EXERCISE 20.1 PAGE NO: 20.9

1. Show that each one of the following progressions is a G.P. Also, find the common ratio in each case:
(i) 4, -2, 1, -1/2, ….
(ii) -2/3, -6, -54, ….
(iii) a, $3a^2/4$, $9a^3/16$, ….
(iv) ½, 1/3, 2/9, 4/27, …
Solution:
(i) 4, -2, 1, -1/2, ….
Let a = 4, b = -2, c = 1
In GP,
$b^2 = ac$
$(-2)^2 = 4(1)$
4 = 4
So, the Common ratio = r = -2/4 = -1/2

(ii) -2/3, -6, -54, ….
Let a = -2/3, b = -6, c = -54
In GP,
$b^2 = ac$
$(-6)^2 = -2/3 \times (-54)$
36 = 36
So, the Common ratio = r = -6/(-2/3) = -6 × 3/-2 = 9

(iii) a, $3a^2/4$, $9a^3/16$, ….
Let a = a, b = $3a^2/4$, c = $9a^3/16$
In GP,
$b^2 = ac$
$(3a^2/4)^2 = 9a^3/16 \times a$
$9a^4/16 = 9a^4/16$
So, the Common ratio = r = $(3a^2/4)/a = 3a^2/4a = 3a/4$

(iv) ½, 1/3, 2/9, 4/27, …
Let a = 1/2, b = 1/3, c = 2/9
In GP,
$b^2 = ac$
$(1/3)^2 = 1/2 \times (2/9)$
1/9 = 1/9

So, the Common ratio = r = (1/3)/(1/2) = (1/3) × 2 = 2/3

2. Show that the sequence defined by $a_n = 2/3^n$, n ∈ N is a G.P.

Solution:

Given:

$a_n = 2/3^n$

Let us consider n = 1, 2, 3, 4, … since n is a natural number.

So,

$a_1 = 2/3$

$a_2 = 2/3^2 = 2/9$

$a_3 = 2/3^3 = 2/27$

$a_4 = 2/3^4 = 2/81$

In GP,

a_3/a_2 = (2/27) / (2/9)

= 2/27 × 9/2

= 1/3

a_2/a_1 = (2/9) / (2/3)

= 2/9 × 3/2

= 1/3

∴ Common ratio of consecutive term is 1/3. Hence n ∈ N is a G.P.

3. Find:

(i) the ninth term of the G.P. 1, 4, 16, 64, ….

(ii) the 10^{th} term of the G.P. -3/4, ½, -1/3, 2/9, ….

(iii) the 8^{th} term of the G.P. 0.3, 0.06, 0.012, ….

(iv) the 12^{th} term of the G.P. $1/a^3x^3$, ax, a^5x^5, ….

(v) nth term of the G.P. √3, 1/√3, 1/3√3, …

(vi) the 10^{th} term of the G.P. √2, 1/√2, 1/2√2, ….

Solution:

(i) the ninth term of the G.P. 1, 4, 16, 64, ….

We know that,

$t_1 = a = 1$, $r = t_2/t_1 = 4/1 = 4$

By using the formula,

$T_n = ar^{n-1}$

$T_9 = 1\ (4)^{9-1}$

$= 1\ (4)^8$

$= 4^8$

(ii) the 10^{th} term of the G.P. -3/4, ½, -1/3, 2/9, ….
We know that,
$t_1 = a = -3/4$, $r = t_2/t_1 = (1/2) / (-3/4) = ½ \times -4/3 = -2/3$
By using the formula,
$T_n = ar^{n-1}$
$T_{10} = -3/4 (-2/3)^{10-1}$
$= -3/4 (-2/3)^9$
$= ½ (2/3)^8$

(iii) the 8^{th} term of the G.P., 0.3, 0.06, 0.012, ….
We know that,
$t_1 = a = 0.3$, $r = t_2/t_1 = 0.06/0.3 = 0.2$
By using the formula,
$T_n = ar^{n-1}$
$T_8 = 0.3 (0.2)^{8-1}$
$= 0.3 (0.2)^7$

(iv) the 12^{th} term of the G.P. $1/a^3x^3$, ax, a^5x^5, ….
We know that,
$t_1 = a = 1/a^3x^3$, $r = t_2/t_1 = ax/(1/a^3x^3) = ax (a^3x^3) = a^4x^4$
By using the formula,
$T_n = ar^{n-1}$
$T_{12} = 1/a^3x^3 (a^4x^4)^{12-1}$
$= 1/a^3x^3 (a^4x^4)^{11}$
$= (ax)^{41}$

(v) nth term of the G.P. √3, 1/√3, 1/3√3, …
We know that,
$t_1 = a = \sqrt{3}$, $r = t_2/t_1 = (1/\sqrt{3})/\sqrt{3} = 1/(\sqrt{3}\times\sqrt{3}) = 1/3$
By using the formula,
$T_n = ar^{n-1}$
$T_n = \sqrt{3} (1/3)^{n-1}$

(vi) the 10^{th} term of the G.P. √2, 1/√2, 1/2√2, ….
We know that,
$t_1 = a = \sqrt{2}$, $r = t_2/t_1 = (1/\sqrt{2})/\sqrt{2} = 1/(\sqrt{2}\times\sqrt{2}) = 1/2$
By using the formula,
$T_n = ar^{n-1}$
$T_{10} = \sqrt{2} (1/2)^{10-1}$

$= \sqrt{2}\ (1/2)^9$
$= 1/\sqrt{2}\ (1/2)^8$

4. Find the 4^{th} term from the end of the G.P. 2/27, 2/9, 2/3, …., 162.
Solution:
The nth term from the end is given by:
$a_n = l\ (1/r)^{n-1}$ where, l is the last term, r is the common ratio, n is the nth term
Given: last term, l = 162
$r = t_2/t_1 = (2/9) / (2/27)$
$= 2/9 \times 27/2$
$= 3$
n = 4
So, $a_n = l\ (1/r)^{n-1}$
$a_4 = 162\ (1/3)^{4-1}$
$= 162\ (1/3)^3$
$= 162 \times 1/27$
$= 6$
∴ 4^{th} term from last is 6.

5. Which term of the progression 0.004, 0.02, 0.1, …. is 12.5?
Solution:
By using the formula,
$T_n = ar^{n-1}$
Given:
a = 0.004
$r = t_2/t_1 = (0.02/0.004)$
$= 5$
$T_n = 12.5$
n = ?
So, $T_n = ar^{n-1}$
$12.5 = (0.004)\ (5)^{n-1}$
$12.5/0.004 = 5^{n-1}$
$3000 = 5^{n-1}$
$5^5 = 5^{n-1}$
5 = n-1
n = 5 + 1
= 6
∴ 6^{th} term of the progression 0.004, 0.02, 0.1, …. is 12.5.

6. Which term of the G.P.:
(i) √2, 1/√2, 1/2√2, 1/4√2, … is 1/512√2 ?
(ii) 2, 2√2, 4, … is 128 ?
(iii) √3, 3, 3√3, … is 729 ?
(iv) 1/3, 1/9, 1/27… is 1/19683 ?
Solution:
(i) √2, 1/√2, 1/2√2, 1/4√2, … is 1/512√2 ?
By using the formula,
$T_n = ar^{n-1}$
a = √2
$r = t_2/t_1 = (1/\sqrt{2}) / (\sqrt{2})$
$= 1/2$
$T_n = 1/512\sqrt{2}$
n = ?
$T_n = ar^{n-1}$
$1/512\sqrt{2} = (\sqrt{2})\ (1/2)^{n-1}$
$1/512\sqrt{2}\times\sqrt{2} = (1/2)^{n-1}$
$1/512\times2 = (1/2)^{n-1}$
$1/1024 = (1/2)^{n-1}$
$(1/2)^{10} = (1/2)^{n-1}$
10 = n – 1
n = 10 + 1
= 11
∴ 11^{th} term of the G.P is 1/512√2

(ii) 2, 2√2, 4, … is 128 ?
By using the formula,
$T_n = ar^{n-1}$
a = 2
$r = t_2/t_1 = (2\sqrt{2}/2)$
$= \sqrt{2}$
$T_n = 128$
n = ?
$T_n = ar^{n-1}$
$128 = 2\ (\sqrt{2})^{n-1}$
$128/2 = (\sqrt{2})^{n-1}$
$64 = (\sqrt{2})^{n-1}$
$2^6 = (\sqrt{2})^{n-1}$
12 = n – 1

$n = 12 + 1$
$= 13$
∴ 13^{th} term of the G.P is 128

(iii) $\sqrt{3}, 3, 3\sqrt{3}, \ldots$ is 729 ?
By using the formula,
$T_n = ar^{n-1}$
$a = \sqrt{3}$
$r = t_2/t_1 = (3/\sqrt{3})$
$= \sqrt{3}$
$T_n = 729$
$n = ?$
$T_n = ar^{n-1}$
$729 = \sqrt{3} (\sqrt{3})^{n-1}$
$729 = (\sqrt{3})^n$
$3^6 = (\sqrt{3})^n$
$(\sqrt{3})^{12} = (\sqrt{3})^n$
$n = 12$
∴ 12^{th} term of the G.P is 729

(iv) 1/3, 1/9, 1/27… is 1/19683 ?
By using the formula,
$T_n = ar^{n-1}$
$a = 1/3$
$r = t_2/t_1 = (1/9) / (1/3)$
$= 1/9 \times 3/1$
$= 1/3$
$T_n = 1/19683$
$n = ?$
$T_n = ar^{n-1}$
$1/19683 = (1/3) (1/3)^{n-1}$
$1/19683 = (1/3)^n$
$(1/3)^9 = (1/3)^n$
$n = 9$
∴ 9^{th} term of the G.P is 1/19683

7. Which term of the progression 18, -12, 8, … is 512/729 ?
Solution:
By using the formula,

$T_n = ar^{n-1}$
a = 18
$r = t_2/t_1 = (-12/18)$
= -2/3
$T_n = 512/729$
n = ?
$T_n = ar^{n-1}$
$512/729 = 18 (-2/3)^{n-1}$
$2^9/(729 \times 18) = (-2/3)^{n-1}$
$2^9/36 \times 1/2\times3^2 = (-2/3)^{n-1}$
$(2/3)^8 = (-1)^{n-1} (2/3)^{n-1}$
8 = n – 1
n = 8 + 1
= 9
∴ 9^{th} term of the Progression is 512/729

8. Find the 4th term from the end of the G.P. ½, 1/6, 1/18, 1/54, … , 1/4374
Solution:
The nth term from the end is given by:
$a_n = l (1/r)^{n-1}$ where, l is the last term, r is the common ratio, n is the nth term
Given: last term, l = 1/4374
$r = t_2/t_1 = (1/6) / (1/2)$
= 1/6 × 2/1
= 1/3
n = 4
So, $a_n = l (1/r)^{n-1}$
$a_4 = 1/4374 (1/(1/3))^{4-1}$
$= 1/4374 (3/1)^3$
$= 1/4374 \times 3^3$
= 1/4374 × 27
= 1/162
∴ 4^{th} term from last is 1/162.

EXERCISE 20.2 PAGE NO: 20.16

1. Find three numbers in G.P. whose sum is 65 and whose product is 3375.
Solution:
Let the three numbers be a/r, a, ar
So, according to the question
a/r + a + ar = 65 … equation (1)
a/r × a × ar = 3375 … equation (2)
From equation (2) we get,
$a^3 = 3375$
a = 15.

From equation (1) we get,
$(a + ar + ar^2)/r = 65$
$a + ar + ar^2 = 65r$ … equation (3)
Substituting a = 15 in equation (3) we get
$15 + 15r + 15r^2 = 65r$
$15r^2 – 50r + 15 = 0$… equation (4)

Dividing equation (4) by 5 we get
$3r^2 – 10r + 3 = 0$
$3r^2 – 9r – r + 3 = 0$
3r(r – 3) – 1(r – 3) = 0
r = 3 or r = 1/3
Now, the equation will be
15/3, 15, 15×3 or
15/(1/3), 15, 15×1/3
So the terms are 5, 15, 45 or 45, 15, 5
∴ The three numbers are 5, 15, 45.

2. Find three number in G.P. whose sum is 38 and their product is 1728.
Solution:
Let the three numbers be a/r, a, ar
So, according to the question
a/r + a + ar = 38 … equation (1)
a/r × a × ar = 1728 … equation (2)
From equation (2) we get,
$a^3 = 1728$
a = 12.

From equation (1) we get,
$(a + ar + ar^2)/r = 38$
$a + ar + ar^2 = 38r$ … equation (3)

Substituting a = 12 in equation (3) we get
$12 + 12r + 12r^2 = 38r$
$12r^2 – 26r + 12 = 0$… equation (4)
Dividing equation (4) by 2 we get
$6r^2 – 13r + 6 = 0$
$6r^2 – 9r – 4r + 6 = 0$
$3r(3r – 3) – 2(3r – 3) = 0$
$r = 3/2$ or $r = 2/3$
Now the equation will be
$12/(3/2) = 8$ or
$12/(2/3) = 18$
So the terms are 8, 12, 18
∴ The three numbers are 8, 12, 18

3. The sum of first three terms of a G.P. is 13/12, and their product is – 1. Find the G.P.

Solution:

Let the three numbers be a/r, a, ar
So, according to the question
$a/r + a + ar = 13/12$ … equation (1)
$a/r \times a \times ar = -1$ … equation (2)
From equation (2) we get,
$a^3 = -1$
$a = -1$

From equation (1) we get,
$(a + ar + ar^2)/r = 13/12$
$12a + 12ar + 12ar^2 = 13r$ … equation (3)
Substituting a = – 1 in equation (3) we get
$12(– 1) + 12(– 1)r + 12(– 1)r^2 = 13r$
$12r^2 + 25r + 12 = 0$
$12r^2 + 16r + 9r + 12 = 0$… equation (4)
$4r (3r + 4) + 3(3r + 4) = 0$
$r = -3/4$ or $r = -4/3$

Now the equation will be

-1/(-3/4), -1, -1×-3/4 or -1/(-4/3), -1, -1×-4/3

4/3, -1, ¾ or ¾, -1, 4/3

∴ The three numbers are 4/3, -1, ¾ or ¾, -1, 4/3

4. The product of three numbers in G.P. is 125 and the sum of their products taken in pairs is 87 ½ . Find them.

Solution:

Let the three numbers be a/r, a, ar

So, according to the question

a/r × a × ar = 125 … equation (1)

From equation (1) we get,

$a^3 = 125$

a = 5

a/r × a + a × ar + ar × a/r = 87 ½

a/r × a + a × ar + ar × a/r = 195/2

$a^2/r + a^2r + a^2 = 195/2$

$a^2 (1/r + r + 1) = 195/2$

Substituting a = 5 in above equation we get,

$5^2 [(1+r^2+r)/r] = 195/2$

$1+r^2+r = (195r/2×25)$

$2(1+r^2+r) = 39r/5$

$10 + 10r^2 + 10r = 39r$

$10r^2 – 29r + 10 = 0$

$10r^2 – 25r – 4r + 10 = 0$

5r(2r-5) – 2(2r-5) = 0

r = 5/2, 2/5

So G.P is 10, 5, 5/2 or 5/2, 5, 10

∴ The three numbers are 10, 5, 5/2 or 5/2, 5, 10

5. The sum of the first three terms of a G.P. is 39/10, and their product is 1. Find the common ratio and the terms.

Solution:

Let the three numbers be a/r, a, ar

So, according to the question

a/r + a + ar = 39/10 … equation (1)

a/r × a × ar = 1 … equation (2)

From equation (2) we get,

$a^3 = 1$

a = 1

From equation (1) we get,
$(a + ar + ar^2)/r = 39/10$
$10a + 10ar + 10ar^2 = 39r$ … equation (3)
Substituting a = 1 in 3 we get
$10(1) + 10(1)r + 10(1)r^2 = 39r$
$10r^2 – 29r + 10 = 0$
$10r^2 – 25r – 4r + 10 = 0$… equation (4)
5r(2r – 5) – 2(2r – 5) = 0
r = 2/5 or 5/2
so now the equation will be,
1/(2/5), 1, 1×2/5 or 1/(5/2), 1, 1×5/2
5/2, 1, 2/5 or 2/5, 1, 5/2
∴ The three numbers are 2/5, 1, 5/2

EXERCISE 20.3 PAGE NO: 20.27

1. Find the sum of the following geometric progressions:
(i) 2, 6, 18, … to 7 terms
(ii) 1, 3, 9, 27, … to 8 terms
(iii) 1, -1/2, ¼, -1/8, …
(iv) $(a^2 – b^2)$, (a - b), (a-b)/(a+b), … to n terms
(v) 4, 2, 1, ½ … to 10 terms
Solution:
(i) 2, 6, 18, … to 7 terms
We know that, sum of GP for n terms = $a(r^n – 1)/(r – 1)$
Given:
$a = 2$, $r = t_2/t_1 = 6/2 = 3$, $n = 7$
Now let us substitute the values in
$a(r^n – 1)/(r – 1) = 2 (3^7 - 1)/(3-1)$
$= 2 (3^7 - 1)/2$
$= 3^7 – 1$
$= 2187 – 1$
$= 2186$

(ii) 1, 3, 9, 27, … to 8 terms
We know that, sum of GP for n terms = $a(r^n – 1)/(r – 1)$
Given:
$a = 1$, $r = t_2/t_1 = 3/1 = 3$, $n = 8$
Now let us substitute the values in
$a(r^n – 1)/(r – 1) = 1 (3^8 - 1)/(3-1)$
$= (3^8 - 1)/2$
$= (6561 – 1)/2$
$= 6560/2$
$= 3280$

(iii) 1, -1/2, ¼, -1/8, …
We know that, sum of GP for infinity = $a/(1 - r)$
Given:
$a = 1$, $r = t_2/t_1 = (-1/2)/1 = -1/2$
Now let us substitute the values in
$a/(1 - r) = 1/(1 – (-1/2))$
$= 1/(1 + 1/2)$
$= 1/((2+1)/2)$

= 1/(3/2)
= 2/3

(iv) $(a^2 – b^2)$, (a - b), (a-b)/(a+b), … to n terms
We know that, sum of GP for n terms = $a(r^n – 1)/(r – 1)$
Given:
$a = (a^2 – b^2)$, $r = t_2/t_1 = (a-b)/(a^2 – b^2) = (a-b)/(a-b)(a+b) = 1/(a+b)$, n = n
Now let us substitute the values in
$a(r^n – 1)/(r – 1) =$

$$= (a^2 - b^2)\left(\frac{1-\left(\frac{1}{a+b}\right)^n}{1-\left(\frac{1}{a+b}\right)}\right)$$

$$= (a^2 - b^2)\left(\frac{\left(\frac{(a+b)^n-1}{(a+b)^n}\right)}{\frac{(a+b)-1}{a+b}}\right)$$

$$= \frac{(a+b)(a-b)}{(a+b)^{n-1}}\left(\frac{(a+b)^n-1}{(a+b)-1}\right)$$

$$= \frac{(a-b)}{(a+b)^{n-2}}\left(\frac{(a+b)^n-1}{(a+b)-1}\right)$$

(v) 4, 2, 1, ½ … to 10 terms
We know that, sum of GP for n terms = $a(r^n – 1)/(r – 1)$
Given:
$a = 4$, $r = t_2/t_1 = 2/4 = 1/2$, n = 10
Now let us substitute the values in
$a(r^n – 1)/(r – 1) = 4 ((1/2)^{10} - 1)/((1/2)-1)$
$= 4 ((1/2)^{10} - 1)/((1-2)/2)$
$= 4 ((1/2)^{10} - 1)/(-1/2)$
$= 4 ((1/2)^{10} - 1) \times -2/1$
= -8 [1/1024 -1]
= -8 [1 - 1024]/1024
= -8 [-1023]/1024
= 1023/128

2. Find the sum of the following geometric series :
(i) 0.15 + 0.015 + 0.0015 + … to 8 terms;
(ii) √2 + 1/√2 + 1/2√2 + …. to 8 terms;
(iii) 2/9 – 1/3 + ½ - ¾ + … to 5 terms;
(iv) $(x + y) + (x^2 + xy + y^2) + (x^3 + x^2 y + xy^2 + y^3)$ + …. to n terms ;

(v) $3/5 + 4/5^2 + 3/5^3 + 4/5^4 + \ldots$ to 2n terms;

Solution:

(i) 0.15 + 0.015 + 0.0015 + … to 8 terms

Given:

a = 0.15

$r = t_2/t_1 = 0.015/0.15 = 0.1 = 1/10$

n = 8

By using the formula,

Sum of GP for n terms $= a(1 - r^n)/(1 - r)$

$a(1 - r^n)/(1 - r) = 0.15\,(1 – (1/10)^8) / (1 – (1/10))$

$= 0.15\,(1 – 1/10^8) / (1- (1/10))$

$= 1/6\,(1 – 1/10^8)$

(ii) √2 + 1/√2 + 1/2√2 + …. to 8 terms;

Given:

a = √2

$r = t_2/t_1 = (1/\sqrt{2})/\sqrt{2} = 1/2$

n = 8

By using the formula,

Sum of GP for n terms $= a(1 - r^n)/(1 - r)$

$a(1 - r^n)/(1 - r) = \sqrt{2}\,(1 – (1/2)^8) / (1 - (1/2))$

$= \sqrt{2}\,(1 – 1/256) / (1/2)$

$= \sqrt{2}\,((256 - 1)/256) \times 2$

$= \sqrt{2}\,(255\times2)/256$

$= (255\sqrt{2})/128$

(iii) 2/9 – 1/3 + ½ - ¾ + … to 5 terms;

Given:

a = 2/9

$r = t_2/t_1 = (-1/3) / (2/9) = -3/2$

n = 5

By using the formula,

Sum of GP for n terms $= a(1 - r^n)/(1 - r)$

$a(1 - r^n)/(1 - r) = (2/9)\,(1 – (-3/2)^5) / (1 – (-3/2))$

$= (2/9)\,(1 + (3/2)^5) / (1 + 3/2)$

$= (2/9)\,(1 + (3/2)^5) / (5/2)$

$= (2/9)\,(1 + 243/32) / (5/2)$

$= (2/9)\,((32+243)/32) / (5/2)$

$= (2/9)\,(275/32) \times 2/5$

= 55/72

(iv) $(x + y) + (x^2 + xy + y^2) + (x^3 + x^2 y + xy^2 + y^3) + \ldots$ to n terms;
Let $S_n = (x + y) + (x^2 + xy + y^2) + (x^3 + x^2 y + xy^2 + y^3) + \ldots$ to n terms
Let us multiply and divide by (x – y) we get,
$S_n = 1/(x - y) [(x + y) (x - y) + (x^2 + xy + y^2) (x - y) \ldots$ upto n terms]
$(x – y) S_n = (x^2 – y^2) + x^3 + x^2y + xy^2 – x^2y – xy^2 – y^3$..upto n terms
$(x – y) S_n = (x^2 + x^3 + x^4+\ldots$n terms$) – (y^2 + y^3 + y^4 +\ldots$n terms)
By using the formula,
Sum of GP for n terms $= a(1 - r^n)/(1 - r)$
We have two G.Ps in above sum, so,
$(x - y) S_n = x^2 [(x^n - 1)/ (x - 1)] – y^2 [(y^n - 1)/ (y - 1)]$
$S_n = 1/(x-y) \{x^2 [(x^n - 1)/ (x - 1)] – y^2 [(y^n - 1)/ (y - 1)]\}$

(v) $3/5 + 4/5^2 + 3/5^3 + 4/5^4 + \ldots$ to 2n terms;
The series can be written as:
$3 (1/5 + 1/5^3 + 1/5^5+ \ldots$ to n terms$) + 4 (1/5^2 + 1/5^4 + 1/5^6 + \ldots$ to n terms)
Firstly let us consider $3 (1/5 + 1/5^3 + 1/5^5+ \ldots$ to n terms)
So, a = 1/5
$r = t_2/t_1 = 1/5^2 = 1/25$
By using the formula,
Sum of GP for n terms $= a(1 - r^n)/(1 - r)$

$$3\left(\frac{1}{5}+\frac{1}{5^3}+\frac{1}{5^5}+\cdots n \text{ terms}\right)=3\cdot\frac{\frac{1}{5}\left(1-\left(\frac{1}{25}\right)^n\right)}{1-\frac{1}{25}}$$
$$=\frac{5}{8}\left(1-\frac{1}{5^{2n}}\right)$$

Now, Let us consider $4 (1/5^2 + 1/5^4 + 1/5^6 + \ldots$ to n terms)
So, a = 1/25
$r = t_2/t_1 = 1/5^2 = 1/25$
By using the formula,
Sum of GP for n terms $= a(1 - r^n)/(1 - r)$

$$4\left(\frac{1}{5^2}+\frac{1}{5^4}+\frac{1}{5^6}+\cdots n \text{ terms}\right)=4\cdot\frac{\frac{1}{25}\left(1-\left(\frac{1}{25}\right)^n\right)}{1-\frac{1}{25}}$$

$$= \frac{1}{6}\left(1 - \frac{1}{5^{2n}}\right)$$

Now,

$$\frac{3}{5} + \frac{4}{5^2} + \frac{3}{5^3} + \cdots \text{ 2n terms} = \frac{5}{8}\left(1 - \frac{1}{5^{2n}}\right) + \frac{1}{6}\left(1 - \frac{1}{5^{2n}}\right)$$

$= 19/24\ (1 – 1/5^{2n})$

3. Evaluate the following:

(i) $\sum_{n=1}^{11} (2 + 3^n)$

(ii) $\sum_{k=1}^{n} (2^k + 3^{k-1})$

(iii) $\sum_{n=2}^{10} 4^n$

Solution:

(i) $\sum_{n=1}^{11} (2 + 3^n)$

$= (2 + 3^1) + (2 + 3^2) + (2 + 3^3) + \ldots + (2 + 3^{11})$

$= 2\times 11 + 3^1 + 3^2 + 3^3 + \ldots + 3^{11}$

$= 22 + 3(3^{11} - 1)/(3 - 1)$ [by using the formula, $a(1 - r^n)/(1 - r)$]

$= 22 + 3(3^{11} - 1)/2$

$= [44 + 3(177147 - 1)]/2$

$= [44 + 3(177146)]/2$

$= 265741$

(ii) $\sum_{k=1}^{n} (2^k + 3^{k-1})$

$= (2 + 3^0) + (2^2 + 3) + (2^3 + 3^2) + \ldots + (2^n + 3^{n-1})$

$= (2 + 2^2 + 2^3 + \ldots + 2^n) + (3^0 + 3^1 + 3^2 + \ldots. + 3^{n-1})$

Firstly let us consider,

$(2 + 2^2 + 2^3 + \ldots + 2^n)$

Where, $a = 2$, $r = 2^2/2 = 4/2 = 2$, $n = n$

By using the formula,

Sum of GP for n terms $= a(r^n - 1)/(r - 1)$

$= 2\ (2^n - 1)/(2 - 1)$

$= 2\ (2^n - 1)$

Now, let us consider

$(3^0 + 3^1 + 3^2 + \dots + 3^n)$

Where, $a = 3^0 = 1$, $r = 3/1 = 3$, $n = n$

By using the formula,

Sum of GP for n terms $= a(r^n - 1)/(r - 1)$

$= 1(3^n - 1)/(3 - 1)$

$= (3^n - 1)/2$

So,

$$\sum_{k=1}^{n} (2^k + 3^{k-1})$$

$= (2 + 2^2 + 2^3 + \dots + 2^n) + (3^0 + 3^1 + 3^2 + \dots + 3^n)$

$= 2(2^n - 1) + (3^n - 1)/2$

$= ½ [2^{n+2} + 3^n – 4 - 1]$

$= ½ [2^{n+2} + 3^n - 5]$

$$(iii) \sum_{n=2}^{10} 4^n$$

$= 4^2 + 4^3 + 4^4 + \dots + 4^{10}$

Where, $a = 4^2 = 16$, $r = 4^3/4^2 = 4$, $n = 9$

By using the formula,

Sum of GP for n terms $= a(r^n - 1)/(r - 1)$

$= 16(4^9 - 1)/(4 - 1)$

$= 16(4^9 - 1)/3$

$= 16/3 [4^9 - 1]$

4. Find the sum of the following series :

(i) 5 + 55 + 555 + … to n terms.

(ii) 7 + 77 + 777 + … to n terms.

(iii) 9 + 99 + 999 + … to n terms.

(iv) 0.5 + 0.55 + 0.555 + …. to n terms

(v) 0.6 + 0.66 + 0.666 + …. to n terms.

Solution:

(i) 5 + 55 + 555 + … to n terms.

Let us take 5 as a common term so we get,

5 [1 + 11 + 111 + … n terms]

Now multiply and divide by 9 we get,

5/9 [9 + 99 + 999 + … n terms]

$5/9 [(10 - 1) + (10^2 - 1) + (10^3 - 1) + \dots$ n terms]

$5/9 [(10 + 10^2 + 10^3 + \dots$ n terms) - n]

So the G.P is
5/9 [(10 + 10^2 + 10^3 + … n terms) - n]
By using the formula,
Sum of GP for n terms = $a(r^n - 1)/(r - 1)$
Where, a = 10, r = $10^2/10$ = 10, n = n
$a(r^n - 1)/(r - 1)$ =

$$= \frac{5}{9}\left\{10 \times \frac{(10^n-1)}{10-1} - n\right\}$$
$$= \frac{5}{9}\left\{\frac{10}{9}(10^n - 1) - n\right\}$$
$$= \frac{5}{81}\left\{10^{n+1} - 9n - 10\right\}$$

(ii) 7 + 77 + 777 + … to n terms.
Let us take 7 as a common term so we get,
7 [1 + 11 + 111 + … to n terms]
Now multiply and divide by 9 we get,
7/9 [9 + 99 + 999 + … n terms]
7/9 [(10 - 1) + (10^2 - 1) + (10^3 - 1) + … + (10^n - 1)]
7/9 [(10 + 10^2 + 10^3 + … +10^n)] – 7/9 [(1 + 1 + 1 + … to n terms)]
So the terms are in G.P
Where, a = 10, r = $10^2/10$ = 10, n = n
By using the formula,
Sum of GP for n terms = $a(r^n - 1)/(r - 1)$
7/9 [10 (10^n - 1)/(10-1)] – n
7/9 [10/9 (10^n - 1) - n]
7/81 [10 (10^n - 1) - n]
7/81 (10^{n+1} – 9n - 10)

(iii) 9 + 99 + 999 + … to n terms.
The given terms can be written as
(10 - 1) + (100 - 1) + (1000 - 1) + … + n terms
(10 + 10^2 + 10^3 + … n terms) – n
By using the formula,
Sum of GP for n terms = $a(r^n - 1)/(r - 1)$
Where, a = 10, r = 10, n = n
$a(r^n - 1)/(r - 1)$ = [10 (10^n - 1)/(10-1)] – n
= 10/9 (10^n - 1) – n
= 1/9 [10^{n+1} – 10 – 9n]
= 1/9 [10^{n+1} – 9n - 10]

(iv) 0.5 + 0.55 + 0.555 + …. to n terms
Let us take 5 as a common term so we get,
5(0.1 + 0.11 + 0.111 + ...n terms)
Now multiply and divide by 9 we get,
5/9 [0.9 + 0.99 + 0.999 + …+ to n terms]
5/9 [9/10 + 9/100 + 9/1000 + … + n terms]
This can be written as
5/9 [(1 – 1/10) + (1 – 1/100) + (1 – 1/1000) + … + n terms]
5/9 [n – {1/10 + $1/10^2$ + $1/10^3$ + … + n terms}]
5/9 [n – 1/10 {1-$(1/10)^n$}/{1 – 1/10}]
5/9 [n – 1/9 (1 – $1/10^n$)]

(v) 0.6 + 0.66 + 0.666 + …. to n terms.
Let us take 6 as a common term so we get,
6(0.1 + 0.11 + 0.111 + ...n terms)
Now multiply and divide by 9 we get,
6/9 [0.9 + 0.99 + 0.999 + …+ n terms]
6/9 [9/10 + 9/100 + 9/1000 + …+ n terms]
This can be written as
6/9 [(1 – 1/10) + (1 – 1/100) + (1 – 1/1000) + … + n terms]
6/9 [n – {1/10 + $1/10^2$ + $1/10^3$ + … + n terms}]
6/9 [n – 1/10 {1-$(1/10)^n$}/{1 – 1/10}]
6/9 [n – 1/9 (1 – $1/10^n$)]

5. How many terms of the G.P. 3, 3/2, ¾, ... Be taken together to make 3069/512 ?
Solution:
Given:
Sum of G.P = 3069/512
Where, a = 3, r = (3/2)/3 = 1/2, n = ?
By using the formula,
Sum of GP for n terms = a(r^n - 1)/(r - 1)
3069/512 = 3 ($(1/2)^n$ - 1)/ (1/2 - 1)
3069/512×3×2 = 1 – $(1/2)^n$
3069/3072 – 1 = - $(1/2)^n$
(3069 – 3072)/3072 = - $(1/2)^n$
-3/3072 = - $(1/2)^n$
1/1024 = $(1/2)^n$
$(1/2)^{10} = (1/2)^n$

10 = n
∴ 10 terms are required to make 3069/512

6. How many terms of the series 2 + 6 + 18 + …. Must be taken to make the sum equal to 728?
Solution:
Given:
Sum of GP = 728
Where, a = 2, r = 6/2 = 3, n = ?
By using the formula,
Sum of GP for n terms = $a(r^n - 1)/(r - 1)$
$728 = 2(3^n – 1)/(3-1)$
$728 = 2(3^n - 1)/2$
$728 = 3^n – 1$
$729 = 3^n$
$3^6 = 3^n$
6 = n
∴ 6 terms are required to make a sum equal to 728

7. How many terms of the sequence √3, 3, 3√3,… must be taken to make the sum 39+ 13√3 ?
Solution:
Given:
Sum of GP = 39 + 13√3
Where, a =√3, r = 3/√3 = √3, n = ?
By using the formula,
Sum of GP for n terms = $a(r^n - 1)/(r - 1)$
$39 + 13\sqrt{3} = \sqrt{3}(\sqrt{3}^n - 1)/(\sqrt{3} - 1)$
$(39 + 13\sqrt{3})(\sqrt{3} - 1) = \sqrt{3}(\sqrt{3}^n - 1)$
Let us simplify we get,
$39\sqrt{3} – 39 + 13(3) - 13\sqrt{3} = \sqrt{3}(\sqrt{3}^n - 1)$
$39\sqrt{3} – 39 + 39 - 13\sqrt{3} = \sqrt{3}(\sqrt{3}^n - 1)$
$39\sqrt{3} – 39 + 39 - 13\sqrt{3} = \sqrt{3}^{n+1} - \sqrt{3}$
$26\sqrt{3} + \sqrt{3} = \sqrt{3}^{n+1}$
$27\sqrt{3} = \sqrt{3}^{n+1}$
$\sqrt{3}^6\sqrt{3} = \sqrt{3}^{n+1}$
6+1 = n + 1
7 = n + 1
7 – 1 = n

$6 = n$

∴ 6 terms are required to make a sum of $39 + 13\sqrt{3}$

8. The sum of n terms of the G.P. 3, 6, 12, … is 381. Find the value of n.

Solution:

Given:

Sum of GP = 381

Where, a = 3, r = 6/3 = 2, n = ?

By using the formula,

Sum of GP for n terms = $a(r^n - 1)/(r - 1)$

$381 = 3(2^n - 1)/(2-1)$

$381 = 3(2^n - 1)$

$381/3 = 2^n - 1$

$127 = 2^n - 1$

$127 + 1 = 2^n$

$128 = 2^n$

$2^7 = 2^n$

$n = 7$

∴ value of n is 7

9. The common ratio of a G.P. is 3, and the last term is 486. If the sum of these terms be 728, find the first term.

Solution:

Given:

Sum of GP = 728

Where, r = 3, a = ?

Firstly,

$T_n = ar^{n-1}$

$486 = a3^{n-1}$

$486 = a3^n/3$

$486(3) = a3^n$

$1458 = a3^n$ …. Equation (i)

By using the formula,

Sum of GP for n terms = $a(r^n - 1)/(r - 1)$

$728 = a(3^n - 1)/2$

$1456 = a3^n - a$ … equation (2)

Subtracting equation (1) from (2) we get

$1458 - 1456 = a.3^n - a.3^n + a$

a = 2.

∴ The first term is 2

10. The ratio of the sum of the first three terms is to that of the first 6 terms of a G.P. is 125 : 152. Find the common ratio.

Solution:

Given:

Sum of G.P of 3 terms is 125

By using the formula,

Sum of GP for n terms = $a(r^n - 1)/(r - 1)$

$125 = a(r^n - 1)/(r-1)$

$125 = a(r^3 - 1)/(r-1)$ … equation (1)

Now,

Sum of G.P of 6 terms is 152

By using the formula,

Sum of GP for n terms = $a(r^n - 1)/(r - 1)$

$152 = a(r^n - 1)/(r-1)$

$152 = a(r^6 - 1)/(r-1)$ … equation (2)

Let us divide equation (i) by (ii) we get,

$125/152 = [a(r^3 - 1)/(r-1)] / [a(r^6 - 1)/(r-1)]$

$125/152 = (r^3 – 1)/(r^6 - 1)$

$125/152 = (r^3 – 1)/[(r^3 - 1)(r^3 + 1)]$

$125/152 = 1/(r^3 + 1)$

$125(r^3 + 1) = 152$

$125r^3 + 125 = 152$

$125r^3 = 152 – 125$

$125r^3 = 27$

$r^3 = 27/125$

$r^3 = 3^3/5^3$

$r = 3/5$

∴ The common ratio is 3/5

EXERCISE 20.4 PAGE NO: 20.39

1. Find the sum of the following series to infinity:
(i) $1 – 1/3 + 1/3^2 – 1/3^3 + 1/3^4 + … \infty$
(ii) $8 + 4\sqrt{2} + 4 + …. \infty$
(iii) $2/5 + 3/5^2 + 2/5^3 + 3/5^4 + …. \infty$
(iv) $10 – 9 + 8.1 – 7.29 + …. \infty$
Solution:
(i) $1 – 1/3 + 1/3^2 – 1/3^3 + 1/3^4 + … \infty$
Given:
$S_\infty = 1 – 1/3 + 1/3^2 – 1/3^3 + 1/3^4 + … \infty$
Where, a = 1, r = -1/3
By using the formula,
$S_\infty = a/(1 - r)$
$= 1 / (1 – (-1/3))$
$= 1/ (1 + 1/3)$
$= 1/ ((3+1)/3)$
$= 1/ (4/3)$
$= 3/4$

(ii) $8 + 4\sqrt{2} + 4 + …. \infty$
Given:
$S_\infty = 8 + 4\sqrt{2} + 4 + …. \infty$
Where, a = 8, $r = 4/4\sqrt{2} = 1/\sqrt{2}$
By using the formula,
$S_\infty = a/(1 - r)$
$= 8 / (1 – (1/\sqrt{2}))$
$= 8 / ((\sqrt{2} – 1)/\sqrt{2})$
$= 8\sqrt{2} /(\sqrt{2} – 1)$
Multiply and divide with $\sqrt{2} + 1$ we get,
$= 8\sqrt{2} /(\sqrt{2} – 1) \times (\sqrt{2} + 1)/(\sqrt{2} + 1)$
$= 8 (2 + \sqrt{2})/(2-1)$
$= 8 (2 + \sqrt{2})$

(iii) $2/5 + 3/5^2 + 2/5^3 + 3/5^4 + …. \infty$
The given terms can be written as,
$(2/5 + 2/5^3 + …) + (3/5^2 + 3/5^4 + …)$
(a = 2/5, r = 1/25) and (a = 3/25, r = 1/25)
By using the formula,

$S_\infty = a/(1 - r)$

$$= \left(\frac{\frac{2}{5}}{1-\frac{1}{25}}\right) + \left(\frac{\frac{3}{5}}{1-\frac{1}{25}}\right)$$

$$= \left(\frac{\frac{2}{5}}{\frac{24}{25}}\right) + \left(\frac{\frac{3}{5}}{\frac{24}{25}}\right)$$

$$= \left(\frac{10}{24} + \frac{3}{24}\right)$$

$$= \frac{13}{24}$$

(iv) 10 – 9 + 8.1 – 7.29 + …. ∞

Given:

$S_\infty = 8 + 4\sqrt{2} + 4 + \ldots \infty$

Where, a = 10, r = -9/10

By using the formula,

$S_\infty = a/(1 - r)$

= 10 / (1 – (-9/10))

= 10 / (1 + 9/10)

= 10 / ((10+9)/10)

= 10 / (19/10)

= 100/19

= 5.263

2. Prove that :

$(9^{1/3} \cdot 9^{1/9} \cdot 9^{1/27} \ldots\infty) = 3$.

Solution:

Let us consider the LHS

$(9^{1/3} \cdot 9^{1/9} \cdot 9^{1/27} \ldots\infty)$

This can be written as

$9^{1/3 + 1/9 + 1/27 + \ldots\infty}$

So let us consider m = 1/3 + 1/9 + 1/27 + … ∞

Where, a = 1/3, r = (1/9) / (1/3) = 1/3

By using the formula,

$S_\infty = a/(1 - r)$

= (1/3) / (1 – (1/3))

= (1/3) / ((3-1)/3)

= (1/3) / (2/3)

= ½

So, $9^m = 9^{1/2} = 3$ = RHS
Hence proved.

3. Prove that :
$(2^{1/4} . 4^{1/8} . 8^{1/16} . 16^{1/32} \ldots \infty) = 2$.
Solution:
Let us consider the LHS
$(2^{1/4} . 4^{1/8} . 8^{1/16} . 16^{1/32} \ldots \infty)$
This can be written as
$2^{1/4} . 2^{2/8} . 2^{3/16} . 2^{1/8} \ldots \infty$
Now,
$2^{1/4 + 2/8 + 3/16 + 1/8 + \ldots \infty}$
So let us consider 2^x, where x = ¼ + 2/8 + 3/16 + 1/8 + … ∞ …. (1)
Multiply both sides of the equation with 1/2, we get
x/2 = ½ (¼ + 2/8 + 3/16 + 1/8 + … ∞)
= 1/8 + 2/16 + 3/32 + … + ∞ …. (2)
Now, subtract (2) from (1) we get,
x – x/2 = (¼ + 2/8 + 3/16 + 1/8 + … ∞) – (1/8 + 2/16 + 3/32 + … + ∞)
By grouping similar terms,
x/2 = ¼ + (2/8 – 1/8) + (3/16 – 2/16) + … ∞
x/2 = ¼ + 1/8 + 1/16 + … ∞
x = ½ + ¼ + 1/8 + 1/16 + … ∞
Where, a = 1/2, r = (1/4) / (1/2) = 1/2
By using the formula,
$S_\infty = a/(1 - r)$
= (1/2) / (1 – 1/2)
= (1/2) / ((2-1)/2)
= (1/2) / (1/2)
= 1
From equation (1), $2^x = 2^1 = 2$ = RHS
Hence proved.

4. If S_p denotes the sum of the series $1 + r^p + r^{2p} + \ldots$ to ∞ and s_p the sum of the series $1 – r^p + r^{2p} - \ldots$ to ∞, prove that $s_p + S_p = 2 S_{2p}$.
Solution:
Given:
$S_p = 1 + r^p + r^{2p} + \ldots \infty$
By using the formula,
$S_\infty = a/(1 - r)$

Where, $a = 1$, $r = r^p$
So,
$S_p = 1 / (1 - r^p)$
Similarly, $s_p = 1 - r^p + r^{2p} - \dots \infty$
By using the formula,
$S_\infty = a/(1 - r)$
Where, $a = 1$, $r = -r^p$
So,
$S_p = 1 / (1 – (-r^p))$
$= 1 / (1 + r^p)$
Now, $S_p + s_p = [1 / (1 - r^p)] + [1 / (1 + r^p)]$
$2S_{2p} = [(1 - r^p) + (1 + r^p)] / (1 – r^{2p})$
$= 2 /(1 – r^{2p})$
$\therefore 2S_{2p} = S_p + S_p$

5. Find the sum of the terms of an infinite decreasing G.P. in which all the terms are positive, the first term is 4, and the difference between the third and fifth term is equal to 32/81.

Solution:

Let ‘a’ be the first term of GP and ‘r’ be the common ratio.
We know that nth term of a GP is given by-
$a_n = ar^{n-1}$
As, a = 4 (given)
And $a_5 – a_3 = 32/81$ (given)
$4r^4 – 4r^2 = 32/81$
$4r^2(r^2 – 1) = 32/81$
$r^2(r^2 – 1) = 8/81$

Let us denote r^2 with y
$81y(y-1) = 8$
$81y^2 – 81y - 8 = 0$
Using the formula of the quadratic equation to solve the equation, we get

$$y = \frac{-(-81)\pm\sqrt{81^2-4(-8)(81)}}{16}$$
$$= \frac{81\pm\sqrt{6561-2592}}{162}$$
$$= \frac{81\pm 63}{162}$$

y = 18/162 = 1/9 or

y = 144/162
= 8/9
So, r^2 = 1/9 or 8/9
= 1/3 or 2√2/3
We know that,
Sum of infinite, S_∞ = a/(1 - r)
Where, a = 4, r = 1/3
S_∞ = 4 / (1 – (1/3))
= 4 / ((3-1)/3)
= 4 / (2/3)
= 12/2
= 6

Sum of infinite, S_∞ = a/(1 - r)
Where, a = 4, r = 2√2/3
S_∞ = 4 / (1 – (2√2/3))
= 12 / (3 - 2√2)

6. Express the recurring decimal 0.125125125 … as a rational number.
Solution:
Given:
0.125125125
So, 0.125125125 = $0.\overline{125}$
= 0.125 + 0.000125 + 0.000000125 + …
This can be written as
$125/10^3 + 125/10^6 + 125/10^9 + \dots$
$125/10^3\ [1 + 1/10^3 + 1/10^6 + \dots]$
By using the formula,
S_∞ = a/(1 - r)
$125/10^3$ [1 / (1 – 1/1000)]
$125/10^3$ [1 / ((1000 – 1)/1000))]
$125/10^3$ [1 / (999/1000)]
125/1000 (1000/999)
125/999
∴ The decimal 0.125125125 can be expressed in rational number as 125/999

EXERCISE 20.5 PAGE NO: 20.45

1. If a, b, c are in G.P., prove that log a, log b, log c are in A.P.
Solution:
It is given that a, b and c are in G.P.
$b^2 = ac$ {using property of geometric mean}
Now, apply log on both the sides we get,
$\log b^2 = \log (ac)$
$\log (b)^2 = \log a + \log c$
$2 \log b = \log a + \log c$
∴ log a, log b, log c are in A.P

2. If a, b, c are in G.P., prove that $1/\log_a m$, $1/\log_b m$, $1/\log_c m$ are in A.P.
Solution:
Given:
a, b and c are in GP
$b^2 = ac$ {property of geometric mean}
Apply log on both sides with base m
$\log_m b^2 = \log_m ac$
$\log_m b^2 = \log_m a + \log_m c$ {using property of log}
$2\log_m b = \log_m a + \log_m c$
$2/\log_b m = 1/\log_a m + 1/\log_c m$
∴ $1/\log_a m$, $1/\log_b m$, $1/\log_c m$ are in A.P.

3. Find k such that k + 9, k – 6 and 4 form three consecutive terms of a G.P.
Solution:
Let a = k + 9; b = k−6; and c = 4;
We know that a, b and c are in GP, then
$b^2 = ac$ {using property of geometric mean}
$(k - 6)^2 = 4(k + 9)$
$k^2 - 12k + 36 = 4k + 36$
$k^2 - 16k = 0$
k = 0 or k = 16

4. Three numbers are in A.P., and their sum is 15. If 1, 3, 9 be added to them respectively, they from a G.P. find the numbers.
Solution:

Let the first term of an A.P. be ‘a’ and its common difference be‘d’.
$a_1 + a_2 + a_3 = 15$
Where, the three number are: a, a + d, and a + 2d
So,
a + a + d + a + 2d = 15
3a + 3d = 15 or a + d = 5
d = 5 – a … (i)

Now, according to the question:
a + 1, a + d + 3, and a + 2d + 9
they are in GP, that is:
(a+d+3)/(a+1) = (a+2d+9)/(a+d+3)
$(a + d + 3)^2 = (a + 2d + 9) (a + 1)$
$a^2 + d^2 + 9 + 2ad + 6d + 6a = a^2 + a + 2da + 2d + 9a + 9$
$(5 – a)^2 – 4a + 4(5 – a) = 0$
$25 + a^2 – 10a – 4a + 20 – 4a = 0$
$a^2 – 18a + 45 = 0$
$a^2 – 15a – 3a + 45 = 0$
a(a – 15) – 3(a – 15) = 0
a = 3 or a = 15
d = 5 – a
d = 5 – 3 or d = 5 – 15
d = 2 or – 10
Then,
For a = 3 and d = 2, the A.P is 3, 5, 7
For a = 15 and d = -10, the A.P is 15, 5, -5
∴ The numbers are 3, 5, 7 or 15, 5, – 5

5. The sum of three numbers which are consecutive terms of an A.P. is 21. If the second number is reduced by 1 and the third is increased by 1, we obtain three consecutive terms of a G.P. Find the numbers.
Solution:
Let the first term of an A.P. be ‘a’ and its common difference be‘d’.
$a_1 + a_2 + a_3 = 21$
Where, the three number are: a, a + d, and a + 2d
So,
3a + 3d = 21 or
a + d = 7.
d = 7 – a …. (i)

Now, according to the question:
a, a + d – 1, and a + 2d + 1
they are now in GP, that is:
(a+d-1)/a = (a+2d+1)/(a+d-1)
$(a + d – 1)^2 = a(a + 2d + 1)$
$a^2 + d^2 + 1 + 2ad – 2d – 2a = a^2 + a + 2da$
$(7 – a)^2 – 3a + 1 – 2(7 – a) = 0$
$49 + a^2 – 14a – 3a + 1 – 14 + 2a = 0$
$a^2 – 15a + 36 = 0$
$a^2 – 12a – 3a + 36 = 0$
a(a – 12) – 3(a – 12) = 0
a = 3 or a = 12
d = 7 – a
d = 7 – 3 or d = 7 – 12
d = 4 or – 5
Then,
For a = 3 and d = 4, the A.P is 3, 7, 11
For a = 12 and d = -5, the A.P is 12, 7, 2
∴ The numbers are 3, 7, 11 or 12, 7, 2

6. The sum of three numbers a, b, c in A.P. is 18. If a and b are each increased by 4 and c is increased by 36, the new numbers form a G.P. Find a, b, c.
Solution:
Let the first term of an A.P. be 'a' and its common difference be'd'.
b = a + d; c = a + 2d.
Given:
a + b + c = 18
3a + 3d = 18 or a + d = 6.
d = 6 – a … (i)

Now, according to the question:
a + 4, a + d + 4, and a + 2d + 36
they are now in GP, that is:
(a+d+4)/(a+4) = (a+2d+36)/(a+d+4)
$(a + d + 4)^2 = (a + 2d + 36)(a + 4)$
$a^2 + d^2 + 16 + 8a + 2ad + 8d = a^2 + 4a + 2da + 36a + 144 + 8d$
$d^2 – 32a – 128 = 0$
$(6 – a)^2 – 32a – 128 = 0$

$36 + a^2 – 12a – 32a – 128 = 0$
$a^2 – 44a – 92 = 0$
$a^2 – 46a + 2a – 92 = 0$
$a(a – 46) + 2(a – 46) = 0$
$a = – 2$ or $a = 46$
$d = 6 – a$
$d = 6 – (– 2)$ or $d = 6 – 46$
$d = 8$ or $– 40$
Then,
For a = -2 and d = 8, the A.P is -2, 6, 14
For a = 46 and d = -40, the A.P is 46, 6, -34
∴ The numbers are – 2, 6, 14 or 46, 6, – 34

7. The sum of three numbers in G.P. is 56. If we subtract 1, 7, 21 from these numbers in that order, we obtain an A.P. Find the numbers.
Solution:
Let the three numbers be a, ar, ar^2
According to the question
$a + ar + ar^2 = 56$ … (1)
Let us subtract 1,7,21 we get,
$(a – 1), (ar – 7), (ar^2 – 21)$
The above numbers are in AP.
If three numbers are in AP, by the idea of the arithmetic mean, we can write 2b = a + c
$2 (ar - 7) = a – 1 + ar^2 – 21$
$= (ar^2 + a) – 22$
$2ar – 14 = (56 - ar) - 22$
$2ar – 14 = 34 – ar$
$3ar = 48$
$ar = 48/3$
$ar = 16$
$a = 16/r$ …. (2)
Now, substitute the value of a in equation (1) we get,
$(16 + 16r + 16r^2)/r = 56$
$16 + 16r + 16r^2 = 56r$
$16r^2 – 40r + 16 = 0$
$2r^2 – 5r + 2 = 0$
$2r^2 – 4r – r + 2 = 0$
$2r(r - 2) – 1(r - 2) = 0$
$(r - 2) (2r - 1) = 0$

r = 2 or 1/2
Substitute the value of r in equation (2) we get,
a = 16/r
= 16/2 or 16/(1/2)
= 8 or 32
∴ The three numbers are (a, ar, ar^2) is (8, 16, 32)

8. if a, b, c are in G.P., prove that:
(i) $a(b^2 + c^2) = c(a^2 + b^2)$
(ii) $a^2b^2c^2 [1/a^3 + 1/b^3 + 1/c^3] = a^3 + b^3 + c^3$
(iii) $(a+b+c)^2 / (a^2 + b^2 + c^2) = (a+b+c) / (a-b+c)$
(iv) $1/(a^2 – b^2) + 1/b^2 = 1/(b^2 – c^2)$
(v) $(a + 2b + 2c) (a – 2b + 2c) = a^2 + 4c^2$
Solution:
(i) $a(b^2 + c^2) = c(a^2 + b^2)$
Given that a, b, c are in GP.
By using the property of geometric mean,
$b^2 = ac$
Let us consider LHS: $a(b^2 + c^2)$
Now, substituting $b^2 = ac$, we get
$a(ac + c^2)$
$a^2c + ac^2$
$c(a^2 + ac)$
Substitute $ac = b^2$ we get,
$c(a^2 + b^2)$ = RHS
∴ LHS = RHS
Hence proved.

(ii) $a^2b^2c^2 [1/a^3 + 1/b^3 + 1/c^3] = a^3 + b^3 + c^3$
Given that a, b, c are in GP.
By using the property of geometric mean,
$b^2 = ac$
Let us consider LHS: $a^2b^2c^2 [1/a^3 + 1/b^3 + 1/c^3]$
$a^2b^2c^2/a^3 + a^2b^2c^2/b^3 + a^2b^2c^2/c^3$
$b^2c^2/a + a^2c^2/b + a^2b^2/c$
$(ac)c^2/a + (b^2)^2/b + a^2(ac)/c$ [by substituting the $b^2 = ac$]
$ac^3/a + b^4/b + a^3c/c$
$c^3 + b^3 + a^3$ = RHS
∴ LHS = RHS

Hence proved.

(iii) $(a+b+c)^2 / (a^2 + b^2 + c^2) = (a+b+c) / (a-b+c)$
Given that a, b, c are in GP.
By using the property of geometric mean,
$b^2 = ac$
Let us consider LHS: $(a+b+c)^2 / (a^2 + b^2 + c^2)$
$(a+b+c)^2 / (a^2 + b^2 + c^2) = (a+b+c)^2 / (a^2 – b^2 + c^2 + 2b^2)$
$= (a+b+c)^2 / (a^2 – b^2 + c^2 + 2ac)$ [Since, $b^2 = ac$]
$= (a+b+c)^2 / (a+b+c)(a-b+c)$ [Since, $(a+b+c)(a-b+c) = a^2 – b^2 + c^2 + 2ac$]
$= (a+b+c) / (a-b+c)$
= RHS
∴ LHS = RHS
Hence proved.

(iv) $1/(a^2 – b^2) + 1/b^2 = 1/(b^2 – c^2)$
Given that a, b, c are in GP.
By using the property of geometric mean,
$b^2 = ac$
Let us consider LHS: $1/(a^2 – b^2) + 1/b^2$
Let us take LCM
$1/(a^2 – b^2) + 1/b^2 = (b^2 + a^2 – b^2)/(a^2 – b^2)b^2$
$= a^2 / (a^2b^2 – b^4)$
$= a^2 / (a^2b^2 – (b^2)^2)$
$= a^2 / (a^2b^2 – (ac)^2)$ [Since, $b^2 = ac$]
$= a^2 / (a^2b^2 – a^2c^2)$
$= a^2 / a^2(b^2 – c^2)$
$= 1/ (b^2 – c^2)$
= RHS
∴ LHS = RHS
Hence proved.

(v) $(a + 2b + 2c) (a – 2b + 2c) = a^2 + 4c^2$
Given that a, b, c are in GP.
By using the property of geometric mean,
$b^2 = ac$
Let us consider LHS: $(a + 2b + 2c) (a – 2b + 2c)$
Upon expansion we get,
$(a + 2b + 2c) (a – 2b + 2c) = a^2 – 2ab + 2ac + 2ab – 4b^2 + 4bc + 2ac – 4bc + 4c^2$

$= a^2 + 4ac – 4b^2 + 4c^2$
$= a^2 + 4ac – 4(ac) + 4c^2$ [Since, $b^2 = ac$]
$= a^2 + 4c^2$
= RHS

∴ LHS = RHS
Hence proved.

9. If a, b, c, d are in G.P., prove that:
(i) $(ab - cd) / (b^2 – c^2) = (a + c) / b$
(ii) $(a + b + c + d)^2 = (a + b)^2 + 2(b + c)^2 + (c + d)^2$
(iii) (b + c) (b + d) = (c + a) (c + d)
Solution:
(i) $(ab - cd) / (b^2 – c^2) = (a + c) / b$
Given that a, b, c are in GP.
By using the property of geometric mean,
$b^2 = ac$
$bc = ad$
$c^2 = bd$
Let us consider LHS: $(ab - cd) / (b^2 – c^2)$
$(ab - cd) / (b^2 – c^2) = (ab - cd) / (ac - bd)$
$= (ab - cd)b / (ac - bd)b$
$= (ab^2 – bcd) / (ac - bd)b$
$= [a(ac) – c(c^2)] / (ac - bd)b$
$= (a^2c – c^3) / (ac - bd)b$
$= [c(a^2 – c^2)] / (ac - bd)b$
$= [(a+c) (ac – c^2)] / (ac - bd)b$
$= [(a+c) (ac - bd)] / (ac - bd)b$
$= (a+c) / b$
= RHS

∴ LHS = RHS
Hence proved.

(ii) $(a + b + c + d)^2 = (a + b)^2 + 2(b + c)^2 + (c + d)^2$
Given that a, b, c are in GP.
By using the property of geometric mean,
$b^2 = ac$
$bc = ad$
$c^2 = bd$
Let us consider RHS: $(a + b)^2 + 2(b + c)^2 + (c + d)^2$

Let us expand

$(a + b)^2 + 2(b + c)^2 + (c + d)^2 = (a + b)^2 + 2 (a+b) (c+d) + (c+d)^2$

$= a^2 + b^2 + 2ab + 2(c^2 + b^2 + 2cb) + c^2 + d^2 + 2cd$

$= a^2 + b^2 + c^2 + d^2 + 2ab + 2(c^2 + b^2 + 2cb) + 2cd$

$= a^2 + b^2 + c^2 + d^2 + 2(ab + bd + ac + cb +cd)$ [Since, $c^2 = bd$, $b^2 = ac$]

You can visualize the above expression by making separate terms for $(a + b + c)^2 + d^2 + 2d(a + b + c) = \{(a + b + c) + d\}^2$

∴ RHS = LHS

Hence proved.

(iii) $(b + c) (b + d) = (c + a) (c + d)$

Given that a, b, c are in GP.

By using the property of geometric mean,

$b^2 = ac$

$bc = ad$

$c^2 = bd$

Let us consider LHS: $(b + c) (b + d)$

Upon expansion we get,

$(b + c) (b + d) = b^2 + bd + cb + cd$

$= ac + c^2 + ad + cd$ [by using property of geometric mean]

$= c (a + c) + d (a + c)$

$= (a + c) (c + d)$

= RHS

∴ LHS = RHS

Hence proved.

10. If a, b, c are in G.P., prove that the following are also in G.P.:

(i) a^2, b^2, c^2

(ii) a^3, b^3, c^3

(iii) $a^2 + b^2, ab + bc, b^2 + c^2$

Solution:

(i) a^2, b^2, c^2

Given that a, b, c are in GP.

By using the property of geometric mean,

$b^2 = ac$

on squaring both the sides we get,

$(b^2)^2 = (ac)^2$

$(b^2)^2 = a^2c^2$

∴ a^2, b^2, c^2 are in G.P.

(ii) a^3, b^3, c^3

Given that a, b, c are in GP.

By using the property of geometric mean,

$b^2 = ac$

on squaring both the sides we get,

$(b^2)^3 = (ac)^3$

$(b^2)^3 = a^3c^3$

$(b^3)^2 = a^3c^3$

∴ a^3, b^3, c^3 are in G.P.

(iii) $a^2 + b^2, ab + bc, b^2 + c^2$

Given that a, b, c are in GP.

By using the property of geometric mean,

$b^2 = ac$

$a^2 + b^2, ab + bc, b^2 + c^2$ or $(ab + bc)^2 = (a^2 + b^2)(b^2 + c^2)$ [by using the property of GM]

Let us consider LHS: $(ab + bc)^2$

Upon expansion we get,

$(ab + bc)^2 = a^2b^2 + 2ab^2c + b^2c^2$

$= a^2b^2 + 2b^2(b^2) + b^2c^2$ [Since, $ac = b^2$]

$= a^2b^2 + 2b^4 + b^2c^2$

$= a^2b^2 + b^4 + a^2c^2 + b^2c^2$ {again using $b^2 = ac$ }

$= b^2(b^2 + a^2) + c^2(a^2 + b^2)$

$= (a^2 + b^2)(b^2 + c^2)$

= RHS

∴ LHS = RHS

Hence $a^2 + b^2, ab + bc, b^2 + c^2$ are in GP.

EXERCISE 20.6 PAGE NO: 20.54

1. Insert 6 geometric means between 27 and 1/81.

Solution:

Let the six terms be $a_1, a_2, a_3, a_4, a_5, a_6$.

A = 27, B = 1/81

Now, these 6 terms are between A and B.

So the GP is: A, $a_1, a_2, a_3, a_4, a_5, a_6$, B.

So we now have 8 terms in GP with the first term being 27 and eighth being 1/81.

We know that, $T_n = ar^{n-1}$

Here, $T_n = 1/81$, a = 27 and

$1/81 = 27r^{8-1}$

$1/(81\times27) = r^7$

r = 1/3

$a_1 = Ar = 27\times1/3 = 9$

$a_2 = Ar^2 = 27\times1/9 = 3$

$a_3 = Ar^3 = 27\times1/27 = 1$

$a_4 = Ar^4 = 27\times1/81 = 1/3$

$a_5 = Ar^5 = 27\times1/243 = 1/9$

$a_6 = Ar^6 = 27\times1/729 = 1/27$

∴ The six GM between 27 and 1/81 are 9, 3, 1, 1/3, 1/9, 1/27

2. Insert 5 geometric means between 16 and 1/4.

Solution:

Let the five terms be a_1, a_2, a_3, a_4, a_5.

A = 16, B = 1/4

Now, these 5 terms are between A and B.

So the GP is: A, a_1, a_2, a_3, a_4, a_5, B.

So we now have 7 terms in GP with the first term being 16 and seventh being 1/4.

We know that, $T_n = ar^{n-1}$

Here, $T_n = 1/4$, a = 16 and

$1/4 = 16r^{7-1}$

$1/(4\times16) = r^6$

r = 1/2

$a_1 = Ar = 16\times1/2 = 8$

$a_2 = Ar^2 = 16\times1/4 = 4$

$a_3 = Ar^3 = 16\times1/8 = 2$

$a_4 = Ar^4 = 16\times1/16 = 1$

$a_5 = Ar^5 = 16\times1/32 = 1/2$

∴ The five GM between 16 and 1/4 are 8, 4, 2, 1, ½

3. Insert 5 geometric means between 32/9 and 81/2.
Solution:
Let the five terms be a_1, a_2, a_3, a_4, a_5.
A = 32/9, B = 81/2
Now, these 5 terms are between A and B.
So the GP is: A, a_1, a_2, a_3, a_4, a_5, B.
So we now have 7 terms in GP with the first term being 32/9 and seventh being 81/2.
We know that, $T_n = ar^{n-1}$
Here, $T_n = 81/2$, a = 32/9 and
$81/2 = 32/9r^{7-1}$
$(81\times9)/(2\times32) = r^6$
r = 3/2
$a_1 = Ar = (32/9)\times3/2 = 16/3$
$a_2 = Ar^2 = (32/9)\times9/4 = 8$
$a_3 = Ar^3 = (32/9)\times27/8 = 12$
$a_4 = Ar^4 = (32/9)\times81/16 = 18$
$a_5 = Ar^5 = (32/9)\times243/32 = 27$
∴ The five GM between 32/9 and 81/2 are 16/3, 8, 12, 18, 27

4. Find the geometric means of the following pairs of numbers:
(i) 2 and 8
(ii) a^3b and ab^3
(iii) –8 and –2
Solution:
(i) 2 and 8
GM between a and b is √ab
Let a = 2 and b =8
GM = √2×8
= √16
= 4

(ii) a^3b and ab^3
GM between a and b is √ab
Let a = a^3b and b = ab^3
GM = $\sqrt{(a^3b \times ab^3)}$
= $\sqrt{a^4b^4}$
= a^2b^2

(iii) –8 and –2
GM between a and b is $\sqrt{ab}$
Let a = –2 and b = –8
GM = $\sqrt{(-2\times-8)}$
= $\sqrt{16}$
= 4, -4

5. If a is the G.M. of 2 and ¼ find a.
Solution:
We know that GM between a and b is $\sqrt{ab}$
Let a = 2 and b = 1/4
GM = $\sqrt{(2\times1/4)}$
= $\sqrt{(1/2)}$
= $1/\sqrt{2}$
∴ value of a is $1/\sqrt{2}$

6. Find the two numbers whose A.M. is 25 and GM is 20.
Solution:
Given: A.M = 25, G.M = 20.
G.M = $\sqrt{ab}$
A.M = (a+b)/2
So,
$\sqrt{ab}$ = 20 ……. (1)
(a+b)/2 = 25……. (2)
a + b = 50
a = 50 – b
Putting the value of 'a' in equation (1), we get,
$\sqrt{[(50-b)b]} = 20$
$50b - b^2 = 400$
$b^2 - 50b + 400 = 0$
$b^2 - 40b - 10b + 400 = 0$
b(b – 40) – 10(b – 40) = 0
b = 40 or b = 10
If b = 40 then a = 10
If b = 10 then a = 40
∴ The numbers are 10 and 40.

7. Construct a quadratic equation in x such that A.M. of its roots is A and G.M. is G.

Solution:

Let the root of the quadratic equation be a and b.

So, according to the given condition,

A.M = (a+b)/2 = A

a + b = 2A ….. (1)

GM = $\sqrt{ab}$ = G

ab = G^2… (2)

The quadratic equation is given by,

x^2 – x (Sum of roots) + (Product of roots) = 0

x^2 – x (2A) + (G^2) = 0

x^2 – 2Ax + G^2 = 0 [Using (1) and (2)]

∴ The required quadratic equation is x^2 – 2Ax + G^2 = 0.