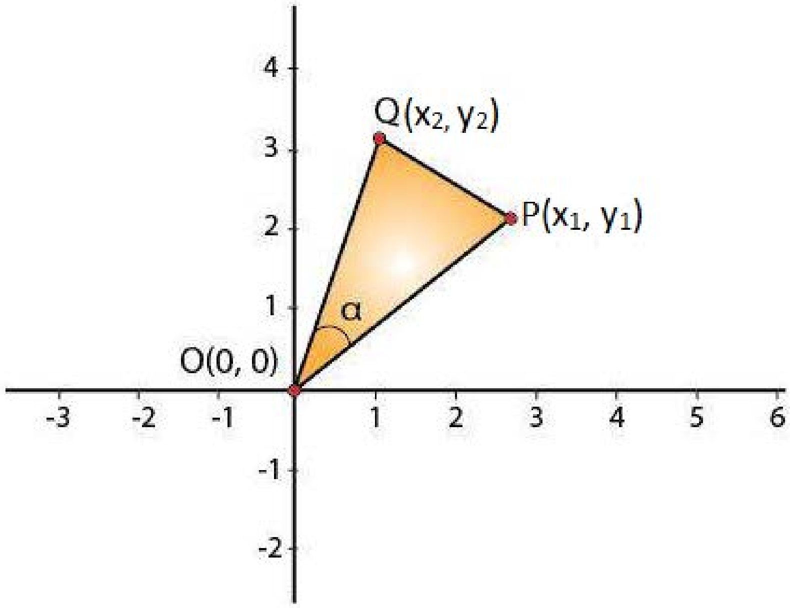
EXERCISE 22.1 PAGE NO: 22.12

1. If the line segment joining the points $P(x_1, y_1)$ and $Q(x_2, y_2)$ subtends an angle α at the origin O, prove that : OP. OQ $\cos \alpha = x_1 x_2 + y_1 y_2$.

Solution:

Given,

Two points P and Q subtends an angle α at the origin as shown in figure:



From figure we can see that points O, P and Q forms a triangle.

Clearly in ΔOPQ we have:

$\cos\alpha = \frac{OP^2+OQ^2-PQ^2}{2OP.OQ}$ {from cosine formula}

$2\ OP.OQ\cos\alpha = OP^2 + OQ^2 - PQ^2$ … equation (1)

We know that the, coordinates of O are (0, 0) $\Rightarrow x_2 = 0$ and $y_2 = 0$

Coordinates of P are $(x_1, y_1) \Rightarrow x_1 = x_1$ and $y_1 = y_1$

By using distance formula we have:

$OP = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$

$= \sqrt{(x_1-0)^2+(y_1-0)^2}$

$= \sqrt{x_1^2+y_1^2}$

Similarly, $OQ = \sqrt{(x_2-0)^2+(y_2-0)^2}$

$= \sqrt{x_2^2+y_2^2}$

And, $PQ = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$

$\therefore OP^2 + OQ^2 - PQ^2 = x_1^2+y_1^2+\ x_2^2+y_2^2-\{(x_2-x_1)^2+(y_2-y_1)^2\}$

By using $(a-b)^2 = a^2 + b^2 - 2ab$

$\therefore OP^2 + OQ^2 - PQ^2 = 2x_1\ x_2 + 2y_1\ y_2$ …. Equation (2)

So now from equation (1) and (2) we have:

$2OP.\,OQ\cos\alpha = 2x_1x_2+2y_1y_2$

$OP.\,OQ\cos\alpha = x_1x_2+y_1y_2$

Hence Proved.

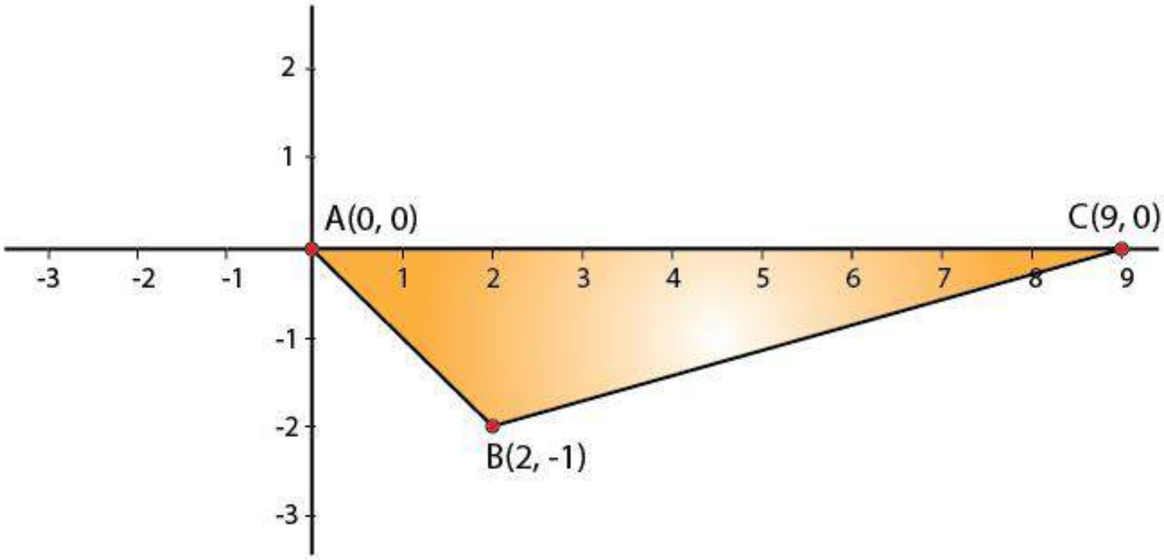
2. The vertices of a triangle ABC are A(0, 0), B (2, -1) and C (9, 0). Find cos B.

Solution:

Given:

The coordinates of triangle.

From the figure,



By using cosine formula,
In ΔABC, we have:

$\cos B = \frac{AB^2+BC^2-AC^2}{2AB.BC}$

Now by using distance formula we have:

$AB = \sqrt{(2-0)^2+(-1-0)^2} = \sqrt{5}$

$BC = \sqrt{(9-2)^2+(0-(-1))^2} = \sqrt{7^2+1^2} = \sqrt{50}$

And, $AC = \sqrt{(9-0)^2+(0-0)^2} = 9$

Now substitute the obtained values in the cosine formula, we get

$\therefore \cos B = \frac{(\sqrt{5})^2+(\sqrt{50})^2-9^2}{2\sqrt{5}\sqrt{50}} = \frac{55-81}{2\sqrt{5}\sqrt{2\times25}} = \frac{-26}{10\sqrt{10}} = \frac{-13}{5\sqrt{10}}$

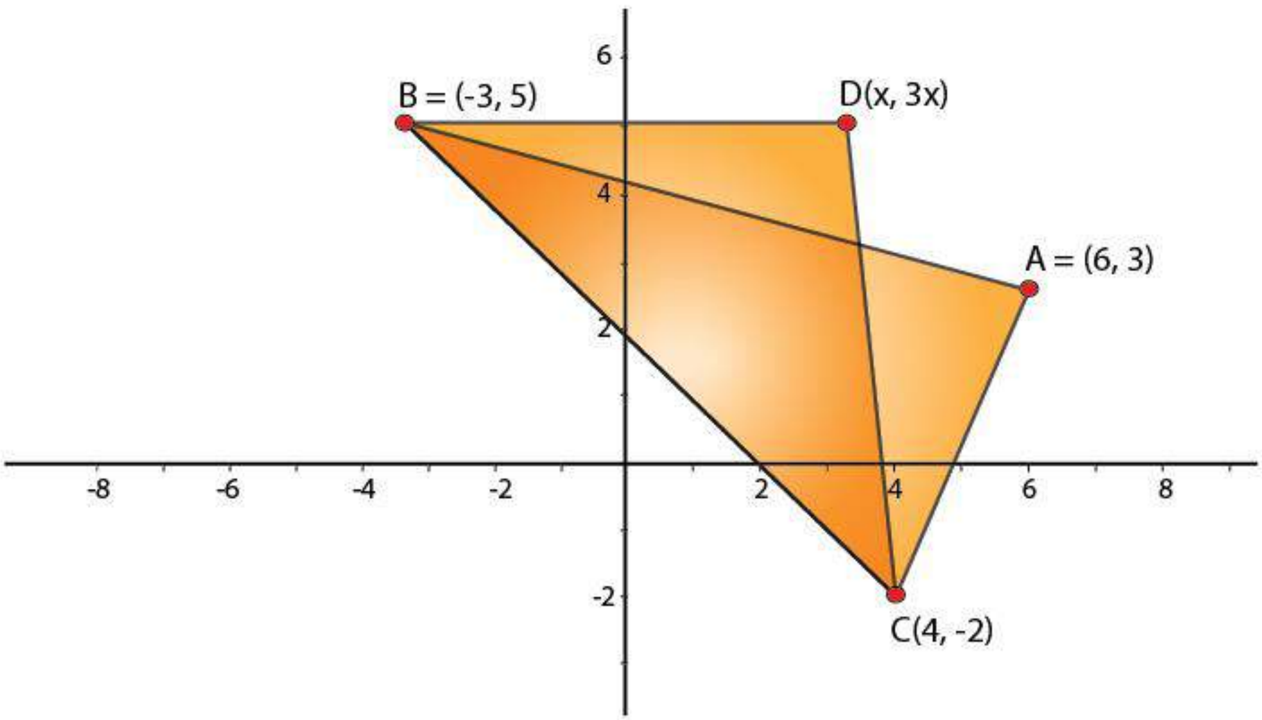
3. Four points A (6, 3), B (-3, 5), C (4, -2) and D (x, 3x) are given in such a way that $\frac{\Delta DBC}{\Delta ABC} = \frac{1}{2}$, find x.

Solution:

Given:

The coordinates of triangle are shown in the below figure.

Also, $\frac{\Delta DBC}{\Delta ABC} = \frac{1}{2}$



Now let us consider Area of a ΔPQR

Where, $P(x_1, y_1)$, $Q(x_2, y_2)$ and $R(x_3, y_3)$ be the 3 vertices of ΔPQR.

So, Area of (ΔPQR) = ½ $[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]$

Area of (ΔDBC) = $\frac{1}{2}[x(5 - (-2)) + (-3)(-2 - 3x) + 4(3x - 5)]$

$= \frac{1}{2}[7x + 6 + 9x + 12x - 20] = 14x - 7$

Similarly, area of (ΔABC) = $\frac{1}{2}[6(5 - (-2)) + (-3)(-2 - 3) + 4(3 - 5)]$

$= \frac{1}{2}[42 + 15 - 8] = \frac{49}{2} = 24.5$

$\therefore \frac{\Delta DBC}{\Delta ABC} = \frac{1}{2} = \frac{14x-7}{24.5}$

24.5 = 28x – 14

28x = 38.5

x = 38.5/28

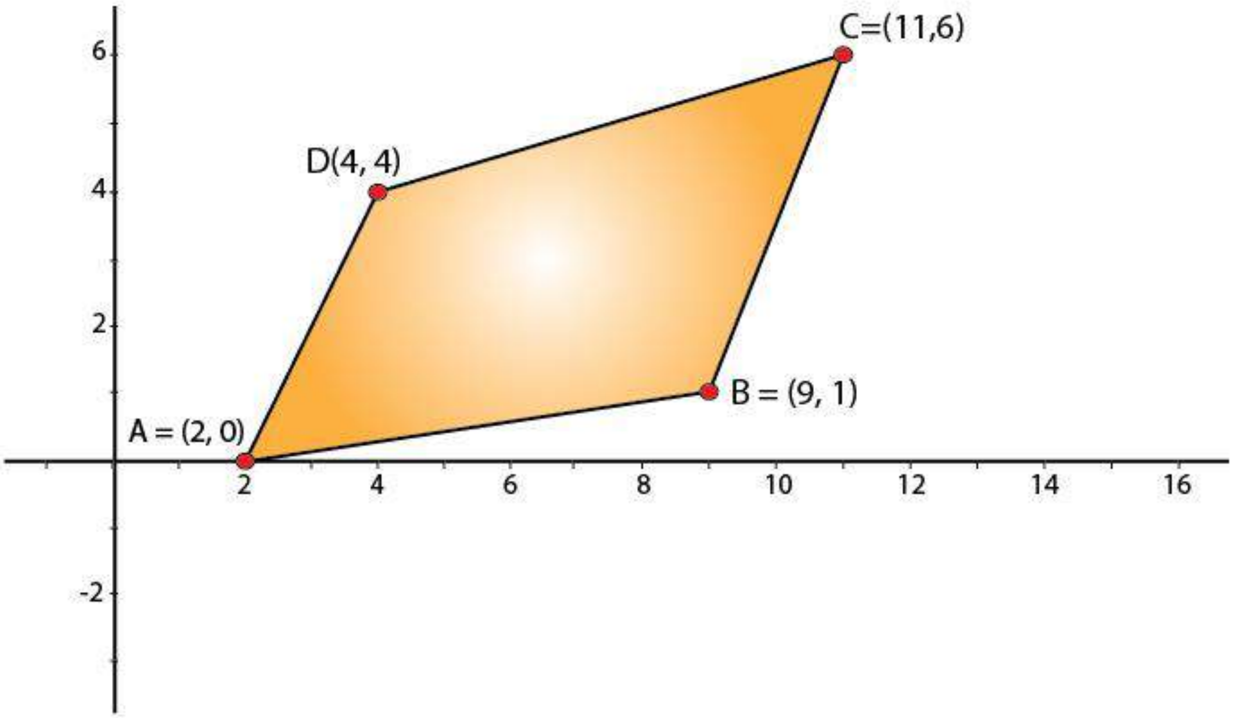
= 1.375

4. The points A (2, 0), B (9, 1), C (11, 6) and D (4, 4) are the vertices of a quadrilateral ABCD. Determine whether ABCD is a rhombus or not.

Solution:

Given:

The coordinates of 4 points that form a quadrilateral is shown in the below figure



Now by using distance formula, we have:

AB = $\sqrt{(9-2)^2+(1-0)^2} = \sqrt{7^2+1} = \sqrt{50}$

BC = $\sqrt{(11-9)^2+(6-1)^2} = \sqrt{2^2+5^2} = \sqrt{29}$

It is clear that, AB ≠ BC [quad ABCD does not have all 4 sides equal.]

∴ ABCD is not a Rhombus

EXERCISE 22.2 PAGE NO: 22.18

1. Find the locus of a point equidistant from the point (2, 4) and the y-axis.

Solution:

Let P (h, k) be any point on the locus and let A (2, 4) and B (0, k).

Then, PA = PB

$PA^2 = PB^2$

By using distance formula:

Distance of (h, k) from (2, 4) = $\sqrt{(h-2)^2+(k-4)^2}$

Distance of (h, k) from (0, k) = $\sqrt{(h-0)^2+(k-k)^2}$

So both the distances are same.

$\therefore \sqrt{(h-2)^2+(k-4)^2} = \sqrt{(h-0)^2+(k-k)^2}$

By squaring on both the sides we get,

$(h-2)^2+(k-4)^2 = (h-0)^2+(k-k)^2$

$h^2+4-4h+k^2-8k+16 = h^2+0$

$k^2-4h-8k+20 = 0$

Replace (h, k) with (x, y)

∴ The locus of point equidistant from (2, 4) and y-axis is

$y^2-4x-8y+20 = 0$

2. Find the equation of the locus of a point which moves such that the ratio of its distance from (2, 0) and (1, 3) is 5: 4.

Solution:

Let P (h, k) be any point on the locus and let A (2, 0) and B (1, 3).

So then, PA/ BP = 5/4

$PA^2 = BP^2 = 25/16$

Distance of (h, k) from (2, 0) = $\sqrt{(h-2)^2+(k-0)^2}$

Distance of (h, k) from (1, 3) = $\sqrt{(h-1)^2+(k-3)^2}$

So,

$$\frac{\sqrt{(h-2)^2+(k-0)^2}}{\sqrt{(h-1)^2+(k-3)^2}} = \frac{5}{4}$$

By squaring on both the sides we get,

$16\{(h-2)^2+k^2\} = 25\{(h-1)^2+(k-3)^2\}$

$16\{h^2+4-4h+k^2\} = 25\{h^2-2h+1+k^2-6k+9\}$

$9h^2 + 9k^2 + 14h - 150k + 186 = 0$

Replace (h, k) with (x, y)

∴ The locus of a point which moves such that the ratio of its distance from (2, 0) and (1, 3) is 5: 4 which is

$9x^2 + 9y^2 + 14x - 150y + 186 = 0$

3. A point moves as so that the difference of its distances from (ae, 0) and (-ae, 0) is 2a, prove that the equation to its locus is

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ **, where $b^2 = a^2(e^2 - 1)$.**

Solution:

Let P (h, k) be any point on the locus and let A (ae, 0) and B (-ae, 0).

Where, PA – PB = 2a

Distance of (h, k) from (ae, 0) $= \sqrt{(h-ae)^2 + (k-0)^2}$

Distance of (h, k) from (-ae, 0) $= \sqrt{(h-(-ae))^2 + (k-0)^2}$

So,

$$\sqrt{(h-ae)^2 + (k-0)^2} - \sqrt{(h-(-ae))^2 + (k-0)^2} = 2a$$

$$\sqrt{(h-ae)^2 + (k-0)^2} = 2a + \sqrt{(h+ae)^2 + (k-0)^2}$$

By squaring on both the sides we get:

$$(h-ae)^2 + (k-0)^2 = \left\{2a + \sqrt{(h+ae)^2 + (k-0)^2}\right\}^2$$

$$\Rightarrow h^2 + a^2e^2 - 2aeh + k^2 = 4a^2 + \{(h+ae)^2 + k^2\} + 4a\sqrt{(h+ae)^2 + (k-0)^2}$$

$$\Rightarrow h^2 + a^2e^2 - 2aeh + k^2$$
$$= 4a^2 + h^2 + 2aeh + a^2e^2 + k^2 + 4a\sqrt{(h+ae)^2 + (k-0)^2}$$

$$-4aeh - 4a^2 = 4a\sqrt{(h+ae)^2 + (k-0)^2}$$

$$-4a(eh + a) = 4a\sqrt{(h+ae)^2 + (k-0)^2}$$

Now again let us square on both the sides we get,

$(eh + a)^2 = (h + ae)^2 + (k - 0)^2$

$e^2h^2 + a^2 + 2aeh = h^2 + a^2e^2 + 2aeh + k^2$

$h^2(e^2 - 1) - k^2 = a^2(e^2 - 1)$

$$\frac{h^2}{a^2} - \frac{k^2}{a^2(e^2-1)} = 1$$

$$\frac{h^2}{a^2} - \frac{k^2}{b^2} = 1 \ [where, b^2 = a^2(e^2-1)]$$

Now let us replace (h, k) with (x, y)
The locus of a point such that the difference of its distances from (ae, 0) and (-ae, 0) is 2a.
$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ Where $b^2 = a^2 (e^2 – 1)$
Hence proved.

4. Find the locus of a point such that the sum of its distances from (0, 2) and (0, -2) is 6.
Solution:
Let P (h, k) be any point on the locus and let A (0, 2) and B (0, -2).
Where, PA – PB = 6

Distance of (h, k) from (0, 2) = $\sqrt{(h-0)^2+(k-2)^2}$

Distance of (h, k) from (0, -2) = $\sqrt{(h-0)^2+(k-(-2))^2}$

So,

$\sqrt{(h)^2+(k-2)^2} + \sqrt{(h)^2+(k+2)^2} = 6$

$\sqrt{(h)^2+(k-2)^2} = 6 - \sqrt{(h)^2+(k+2)^2}$

By squaring on both the sides we get,

$h^2+(k-2)^2 = \left\{6 - \sqrt{h^2+(k+2)^2}\right\}^2$

$\Rightarrow h^2+4-4k+k^2 = 36+\{h^2+k^2+4k+4\} - 12\sqrt{h^2+(k+2)^2}$

$\Rightarrow -8k-36 = -12\sqrt{h^2+(k+2)^2}$

$\Rightarrow -4(2k+9) = -12\sqrt{h^2+(k+2)^2}$

Now, again let us square on both the sides we get,

$(2k+9)^2 = \left\{3\sqrt{h^2+(k+2)^2}\right\}^2$

$4k^2+81+36k = 9(h^2+k^2+4k+4)$

$9h^2+5k^2 = 45$

By replacing (h, k) with (x, y)

∴ The locus of a point is

$9x^2+5y^2 = 45$

5. Find the locus of a point which is equidistant from (1, 3) and x-axis.
Solution:
Let P (h, k) be any point on the locus and let A (1, 3) and B (h, 0).
Where, PA = PB

Distance of (h, k) from (1, 3) = $\sqrt{(h-1)^2+(k-3)^2}$

Distance of (h, k) from (h, 0) = $\sqrt{(h-h)^2+(k-0)^2}$

It is given that both distance are same.

So,

$\sqrt{(h-1)^2+(k-3)^2} = \sqrt{(h-h)^2+(k-0)^2}$

Now, let us square on both the sides we get,

$(h-1)^2+(k-3)^2 = (h-h)^2+(k-0)^2$

$h^2+1-2h+k^2-6k+9 = k^2+0$

$h^2-2h-6k+10 = 0$

By replacing (h, k) with (x, y),

∴ The locus of point equidistant from (1, 3) and x-axis is

$x^2-2x-6y+10 = 0$

6. Find the locus of a point which moves such that its distance from the origin is three times is distance from x-axis.

Solution:

Let P (h, k) be any point on the locus and let A (0, 0) and B (h, 0).

Where, PA = 3PB

Distance of (h, k) from (0, 0) = $\sqrt{(h-0)^2+(k-0)^2}$

Distance of (h, k) from (h, 0) = $\sqrt{(h-h)^2+(k-0)^2}$

So, where PA = 3PB

$\therefore \sqrt{(h-0)^2+(k-0)^2} = 3\sqrt{(h-h)^2+(k-0)^2}$

Now by squaring on both the sides we get,

$h^2 + k^2 = 9k^2$

$h^2 = 8k^2$

By replacing (h, k) with (x, y)

∴ The locus of point is $x^2 = 8y^2$

EXERCISE 22.3 PAGE NO: 22.21

1. What does the equation $(x – a)^2 + (y – b)^2 = r^2$ become when the axes are transferred to parallel axes through the point (a-c, b)?

Solution:

Given:

The equation, $(x – a)^2 + (y – b)^2 = r^2$

The given equation $(x – a)^2 + (y – b)^2 = r^2$ can be transformed into the new equation by changing x by x – a + c and y by y – b, i.e. substitution of x by x + a and y by y + b.

$((x + a - c) – a)^2 + ((y + b) - b)^2 = r^2$

$(x – c)^2 + y^2 = r^2$

$x^2 + c^2 – 2cx + y^2 = r^2$

$x^2 + y^2 - 2cx = r^2 - c^2$

Hence, the transformed equation is $x^2 + y^2 - 2cx = r^2 - c^2$

2. What does the equation $(a – b)(x^2 + y^2) – 2abx = 0$ become if the origin is shifted to the point (ab / (a-b), 0) without rotation?

Solution:

Given:

The equation $(a – b)(x^2 + y^2) – 2abx = 0$

The given equation $(a – b)(x^2 + y^2) – 2abx = 0$ can be transformed into new equation by changing x by [X + ab / (a-b)] and y by Y

$$(a-b)\left[\left(X+\frac{ab}{a-b}\right)^2+Y^2\right]-2ab\times\left(X+\frac{ab}{a-b}\right)=0$$

Upon expansion we get,

$$(a-b)\left(X^2+\frac{a^2b^2}{(a-b)^2}+\frac{2abX}{a-b}+Y^2\right)-2abX-\frac{2a^2b^2}{a-b}=0$$

Now let us simplify,

$$(a-b)\left(X^2+Y^2\right)+\frac{a^2b^2}{a-b}+2abX-2abX-\frac{2a^2b^2}{a-b}=0$$

$$(a-b)\left(X^2+Y^2\right)-\frac{a^2b^2}{a-b}=0$$

By taking LCM we get,

$$(a-b)^2\left(X^2+Y^2\right)=a^2b^2$$

Hence, the transformed equation is $(a – b)^2 (X^2 + Y^2) = a^2 b^2$

3. Find what the following equations become when the origin is shifted to the point

(1, 1)?
(i) $x^2 + xy - 3x - y + 2 = 0$
(ii) $x^2 - y^2 - 2x + 2y = 0$
(iii) $xy - x - y + 1 = 0$
(iv) $xy - y^2 - x + y = 0$
Solution:
(i) $x^2 + xy - 3x - y + 2 = 0$
Firstly let us substitute the value of x by x + 1 and y by y + 1
Then,
$(x + 1)^2 + (x + 1)(y + 1) - 3(x + 1) - (y + 1) + 2 = 0$
$x^2 + 1 + 2x + xy + x + y + 1 - 3x - 3 - y - 1 + 2 = 0$
Upon simplification we get,
$x^2 + xy = 0$
∴ The transformed equation is $x^2 + xy = 0$.

(ii) $x^2 - y^2 - 2x + 2y = 0$
Let us substitute the value of x by x + 1 and y by y + 1
Then,
$(x + 1)^2 - (y + 1)^2 - 2(x + 1) + 2(y + 1) = 0$
$x^2 + 1 + 2x - y^2 - 1 - 2y - 2x - 2 + 2y + 2 = 0$
Upon simplification we get,
$x^2 - y^2 = 0$
∴ The transformed equation is $x^2 - y^2 = 0$.

(iii) $xy - x - y + 1 = 0$
Let us substitute the value of x by x + 1 and y by y + 1
Then,
$(x + 1)(y + 1) - (x + 1) - (y + 1) + 1 = 0$
$xy + x + y + 1 - x - 1 - y - 1 + 1 = 0$
Upon simplification we get,
$xy = 0$
∴ The transformed equation is $xy = 0$.

(iv) $xy - y^2 - x + y = 0$
Let us substitute the value of x by x + 1 and y by y + 1
Then,
$(x + 1)(y + 1) - (y + 1)^2 - (x + 1) + (y + 1) = 0$
$xy + x + y + 1 - y^2 - 1 - 2y - x - 1 + y + 1 = 0$
Upon simplification we get,

$xy - y^2 = 0$

∴ The transformed equation is $xy - y^2 = 0$.

4. At what point the origin be shifted so that the equation $x^2 + xy – 3x + 2 = 0$ does not contain any first-degree term and constant term?

Solution:

Given:

The equation $x^2 + xy – 3x + 2 = 0$

We know that the origin has been shifted from (0, 0) to (p, q)

So any arbitrary point (x, y) will also be converted as (x + p, y + q).

The new equation is:

$(x + p)^2 + (x + p)(y + q) – 3(x + p) + 2 = 0$

Upon simplification,

$x^2 + p^2 + 2px + xy + py + qx + pq – 3x – 3p + 2 = 0$

$x^2 + xy + x(2p + q – 3) + y(q – 1) + p^2 + pq – 3p – q + 2 = 0$

For no first degree term, we have $2p + q - 3 = 0$ and $p – 1 = 0$, and

For no constant term we have $p^2 + pq – 3p - q + 2 = 0$.

By solving these simultaneous equations we have p = 1 and q = 1 from first equation.

The values p = 1 and q = 1 satisfies $p^2 + pq – 3p - q + 2 = 0$.

Hence, the point to which origin must be shifted is (p, q) = (1, 1).

5. Verify that the area of the triangle with vertices (2, 3), (5, 7) and (-3 -1) remains invariant under the translation of axes when the origin is shifted to the point (-1, 3).

Solution:

Given:

The points (2, 3), (5, 7), and (-3, -1).

The area of triangle with vertices (x_1, y_1), (x_2, y_2), and (x_3, y_3) is

$= ½ [x_1(y_2 – y_3) + x_2(y_3 - y_1) + x_3(y_1 – y_2)]$

The area of given triangle $= ½ [2(7+1) + 5(-1-3) – 3(3-7)]$

$= ½ [16 – 20 + 12]$

$= ½ [8]$

$= 4$

Origin shifted to point (-1, 3), the new coordinates of the triangle are (3, 0), (6, 4), and (-2, -4) obtained from subtracting a point (-1, 3).

The new area of triangle $= ½ [3(4-(-4)) + 6(-4-0) – 2(0-4)]$

$= ½ [24-24+8]$

$= ½ [8]$

$= 4$

Since the area of the triangle before and after the translation after shifting of origin

remains same, i.e. 4.
∴ We can say that the area of a triangle is invariant to shifting of origin.