) = 3^4 - 1 = 80 \text{ which is divisible by } 8.$$

$$P(3) = 3^6 - 1 = 728 \text{ which is divisible by } 8.$$

Let $P(k) = 3^{2k} - 1$ be divisible by 8

So, we get,

$$\Rightarrow 3^{2k} - 1 = 8x.$$

Now, we also get that,

$$\begin{aligned} \Rightarrow P(k+1) &= 3^{2(k+1)} - 1 \\ &= 3^2(8x + 1) - 1 \\ &= 72x + 8 \text{ is divisible by } 8. \end{aligned}$$

$\Rightarrow P(k+1)$ is true when $P(k)$ is true.

Therefore, by Mathematical Induction,

$P(n) = 3^{2n} - 1$ is divisible by 8, for all natural numbers n .

7. For any natural number n , $7^n - 2^n$ is divisible by 5.

Solution:

According to the question,

$$P(n) = 7^n - 2^n \text{ is divisible by } 5.$$

So, substituting different values for n , we get,

$$P(0) = 7^0 - 2^0 = 0 \text{ Which is divisible by } 5.$$

$$P(1) = 7^1 - 2^1 = 5 \text{ Which is divisible by } 5.$$

$$P(2) = 7^2 - 2^2 = 45 \text{ Which is divisible by } 5.$$

$$P(3) = 7^3 - 2^3 = 335 \text{ Which is divisible by } 5.$$

Let $P(k) = 7^k - 2^k$ be divisible by 5

So, we get,

$$\Rightarrow 7^k - 2^k = 5x.$$

Now, we also get that,

$$\begin{aligned} \Rightarrow P(k+1) &= 7^{k+1} - 2^{k+1} \\ &= (5 + 2)7^k - 2(2^k) \\ &= 5(7^k) + 2(7^k - 2^k) \\ &= 5(7^k) + 2(5x) \text{ Which is divisible by 5.} \end{aligned}$$

$\Rightarrow P(k+1)$ is true when $P(k)$ is true.

Therefore, by Mathematical Induction,

$P(n) = 7^n - 2^n$ is divisible by 5 is true for each natural number n .

8. For any natural number n , $x^n - y^n$ is divisible by $x - y$, where x integers with $x \neq y$.

Solution:

According to the question,

$P(n) = x^n - y^n$ is divisible by $x - y$, x integers with $x \neq y$.

So, substituting different values for n , we get,

$$P(0) = x^0 - y^0 = 0 \text{ Which is divisible by } x - y.$$

$$P(1) = x - y \text{ Which is divisible by } x - y.$$

$$\begin{aligned} P(2) &= x^2 - y^2 \\ &= (x + y)(x - y) \text{ Which is divisible by } x - y. \end{aligned}$$

$$\begin{aligned} P(3) &= x^3 - y^3 \\ &= (x - y)(x^2 + xy + y^2) \text{ Which is divisible by } x - y. \end{aligned}$$

Let $P(k) = x^k - y^k$ be divisible by $x - y$;

So, we get,

$$\Rightarrow x^k - y^k = a(x - y).$$

Now, we also get that,

$$\begin{aligned} \Rightarrow P(k+1) &= x^{k+1} - y^{k+1} \\ &= x^k(x - y) + y(x^k - y^k) \\ &= x^k(x - y) + y a(x - y) \text{ Which is divisible by } x - y. \end{aligned}$$

$\Rightarrow P(k+1)$ is true when $P(k)$ is true.

Therefore, by Mathematical Induction,

$P(n) x^n - y^n$ is divisible by $x - y$, where x integers with $x \neq y$ which is true for any natural number n .

9. $n^3 - n$ is divisible by 6, for each natural number $n \geq 2$.

Solution:

According to the question,

$P(n) = n^3 - n$ is divisible by 6.

So, substituting different values for n , we get,

$$P(0) = 0^3 - 0 = 0 \text{ Which is divisible by 6.}$$

$$P(1) = 1^3 - 1 = 0 \text{ Which is divisible by 6.}$$

$$P(2) = 2^3 - 2 = 6 \text{ Which is divisible by 6.}$$

Let $P(k) = k^3 - k$ be divisible by 6.

So, we get,

$$\Rightarrow k^3 - k = 6x.$$

Now, we also get that,

$$\begin{aligned} \Rightarrow P(k+1) &= (k+1)^3 - (k+1) \\ &= (k+1)(k^2+2k+1-1) \\ &= k^3 + 3k^2 + 2k \\ &= 6x+3k(k+1) \text{ [}n(n+1)\text{ is always even and divisible by 2]} \\ &= 6x + 3 \times 2y \text{ Which is divisible by 6.} \end{aligned}$$

$\Rightarrow P(k+1)$ is true when $P(k)$ is true.

Therefore, by Mathematical Induction,

$P(n) = n^3 - n$ is divisible by 6, for each natural number $n \geq 2$.

10. $n(n^2 + 5)$ is divisible by 6, for each natural number n .

Solution:

According to the question,

$P(n) = n(n^2 + 5)$ is divisible by 6.

So, substituting different values for n , we get,

$P(0) = 0(0^2 + 5) = 0$ Which is divisible by 6.

$P(1) = 1(1^2 + 5) = 6$ Which is divisible by 6.

$P(2) = 2(2^2 + 5) = 18$ Which is divisible by 6.

$P(3) = 3(3^2 + 5) = 42$ Which is divisible by 6.

Let $P(k) = k(k^2 + 5)$ be divisible by 6.

So, we get,

$$\Rightarrow k(k^2 + 5) = 6x.$$

Now, we also get that,

$$\begin{aligned} \Rightarrow P(k+1) &= (k+1)((k+1)^2 + 5) = (k+1)(k^2+2k+6) \\ &= k^3 + 3k^2 + 8k + 6 \\ &= 6x+3k^2+3k+6 \\ &= 6x+3k(k+1)+6[n(n+1) \text{ is always even and divisible by 2}] \\ &= 6x + 3 \times 2y + 6 \text{ Which is divisible by 6.} \end{aligned}$$

$\Rightarrow P(k+1)$ is true when $P(k)$ is true.

Therefore, by Mathematical Induction,

$P(n) = n(n^2 + 5)$ is divisible by 6, for each natural number n .

11. $n^2 < 2^n$ for all natural numbers $n \geq 5$.

Solution:

According to the question,

$P(n)$ is $n^2 < 2^n$ for $n \geq 5$

Let $P(k) = k^2 < 2^k$ be true;

$$\Rightarrow P(k+1) = (k+1)^2$$

$$= k^2 + 2k + 1$$

$$2^{k+1} = 2(2^k) > 2k^2$$

Since, $n^2 > 2n + 1$ for $n \geq 3$

We get that,

$$k^2 + 2k + 1 < 2k^2$$

$$\Rightarrow (k+1)^2 < 2^{(k+1)}$$

$\Rightarrow P(k+1)$ is true when $P(k)$ is true.

Therefore, by Mathematical Induction,

$P(n) = n^2 < 2^n$ is true for all natural numbers $n \geq 5$.

12. $2n < (n + 2)!$ for all natural number n .

Solution:

According to the question,

$P(n)$ is $2n < (n + 2)!$

So, substituting different values for n , we get,

$$P(0) \Rightarrow 0 < 2!$$

$$P(1) \Rightarrow 2 < 3!$$

$$P(2) \Rightarrow 4 < 4!$$

$$P(3) \Rightarrow 6 < 5!$$

Let $P(k) = 2k < (k + 2)!$ is true;

Now, we get that,

$$\Rightarrow P(k+1) = 2(k+1) ((k+1)+2)!$$

We know that,

$$[(k+1)+2)! = (k+3)! = (k+3)(k+2)(k+1)\dots\dots\dots 3 \times 2 \times 1]$$

But, we also know that,

$$= 2(k+1) \times (k+3)(k+2)\dots\dots\dots 3 \times 1 > 2(k+1)$$

Therefore, $2(k+1) < ((k+1) + 2)!$

$\Rightarrow P(k+1)$ is true when $P(k)$ is true.

Therefore, by Mathematical Induction,

$P(n) = 2n < (n + 2)!$ Is true for all natural number n .

13. $\sqrt{n} < 1/\sqrt{1} + 1/\sqrt{2} + \dots 1/\sqrt{n}$, for all natural numbers $n \geq 2$.

Solution:

According to the question,

$$P(n) \text{ is } \sqrt{n} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \dots\dots\dots + \frac{1}{\sqrt{n}} ; n \geq 2$$

$$P(n) \text{ is } \sqrt{n} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \dots\dots\dots + \frac{1}{\sqrt{n}} ; n \geq 2$$

$$P(2) \text{ is } \sqrt{2} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} \Rightarrow 1.414 < 1.707 \text{ It's true}$$

$$P(3) \text{ is } \sqrt{3} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} \Rightarrow 1.732 < 2.284 \text{ It's true}$$

$$\text{Let } P(k) = \sqrt{k} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \dots\dots\dots + \frac{1}{\sqrt{k}} ; \text{ is true}$$

Adding $\sqrt{k+1} - \sqrt{k}$ on both sides.

$$\Rightarrow \sqrt{k} + \sqrt{k+1} - \sqrt{k} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \dots\dots\dots + \frac{1}{\sqrt{k}} + \sqrt{k+1} - \sqrt{k}$$

$$\left[\because \sqrt{k+1} - \sqrt{k} = \frac{(\sqrt{k+1} - \sqrt{k})(\sqrt{k+1} + \sqrt{k})}{(\sqrt{k+1} + \sqrt{k})} = \frac{1}{(\sqrt{k+1} + \sqrt{k})} \leq \frac{1}{\sqrt{k+1}} \right]$$

$$\Rightarrow \sqrt{k+1} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \dots\dots\dots + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}}$$

$\Rightarrow P(k+1)$ is true when $P(k)$ is true.

Therefore, by Mathematical Induction,

$$\sqrt{n} < 1/\sqrt{1} + 1/\sqrt{2} + \dots + 1/\sqrt{n}, \text{ for all natural numbers } n \geq 2$$

14. $2 + 4 + 6 + \dots + 2n = n^2 + n$ for all natural numbers n .

Solution:

According to the question,

$$P(n) \text{ is } 2 + 4 + 6 + \dots + 2n = n^2 + n.$$

So, substituting different values for n , we get,

$$P(0) = 0 = 0^2 + 0 \text{ Which is true.}$$

$$P(1) = 2 = 1^2 + 1 \text{ Which is true.}$$

$$P(2) = 2 + 4 = 2^2 + 2 \text{ Which is true.}$$

$$P(3) = 2 + 4 + 6 = 3^2 + 2 \text{ Which is true.}$$

Let $P(k) = 2 + 4 + 6 + \dots + 2k = k^2 + k$ be true;

So, we get,

$$\begin{aligned} \Rightarrow P(k+1) \text{ is } 2 + 4 + 6 + \dots + 2k + 2(k+1) &= k^2 + k + 2k + 2 \\ &= (k^2 + 2k + 1) + (k+1) \\ &= (k+1)^2 + (k+1) \end{aligned}$$

$\Rightarrow P(k+1)$ is true when $P(k)$ is true.

Therefore, by Mathematical Induction,

$2 + 4 + 6 + \dots + 2n = n^2 + n$ is true for all natural numbers n .

15. $1 + 2 + 2^2 + \dots + 2^n = 2^{n+1} - 1$ for all natural numbers n .

Solution:

According to the question,

$$P(n) \text{ is } 1 + 2 + 2^2 + \dots + 2^n = 2^{n+1} - 1.$$

So, substituting different values for n , we get,

$$P(0) = 1 = 2^{0+1} - 1 \text{ Which is true.}$$

$$P(1) = 1 + 2 = 3 = 2^{1+1} - 1 \text{ Which is true.}$$

$$P(2) = 1 + 2 + 2^2 = 7 = 2^{2+1} - 1 \text{ Which is true.}$$

$$P(3) = 1 + 2 + 2^2 + 2^3 = 15 = 2^{3+1} - 1 \text{ Which is true.}$$

Let $P(k) = 1 + 2 + 2^2 + \dots + 2^k = 2^{k+1} - 1$ be true;

So, we get,

$$\begin{aligned} \Rightarrow P(k+1) \text{ is } 1 + 2 + 2^2 + \dots + 2^k + 2^{k+1} &= 2^{k+1} - 1 + 2^{k+1} \\ &= 2 \times 2^{k+1} - 1 \\ &= 2^{(k+1)+1} - 1 \end{aligned}$$

$\Rightarrow P(k+1)$ is true when $P(k)$ is true.

Therefore, by Mathematical Induction,

$1 + 2 + 2^2 + \dots + 2^n = 2^{n+1} - 1$ is true for all natural numbers n .