EXERCISE 29.1 PAGE NO: 29.11

1. Show that $\lim_{x\to 0}\frac{x}{|x|}$ does not exist.

Solution:

Firstly let us consider LHS:

$$\lim_{x\to 0^-}\left(\frac{x}{|x|}\right)$$

So, let x = 0 – h, where, h = 0

$$\lim_{x\to 0}\frac{x}{|x|} = \lim_{h\to 0}\left(\frac{0-h}{|0-h|}\right)$$

$$= \lim_{h\to 0}\left(\frac{-h}{h}\right)$$

$$= -1$$

Now, let us consider RHS:

$$\lim_{x\to 0^+}\frac{(x)}{|x|}$$

So, let x = 0 + h, where, h = 0

$$\lim_{x\to 0}\frac{x}{|x|} = \lim_{h\to 0}\left(\frac{0+h}{|0+h|}\right)$$

$$= \lim_{h\to 0}\left(\frac{h}{h}\right)$$

$$= 1$$

Since LHS ≠ RHS

∴ Limit does not exist.

2. Find k so that $\lim_{x\to 2} f(x)$ may exist, where $f(x) = \begin{cases} 2x+3, x \le 2 \\ x+k,\ x > 2 \end{cases}$

Solution:

Firstly let us consider LHS:

$$\lim_{x\to 2^-} f(x)$$

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-}(2x+3)$$

So, let x = 2 – h, where h = 0

Substituting the value of x, we get

$$\lim_{h\to 0}[2(2-h)+3]$$

=> 2 (2 - 0) + 3 = 7

Now let us consider RHS:

$$\lim_{x\to 2^+} f(x)$$

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+}(x+k)$$

So, let x = 2 + h, where, h = 0

$$\lim_{h\to 0}(2+h+k)$$

=> 2 + 0 + k = 2 + k

Since, Limit exists, LHS = RHS

7 = 2 + k

k = 7 – 2

= 5

∴ Value of k is 5.

3. Show that $\lim_{x\to 0}\frac{1}{x}$ does not exist.

Solution:

Firstly let us consider LHS:

$$\lim_{x\to 0^-}\left(\frac{1}{x}\right)$$

So, let x = 0 – h, where h = 0

$$\lim_{x\to 0^-}\left(\frac{1}{x}\right) = \lim_{h\to 0}\left(\frac{1}{0-h}\right)$$

$$= -\infty$$

Now, let us consider RHS:

$$\lim_{x\to 0^+}\left(\frac{1}{x}\right)$$

So, let x = 0 + h, where h = 0

$$\lim_{x\to 0^+}\left(\frac{1}{x}\right) = \lim_{h\to 0}\left(\frac{1}{0+h}\right)$$

$$= \infty$$

Since, LHS ≠ RHS

∴ Limit does not exist.

4. Let f(x) be a function defined by $f(x)=\begin{cases}\dfrac{3x}{|x|+2x}, & x\neq 0\\ 0, & x=0\end{cases}$

Show that $\lim_{x\to 0} f(x)$ **does not exist.**

Solution:

Firstly let us consider LHS:

$$\lim_{x\to 0^-}\left[\frac{3x}{|x|+2x}\right]$$

So, let x = 0 – h, where h = 0

Substituting the value of x, we get

$$\lim_{x\to 0^-}\left[\frac{3x}{|x|+2x}\right]=\lim_{h\to 0}\left[\frac{3(-h)}{|-h|+2(-h)}\right]$$

$$=\lim_{h\to 0}\left[\frac{-3h}{h-2h}\right]$$

$$=\lim_{h\to 0}\left[\frac{-3h}{-h}\right]$$

$= 3$

Now, let us consider RHS:

$$\lim_{x\to 0^+}\left(\frac{3x}{|x|+2x}\right)$$

So, let x = 0 + h, where h = 0

Substituting the value of x, we get

$$\lim_{x\to 0^+}\left(\frac{3x}{|x|+2x}\right)=\lim_{h\to 0}\left(\frac{3h}{|h|+2h}\right)$$

$$=\lim_{h\to 0}\left(\frac{3h}{h+2h}\right)$$

$= 1$

Since, LHS ≠ RHS

∴ Limit does not exist.

5. Let $f(x)=\begin{cases}x+1, \text{if } x>0\\ x-1, \text{if } x<0\end{cases}$ **. Prove that** $\lim_{x\to 0} f(x)$ **does not exist.**

Solution:

Firstly let us consider LHS:

$\lim_{x\to 0^-} f(x)$

So, let x = 0 – h, where h = 0

Substituting the value of x, we get

$\lim_{x\to 0^-} f(x) = \lim_{h\to 0}(0-h-1)$

$= -1$

Now, let us consider RHS

$\lim_{x\to 0^+} f(x)$

So, let x = 0 + h, where h = 0

Substituting the value of x, we get

$\lim_{x\to 0^+} f(x) = \lim_{x\to 0}(x+1)$

$= \lim_{h\to 0}(0+h+1)$

$= 1$

Since, LHS ≠ RHS

∴ Limit does not exist.

EXERCISE 29.2

Evaluate the following limits:

1. $\lim_{x \to 1} \frac{x^2 + 1}{x + 1}$

Solution:

Given:

$\lim_{x \to 1} \frac{x^2 + 1}{x + 1}$

Let us substitute the value of x directly in the given limit, we get

$\lim_{x \to 1} \frac{x^2 + 1}{x + 1} = \frac{1^2 + 1}{1 + 1}$

$= 2 / 2$

$= 1$

∴ The value of the given limit is 1.

2. $\lim_{x \to 0} \frac{2x^2 + 3x + 4}{x^2 + 3x + 2}$

Solution:

Given:

$\lim_{x \to 0} \frac{2x^2 + 3x + 4}{x^2 + 3x + 2}$

Let us substitute the value of x directly in the given limit, we get

$\lim_{x \to 0} \frac{2x^2 + 3x + 4}{x^2 + 3x + 2} = \frac{2(0^2) + 3(0) + 4}{0^2 + 3(0) + 2}$

$= 4 / 2$

$= 2$

∴ The value of the given limit is 2.

3. $\lim_{x \to 3} \frac{\sqrt{2x + 3}}{x + 3}$

Solution:

Given:

$$\lim_{x \to 3} \frac{\sqrt{2x+3}}{x+3}$$

Let us substitute the value of x directly in the given limit, we get

$$\lim_{x \to 3} \frac{\sqrt{2x+3}}{x+3} = \frac{\sqrt{2(3)+3}}{3+3}$$

$= \sqrt{9} / 6$

$= 3 / 6$

$= 1 / 2$

∴ The value of the given limit is 1/2.

4. $\lim_{x \to 1} \frac{\sqrt{x+8}}{\sqrt{x}}$

Solution:

Given:

$$\lim_{x \to 1} \frac{\sqrt{x+8}}{\sqrt{x}}$$

Let us substitute the value of x directly in the given limit, we get

$$\lim_{x \to 1} \frac{\sqrt{x+8}}{\sqrt{x}} = \frac{\sqrt{1+8}}{1}$$

$$= \frac{\sqrt{9}}{1}$$

$= 3$

∴ The value of the given limit is 3.

5. $\lim_{x \to a} \frac{\sqrt{x} + \sqrt{a}}{x + a}$

Solution:

Given:

$$\lim_{x \to a} \frac{\sqrt{x} + \sqrt{a}}{x + a}$$

Let us substitute the value of x directly in the given limit, we get

$$\lim_{x \to a} \frac{\sqrt{x} + \sqrt{a}}{x + a} = \frac{\sqrt{a} + \sqrt{a}}{a + a}$$

$$= \frac{2\sqrt{a}}{2a}$$

$$= \frac{1}{\sqrt{a}}$$

∴ The value of the given limit is $1/\sqrt{a}$.

EXERCISE 29.3 PAGE NO: 29.23

Evaluate the following limits:

1. $\lim_{x\to -5} \frac{2x^2+9x-5}{x+5}$

Solution:

Given:

The limit $\lim_{x\to -5} \frac{2x^2+9x-5}{x+5}$

By substituting the value of x, we get

$$\lim_{x\to -5} \frac{2x^2+9x-5}{x+5} = \frac{2(-5)^2+9(-5)-5}{(-5)+5}$$
$$= \frac{50-50}{(-5)+5}$$
$$= \frac{0}{0}$$ [Since, it is of the form indeterminate]

By using factorization method:

$$\lim_{x\to -5} \frac{2x^2+9x-5}{x+5} = \lim_{x\to -5} \frac{2x^2+9x-5}{x+5}$$
$$= \lim_{x\to -5} \frac{2x^2+10x-x-5}{x+5}$$
$$= \lim_{x\to -5} \frac{2x(x+5)-(x+5)}{x+5}$$
$$= \lim_{x\to -5} \frac{(2x-1)(x+5)}{x+5}$$
$$= \lim_{x\to -5} 2x-1$$
$$= 2(-5) - 1$$
$$= -11$$

∴ The value of the given limit is – 11.

2. $\lim_{x\to 3} \frac{x^2 - 4x + 3}{x^2 - 2x - 3}$

Solution:

Given:

The limit $\lim_{x\to 3} \frac{x^2 - 4x + 3}{x^2 - 2x - 3}$

By substituting the value of x, we get

$$\lim_{x\to 3} \frac{x^2 - 4x + 3}{x^2 - 2x - 3} = \frac{(3)^2 - 4(3) + 3}{(3)^2 - 2(3) - 3}$$

$$= \frac{12 - 12}{(-9) + 9}$$

$$= \frac{0}{0}$$ [Since, it is of the form indeterminate]

By using factorization method:

$$\lim_{x\to 3} \frac{x^2 - 4x + 3}{x^2 - 2x - 3} = \lim_{x\to 3} \frac{(x^2 - 4x + 3)}{(x^2 - 2x - 3)}$$

$$= \lim_{x\to 3} \frac{(x^2 - 3x - x + 3)}{(x^2 - 3x + x - 3)}$$

$$= \lim_{x\to 3} \frac{x(x - 3) - 1(x - 3)}{x(x - 3) + 1(x - 3)}$$

$$= \lim_{x\to 3} \frac{(x - 3)(x - 1)}{(x - 3)(x + 1)}$$

$$= \lim_{x\to 3} \frac{(x - 1)}{(x + 1)}$$

$$= \frac{(3 - 1)}{(3 + 1)}$$

$$= 2/4$$

$$= 1/2$$

∴ The value of the given limit is ½.

3. $\lim_{x\to 3} \frac{x^4 - 81}{x^2 - 9}$

Solution:

Given:

The limit $\lim_{x\to 3} \frac{x^4 - 81}{x^2 - 9}$

By substituting the value of x, we get

$$\lim_{x\to 3} \frac{x^4 - 81}{x^2 - 9} = \frac{(3)^4 - 81}{(3)^2 - 9}$$

$$= \frac{81 - 81}{(-9) + 9}$$

$$= \frac{0}{0}$$ [Since, it is of the form indeterminate]

By using factorization method:

$$\lim_{x\to 3} \frac{x^4 - 81}{x^2 - 9} = \lim_{x\to 3} \frac{(x^4 - 81)}{(x^2 - 9)}$$

$$= \lim_{x\to 3} \frac{(x^4 - 3^4)}{(x^2 - 3^2)}$$

$$= \lim_{x\to 3} \frac{((x^2)^2 - (3^2)^2)}{(x^2 - 3^2)}$$ [Since $a^2 - b^2 = (a + b)(a - b)$]

So,

$$= \lim_{x\to 3} \frac{(x^2 - 3^2)(x^2 + 3^2)}{(x^2 - 3^2)}$$

$$= \lim_{x\to 3} (x^2 + 3^2)$$

$$= 3^2 + 3^2$$

$$= 18$$

∴ The value of the given limit is 18.

4. $\lim_{x\to 2} \frac{x^3 - 8}{x^2 - 4}$

Solution:

Given:

The limit $\lim_{x\to 2} \frac{x^3 - 8}{x^2 - 4}$

By substituting the value of x, we get

$$\lim_{x\to 2}\frac{x^3-8}{x^2-4} = \frac{(2)^3-8}{(2)^2-4}$$
$$= \frac{8-8}{(4)-4}$$
$$= \frac{0}{0}$$ [Since, it is of the form indeterminate]

By using factorization method:

$$\lim_{x\to 2}\frac{x^3-8}{x^2-4} = \lim_{x\to 2}\frac{(x^3-8)}{(x^2-4)}$$
$$= \lim_{x\to 2}\frac{(x^3-2^3)}{(x^2-2^2)}$$
$$= \lim_{x\to 2}\frac{(x-2)(x^2+2^2+2x)}{(x+2)(x-2)}$$

[By using the formula, $(a^3 - b^3) = (a - b)(a^2 + b^2 + ab)$ & $(a^2 - b^2) = (a + b)(a - b)$]

$$= \lim_{x\to 2}\frac{(x^2+2^2+2x)}{(x+2)}$$
$$= \frac{(2^2+2^2+2(2))}{(2+2)}$$
$$= \frac{3.4}{(4)}$$
$$= 3$$

∴ The value of the given limit is 3.

5. $\lim_{x\to -1/2}\frac{8x^3+1}{2x+1}$

Solution:

Given:

The limit $\lim_{x\to -1/2}\frac{8x^3+1}{2x+1}$

By substituting the value of x, we get

$$\lim_{x\to -1/2}\frac{8x^3+1}{2x+1} = \frac{8\left(-\frac{1}{2}\right)^3+1}{2\left(-\frac{1}{2}\right)+1}$$
$$= \frac{-1+1}{-1+1}$$
$$= \frac{0}{0}$$ [Since, it is of the form indeterminate]

By using factorization method:

$$\lim_{x \to -1/2} \frac{8x^3+1}{2x+1} = \lim_{x \to -\frac{1}{2}} \frac{8x^3 + 1}{2x + 1}$$

$$= \lim_{x \to -\frac{1}{2}} \frac{(2x)^3 + (1)^3}{2x + 1}$$

[By using the formula, $a^3 + b^3 = (a + b)(a^2 + b^2 - ab)$]

$$= \lim_{x \to -\frac{1}{2}} \frac{(2x + 1)((2x)^2 + (1)^2 - 2x)}{2x + 1}$$

$$= \lim_{x \to -\frac{1}{2}} (2x)^2 + (1)^2 - 2x$$

$$= (2(\frac{-1}{2}))^2 + (1)^2 - 2(-\frac{1}{2})$$

$$= 1 + 1 + 1$$

$$= 3$$

∴ The value of the given limit is 3.

EXERCISE 29.4

PAGE NO: 29.28

Evaluate the following limits:

1. $\lim_{x\to 0} \frac{\sqrt{1+x+x^2}-1}{x}$

Solution:

Given:

The limit $\lim_{x\to 0} \frac{\sqrt{1+x+x^2}-1}{x}$

We need to find the limit of the given equation when x => 0

Now let us substitute the value of x as 0, we get an indeterminate form of 0/0.

Let us rationalizing the given equation, we get

$$\lim_{x\to 0} \frac{\sqrt{1+x+x^2}-1}{x} = \lim_{x\to 0} \frac{(\sqrt{1+x+x^2}-1)}{x} \frac{(\sqrt{1+x+x^2}+1)}{(\sqrt{1+x+x^2}+1)}$$

[By using the formula: $(a + b)(a - b) = a^2 - b^2$]

$$= \lim_{x\to 0} \frac{1+x+x^2-1}{x\left(\sqrt{1+x+x^2}+1\right)}$$

$$= \lim_{x\to 0} \frac{x(1+x)}{x\left(\sqrt{1+x+x^2}+1\right)}$$

$$= \lim_{x\to 0} \frac{(1+x)}{\left(\sqrt{1+x+x^2}+1\right)}$$

Now we can see that the indeterminate form is removed,

So, now we can substitute the value of x as 0, we get

$$\lim_{x\to 0} \frac{\sqrt{1+x+x^2}-1}{x} = \frac{1}{1+1}$$

$$= \frac{1}{2}$$

∴ The value of the given limit is ½.

2. $\lim_{x\to 0} \frac{2x}{\sqrt{a+x}-\sqrt{a-x}}$

Solution:

Given:

The limit $\lim_{x\to 0} \frac{2x}{\sqrt{a+x}-\sqrt{a-x}}$

We need to find the limit of the given equation when x => 0

Now let us substitute the value of x as 0, we get an indeterminate form of 0/0.

Let us rationalizing the given equation, we get

$$\lim_{x\to 0}\frac{2x}{\sqrt{a+x}-\sqrt{a-x}} = \lim_{x\to 0}\frac{2x}{(\sqrt{a+x}-\sqrt{a-x})}\frac{(\sqrt{a+x}+\sqrt{a-x})}{(\sqrt{a+x}+\sqrt{a-x})}$$

[By using the formula: $(a+b)(a-b) = a^2 - b^2$]

$$= \lim_{x\to 0}\frac{2x(\sqrt{a+x}+\sqrt{a-x})}{a+x-a+x}$$

$$= \lim_{x\to 0}\frac{2x(\sqrt{a+x}+\sqrt{a-x})}{2x}$$

$$= \lim_{x\to 0}\frac{(\sqrt{a+x}+\sqrt{a-x})}{1}$$

Now we can see that the indeterminate form is removed,
So, now we can substitute the value of x as 0, we get

$$\lim_{x\to 0}\frac{2x}{\sqrt{a+x}-\sqrt{a-x}} = \sqrt{a}+\sqrt{a}$$

$$= 2\sqrt{a}$$

∴ The value of the given limit is $2\sqrt{a}$

3. $\lim_{x\to 0}\frac{\sqrt{a^2+x^2}-a}{x^2}$

Solution:

Given:
The limit $\lim_{x\to 0}\frac{\sqrt{a^2+x^2}-a}{x^2}$

We need to find the limit of the given equation when x => 0
Now let us substitute the value of x as 0, we get an indeterminate form of 0/0.
Let us rationalizing the given equation, we get

$$\lim_{x\to 0}\frac{\sqrt{a^2+x^2}-a}{x^2} = \lim_{x\to 0}\frac{(\sqrt{a^2+x^2}-a)}{x^2}\frac{(\sqrt{a^2+x^2}+a)}{(\sqrt{a^2+x^2}+a)}$$

[By using the formula: $(a+b)(a-b) = a^2 - b^2$]

$$= \lim_{x\to 0}\frac{(a^2+x^2-a^2)}{x^2(\sqrt{a^2+x^2}+a)}$$

$$= \lim_{x\to 0}\frac{x^2}{x^2(\sqrt{a^2+x^2}+a)}$$

$$= \lim_{x\to 0}\frac{1}{(\sqrt{a^2+x^2}+a)}$$

Now we can see that the indeterminate form is removed,
So, now we can substitute the value of x as 0, we get

$$\lim_{x\to 0}\frac{\sqrt{a^2+x^2}-a}{x^2}=\frac{1}{a+a}$$

$$=\frac{1}{2a}$$

∴ The value of the given limit is $\frac{1}{2a}$.

4. $\lim_{x\to 0}\frac{\sqrt{1+x}-\sqrt{1-x}}{2x}$

Solution:

Given:

The limit $\lim_{x\to 0}\frac{\sqrt{1+x}-\sqrt{1-x}}{2x}$

We need to find the limit of the given equation when x => 0

Now let us substitute the value of x as 0, we get an indeterminate form of 0/0.

Let us rationalizing the given equation, we get

$$\lim_{x\to 0}\frac{\sqrt{1+x}-\sqrt{(1-x)}}{2x}=\lim_{x\to 0}\frac{\left(\sqrt{1+x}-\sqrt{(1-x)}\right)}{2x}\frac{\left(\sqrt{1+x}+\sqrt{(1-x)}\right)}{\left(\sqrt{1+x}+\sqrt{(1-x)}\right)}$$

[By using the formula: (a + b) (a - b) = a^2 - b^2]

$$=\lim_{x\to 0}\frac{1+x-1+x}{2x\left(\sqrt{1+x}+\sqrt{(1-x)}\right)}$$

$$=\lim_{x\to 0}\frac{2x}{2x\left(\sqrt{1+x}+\sqrt{(1-x)}\right)}$$

$$=\lim_{x\to 0}\frac{1}{\left(\sqrt{1+x}+\sqrt{(1-x)}\right)}$$

Now we can see that the indeterminate form is removed,
So, now we can substitute the value of x as 0, we get

$$\lim_{x\to 0}\frac{\sqrt{1+x}-\sqrt{(1-x)}}{2x}=\frac{1}{1+1}$$

$$=\frac{1}{2}$$

∴ The value of the given limit is ½.

5. $\lim_{x\to 2} \frac{\sqrt{3-x}-1}{2-x}$

Solution:

Given:

The limit $\lim_{x\to 2} \frac{\sqrt{3-x}-1}{2-x}$

We need to find the limit of the given equation when x => 0

Now let us substitute the value of x as 0, we get an indeterminate form of 0/0.

Let us rationalizing the given equation, we get

$$\lim_{x\to 2} \frac{\sqrt{3-x}-1}{2-x} = \lim_{x\to 2} \frac{(\sqrt{3-x}-1)}{(2-x)} \frac{(\sqrt{3-x}+1)}{(\sqrt{3-x}+1)}$$

[By using the formula: $(a + b)(a - b) = a^2 - b^2$]

$$= \lim_{x\to 2} \frac{(3-x-1)}{(2-x)(\sqrt{3-x}+1)}$$

$$= \lim_{x\to 2} \frac{(2-x)}{(2-x)(\sqrt{3-x}+1)}$$

$$= \lim_{x\to 2} \frac{1}{(\sqrt{3-x}+1)}$$

Now we can see that the indeterminate form is removed,

So, now we can substitute the value of x as 0, we get

$$\lim_{x\to 2} \frac{\sqrt{3-x}-1}{2-x} = \frac{1}{1+1}$$

$$= \frac{1}{2}$$

∴ The value of the given limit is ½.

EXERCISE 29.5

PAGE NO: 29.33

Evaluate the following limits:

1. $\lim_{x \to a} \frac{(x+2)^{5/2} - (a+2)^{5/2}}{x-a}$

Solution:

Given: $\lim_{x \to a} \frac{(x+2)^{5/2} - (a+2)^{5/2}}{x-a}$

The limit

When $x = a$, the expression $\lim_{x \to a} \frac{(x+2)^{5/2} - (a+2)^{5/2}}{x-a}$ assumes the form (0/0).

So let, $Z = \lim_{x \to a} \frac{(x+2)^{5/2} - (a+2)^{5/2}}{x-a}$

By using the formula: $\lim_{x \to a} \frac{(x)^n - (a)^n}{x-a} = na^{n-1}$

Since, Z is not of the form as described above.

Let us simplify, we get

$Z = \lim_{x \to a} \frac{(x+2)^{5/2} - (a+2)^{5/2}}{x-a}$

$Z = \lim_{x \to a} \frac{(x+2)^{5/2} - (a+2)^{5/2}}{x+2-(a+2)}$

Let $x + 2 = y$ and $a + 2 = k$

As $x \to a$; $y \to k$

So,

$Z = \lim_{y \to k} \frac{(y)^{5/2} - (k)^{5/2}}{y-k}$

By using the formula: $\lim_{x \to a} \frac{(x)^n - (a)^n}{x-a} = na^{n-1}$

$Z = \frac{5}{2} k^{\frac{5}{2}-1}$

$= \frac{5}{2} k^{\frac{3}{2}}$

$= \frac{5}{2} (a+2)^{\frac{3}{2}}$

$\therefore \lim_{x \to a} \frac{(x+2)^{5/2} - (a+2)^{5/2}}{x-a} = \frac{5}{2} (a+2)^{\frac{3}{2}}$

2. $\lim_{x\to a} \frac{(x+2)^{3/2} - (a+2)^{3/2}}{x-a}$

Solution:

Given:

The limit $\lim_{x\to a} \frac{(x+2)^{3/2} - (a+2)^{3/2}}{x-a}$

When x = a, the expression $\lim_{x\to a} \frac{(x+2)^{3/2} - (a+2)^{3/2}}{x-a}$ assumes the form (0/0).

So let, $Z = \lim_{x\to a} \frac{(x+2)^{3/2} - (a+2)^{3/2}}{x-a}$

By using the formula: $\lim_{x\to a} \frac{(x)^n - (a)^n}{x-a} = na^{n-1}$

Since, Z is not of the form as described above.

Let us simplify, we get

$Z = \lim_{x\to a} \frac{(x+2)^{3/2} - (a+2)^{3/2}}{x-a}$

$Z = \lim_{x\to a} \frac{(x+2)^{3/2} - (a+2)^{3/2}}{x+2-(a+2)}$

Let x + 2 = y and a+2 = k

As $x \to a$; $y \to k$

$Z = \lim_{y\to k} \frac{(y)^{3/2} - (k)^{3/2}}{y-k}$

By using the formula: $\lim_{x\to a} \frac{(x)^n - (a)^n}{x-a} = na^{n-1}$

$Z = \frac{3}{2} k^{\frac{3}{2}-1}$

$= \frac{3}{2} k^{\frac{1}{2}}$

$= \frac{3}{2} (a+2)^{\frac{1}{2}}$

$\therefore \lim_{x\to a} \frac{(x+2)^{3/2} - (a+2)^{3/2}}{x-a} = \frac{3}{2}\sqrt{a+2}$

3. $\lim_{x\to a} \frac{(1+x)^6 - 1}{(1+x)^2 - 1}$

Solution:

Given:

The limit $\lim_{x\to a} \frac{(1+x)^6 - 1}{(1+x)^2 - 1}$

When x = a, the expression $\lim\limits_{x\to a}\frac{(1+x)^6-1}{(1+x)^2-1}$ assumes the form (0/0).

So let, Z = $\lim\limits_{x\to a}\frac{(1+x)^6-1}{(1+x)^2-1}$

Z = $\frac{(1+a)^6-1}{(1+a)^2-1} = \frac{\{(1+a)^2\}^3-1}{(1+a)^2-1}$

[This can be further simplified using the formula: $a^3 - 1 = (a-1)(a^2 + a + 1)$]

Z = $\frac{\{(1+a)^2-1\}((1+a)^4+(1+a)^2+1)}{(1+a)^2-1}$

= $(1+a)^4 + (1+a)^2 + 1$

= 1 + 1 + 1

= 3

∴ $\lim\limits_{x\to a}\frac{(1+x)^6-1}{(1+x)^2-1} = 3$

4. $\lim\limits_{x\to a}\frac{x^{2/7}-a^{2/7}}{x-a}$

Solution:

Given: $\lim\limits_{x\to a}\frac{x^{2/7}-a^{2/7}}{x-a}$

The limit

When x = a, the expression $\lim\limits_{x\to a}\frac{x^{2/7}-a^{2/7}}{x-a}$ assumes the form (0/0).

So let, Z = $\lim\limits_{x\to a}\frac{x^{2/7}-a^{2/7}}{x-a}$

By using the formula: $\lim\limits_{x\to a}\frac{(x)^n-(a)^n}{x-a} = na^{n-1}$

Since, Z is not of the form as described above.

Let us simplify, we get

Z = $\lim\limits_{x\to a}\frac{x^{2/7}-a^{2/7}}{x-a}$

By using the formula: $\lim\limits_{x\to a}\frac{(x)^n-(a)^n}{x-a} = na^{n-1}$

Z = $\frac{2}{7}\,a^{\frac{2}{7}-1}$

= $\frac{2}{7}\,a^{-\frac{5}{7}}$

∴ $\lim\limits_{x\to a}\frac{x^{2/7}-a^{2/7}}{x-a} = \frac{2}{7}\,a^{-\frac{5}{7}}$

5. $\lim_{x\to a}\frac{x^{5/7}-a^{5/7}}{x^{2/7}-a^{2/7}}$

Solution:

Given: $\lim_{x\to a}\frac{x^{5/7}-a^{5/7}}{x^{2/7}-a^{2/7}}$

The limit

When x = a, the expression $\lim_{x\to a}\frac{x^{5/7}-a^{5/7}}{x^{2/7}-a^{2/7}}$ assumes the form (0/0).

So let, $Z = \lim_{x\to a}\frac{x^{5/7}-a^{5/7}}{x^{2/7}-a^{2/7}}$

By using the formula: $\lim_{x\to a}\frac{(x)^n-(a)^n}{x-a} = na^{n-1}$

Since, Z is not of the form as described above.

Let us simplify, we get

$Z = \lim_{x\to a}\frac{x^{\frac{5}{7}}-a^{\frac{5}{7}}}{x^{\frac{2}{7}}-a^{\frac{2}{7}}}$

Let us divide the numerator and denominator by (x – a), we get

$Z = \lim_{x\to a}\frac{\frac{x^{\frac{5}{7}}-a^{\frac{5}{7}}}{x-a}}{\frac{x^{\frac{2}{7}}-a^{\frac{2}{7}}}{x-a}}$

By using algebra of limits, we have

$Z = \frac{\lim_{x\to a}\frac{x^{\frac{5}{7}}-a^{\frac{5}{7}}}{x-a}}{\lim_{x\to a}\frac{x^{\frac{2}{7}}-a^{\frac{2}{7}}}{x-a}}$

So now again, by using the formula: $\lim_{x\to a}\frac{(x)^n-(a)^n}{x-a} = na^{n-1}$

$Z = \frac{\frac{5}{7}a^{\frac{5}{7}-1}}{\frac{2}{7}a^{\frac{2}{7}-1}}$

$= \frac{5a^{-\frac{2}{7}}}{2a^{-\frac{5}{7}}}$

$= \frac{5}{2}a^{\frac{3}{7}}$

$\therefore \lim_{x\to a}\frac{x^{\frac{5}{7}}-a^{\frac{5}{7}}}{x^{\frac{2}{7}}-a^{\frac{2}{7}}} = \frac{5}{2}a^{\frac{3}{7}}$

EXERCISE 29.6

Evaluate the following limits:

1. $\lim_{x\to\infty}\frac{(3x-1)(4x-2)}{(x+8)(x-1)}$

Solution:

Given:

The limit $\lim_{x\to\infty}\frac{(3x-1)(4x-2)}{(x+8)(x-1)}$

Let us simplify the expression, we get

$$\lim_{x\to\infty}\frac{(3x-1)(4x-2)}{(x+8)(x-1)} = \lim_{x\to\infty}\frac{(12x^2-10x+2)}{(x^2+9x-8)}$$

$$= \lim_{x\to\infty}\left(\frac{12-\frac{10}{x}+\frac{2}{x^2}}{1+\frac{9}{x}-\frac{8}{x^2}}\right)$$

When substituting the value of x as $x\to\infty$ and $\frac{1}{x}\to 0$ then,

$$= \frac{12-0+0}{1}$$

$$= 12$$

$$\therefore \lim_{x\to\infty}\frac{(3x-1)(4x-2)}{(x+8)(x-1)} = 12$$

2. $\lim_{x\to\infty}\frac{3x^3-4x^2+6x-1}{2x^3+x^2-5x+7}$

Solution:

Given:

The limit $\lim_{x\to\infty}\frac{3x^3-4x^2+6x-1}{2x^3+x^2-5x+7}$

Let us simplify the expression, we get

$$\lim_{x\to\infty}\frac{3x^3-4x^2+6x-1}{2x^3+x^2-5x+7} = \lim_{x\to\infty}\frac{3-\frac{4}{x}+\frac{6}{x^2}-\frac{1}{x^3}}{2+\frac{1}{x}-\frac{5}{x^2}+\frac{7}{x^3}}$$

When substituting the value of x as $x\to\infty$ and $\frac{1}{x}\to 0$ then,

$$= \frac{3-0+0-0}{2+0-0+0}$$

$= 3 / 2$

$\therefore \lim_{x\to\infty} \frac{3x^3-4x^2+6x-1}{2x^3+x^2-5x+7} = \frac{3}{2}$

3. $\lim_{x\to\infty} \frac{5x^3-6}{\sqrt{9+4x^6}}$

Solution:

Given:

The limit $\lim_{x\to\infty} \frac{5x^3-6}{\sqrt{9+4x^6}}$

Let us simplify the expression, we get

$$\lim_{x\to\infty} \frac{5x^3-6}{\sqrt{(9+4x^6)}} = \lim_{x\to\infty} \frac{5-\frac{6}{x^3}}{\sqrt{(\frac{9}{x^6}+\frac{4x^6}{x^6}}}$$

$$= \lim_{x\to\infty} \frac{\left(5-\frac{6}{x^3}\right)}{\sqrt{\frac{9}{x^6}+4}}$$

When substituting the value of x as $x \to \infty$ and $\frac{1}{x} \to 0$ then,

$$= \frac{5}{\sqrt{4}}$$

$$= 5/2$$

$\therefore \lim_{x\to\infty} \frac{5x^3-6}{\sqrt{(9+4x^6)}} = \frac{5}{2}$

4. $\lim_{x\to\infty} \sqrt{x^2+cx} - x$

Solution:

Given:

The limit $\lim_{x\to\infty} \sqrt{x^2+cx} - x$

Let us simplify the expression by rationalizing the numerator, we get

$$\lim_{x\to\infty} \sqrt{x^2+cx} - x = \lim_{x\to\infty} \left(\sqrt{x^2+cx} - x\right).\frac{\sqrt{x^2+cx}+x}{\sqrt{x^2+cx}+x}$$

$$= \lim_{x\to\infty} \frac{(x^2+cx-x^2)}{\sqrt{x^2+cx}+x}$$

$$= \lim_{x\to\infty} \frac{cx}{\sqrt{x^2+cx}+x}$$

By taking 'x' as common from both numerator and denominator, we get

$$= \lim_{x\to\infty} \frac{c}{\sqrt{1+\frac{c}{x}}+1}$$

When substituting the value of x as $x \to \infty$ and $\frac{1}{x} \to 0$ then,

$$= \frac{c}{1+1}$$

$$= \frac{c}{2}$$

$$\therefore \lim_{x\to\infty} \sqrt{x^2+cx} - x = \frac{c}{2}$$

5. $\lim_{x\to\infty} \sqrt{x+1} - \sqrt{x}$

Solution:

Given: $\lim_{x\to\infty} \sqrt{x+1} - \sqrt{x}$

The limit

Let us simplify the expression by rationalizing the numerator, we get

On rationalizing the numerator we get,

$$\lim_{x\to\infty} \sqrt{x+1} - \sqrt{x} = \lim_{x\to\infty} \sqrt{x+1} - \sqrt{x}.\frac{\sqrt{x+1}+\sqrt{x}}{(\sqrt{x+1}+\sqrt{x})}$$

$$= \lim_{x\to\infty} \frac{(x+1-x)}{\sqrt{x+1}+\sqrt{x}}$$

$$= \lim_{x\to\infty} \left(\frac{1}{\sqrt{x+1}+\sqrt{x}}\right)$$

When substituting the value of x as $x \to \infty$ and $\frac{1}{x} \to 0$ then,

$$= \frac{1}{\infty}$$

$$= 0$$

$$\therefore \lim_{x\to\infty} \sqrt{x+1} - \sqrt{x} = 0$$

EXERCISE 29.7

Evaluate the following limits:

1. $\lim_{x \to 0} \frac{\sin 3x}{5x}$

Solution:

Given:

The limit $\lim_{x \to 0} \frac{\sin 3x}{5x}$

Let us consider the limit:

$$\lim_{x \to 0} \frac{\sin 3x}{5x} = \frac{1}{5} \lim_{x \to 0} \frac{\sin 3x}{x}$$

Now let us multiply and divide the expression by 3, we get

$$= \frac{1}{5} \lim_{x \to 0} \frac{\sin 3x}{3x} \times 3$$

$$= \frac{3}{5} \lim_{x \to 0} \frac{\sin 3x}{3x}$$

Now, put $3x = y$

$$= \frac{3}{5} \lim_{y \to 0} \frac{\sin y}{y} \quad [\text{We know that, } \lim_{y \to 0} \frac{\sin y}{y} = 1]$$

So,

$$\lim_{x \to 0} \frac{\sin 3x}{5x} = \frac{3}{5} \lim_{y \to 0} \frac{\sin y}{y}$$

$$= \frac{3}{5} \times 1$$

$$= \frac{3}{5}$$

$\therefore$ The value of $\lim_{x \to 0} \frac{\sin 3x}{5x} = \frac{3}{5}$

2. $\lim_{x \to 0} \frac{\sin x^0}{x}$

Solution:

Given:

The limit $\lim_{x \to 0} \frac{\sin x^0}{x}$

We know, $1° = \frac{\pi}{180}$ radians

So,

$x^\circ = \frac{\pi x}{180}$ radians

Let us consider the limit,

$$\lim_{x\to 0}\frac{\sin x^\circ}{x} = \lim_{x\to 0}\frac{\sin\frac{\pi x}{180}}{x}$$

Now let us multiply and divide the expression by $\frac{\pi}{180}$, we get

$$= \lim_{x\to 0}\frac{\sin\frac{\pi x}{180}\times\frac{\pi}{180}}{x\times\frac{\pi}{180}}$$

$$= \frac{\pi}{180}\lim_{x\to 0}\frac{\sin\frac{\pi x}{180}}{\frac{\pi x}{180}}$$

Now, put $\frac{\pi x}{180} = y$

$$= \frac{\pi}{180}\lim_{y\to 0}\frac{\sin y}{y}\ \left[\text{We know that, } \lim_{y\to 0}\frac{\sin y}{y} = 1\right]$$

$$\lim_{x\to 0}\frac{\sin x^\circ}{x} = \frac{\pi}{180}\lim_{y\to 0}\frac{\sin y}{y}$$

$$= \frac{\pi}{180}\times 1$$

$$= \frac{\pi}{180}$$

∴ The value of $\lim_{x\to 0}\frac{\sin x^\circ}{x} = \frac{\pi}{180}$

3. $\lim_{x\to 0}\frac{x^2}{\sin x^2}$

Solution:

Given:

The limit $\lim_{x\to 0}\frac{x^2}{\sin x^2}$

Let us consider the limit and divide the expression by x^2, we get

$$\lim_{x\to 0}\frac{x^2}{\sin x^2} = \lim_{x\to 0}\frac{1}{\frac{\sin x^2}{x^2}}$$

Now, put $x^2 = y$

$$\lim_{x\to 0}\frac{1}{\frac{\sin x^2}{x^2}} = \frac{1}{\lim_{y\to 0}\frac{\sin y}{y}}$$ [We know that, $\lim_{y\to 0}\frac{\sin y}{y} = 1$]

$$= \frac{1}{1}$$

$$=1$$

∴ The value of $\lim_{x\to 0}\frac{x^2}{\sin x^2} = 1$

4. $\lim_{x\to 0}\frac{\sin x \cos x}{3x}$

Solution:

Given:

The limit $\lim_{x\to 0}\frac{\sin x \cos x}{3x}$

Let us consider the limit

$$\lim_{x\to 0}\frac{\sin x \cos x}{3x} = \frac{1}{3}\lim_{x\to 0}\frac{\sin x\cos x}{x}$$

$$= \frac{1}{3}\lim_{x\to 0}\left(\frac{\sin x}{x}\right)\cos x$$

We know,

$$\lim_{x\to 0} A(x).B(x) = \lim_{x\to 0} A(x) \times \lim_{x\to 0} B(x)$$

So,

$$= \frac{1}{3}\lim_{x\to 0}\frac{\sin x}{x} \times \lim_{x\to 0}\cos x$$ [We know that, $\lim_{x\to 0}\frac{\sin x}{x} = 1$]

$$\lim_{x\to 0}\frac{\sin x\cos x}{3x} = \frac{1}{3}\times 1\times \cos 0$$

$$= \frac{1}{3}\times 1\times 1$$ [Since, $\cos 0 = 1$]

$$= \frac{1}{3}$$

∴ The value of $\lim_{x\to 0}\frac{\sin x\cos x}{3x} = \frac{1}{3}$

5. $\lim_{x\to 0} \frac{3\sin x - 4\sin^3 x}{x}$

Solution:

Given:

The limit $\lim_{x\to 0} \frac{3\sin x - 4\sin^3 x}{x}$

We know that, Sin3x = 3sinx – 4 $\sin^3$x

So,

$$\lim_{x\to 0} \frac{3\sin x - 4\sin^3 x}{x} = \lim_{x\to 0} \frac{\sin 3x}{x}$$

Now multiply and divide the expression by 3, we get

$$\lim_{x\to 0} \frac{\sin 3x}{x} = \lim_{x\to 0} \frac{\sin 3x \times 3}{3x}$$

$$= 3\lim_{x\to 0} \frac{\sin 3x}{3x}$$

Now, put 3x = y

$$= 3\lim_{y\to 0} \frac{\sin y}{y}$$ [We know that, $\lim_{y\to 0} \frac{\sin y}{y} = 1$]

$$\lim_{x\to 0} \frac{3\sin x - 4\sin^3 x}{x} = 3\lim_{y\to 0} \frac{\sin y}{y}$$

$$= 3 \times 1$$

$$= 3$$

∴ The value of $\lim_{x\to 0} \frac{3\sin x - 4\sin^3 x}{x} = 3$

EXERCISE 29.8

Evaluate the following limits:

1. $\lim_{x\to\pi/2}\left(\frac{\pi}{2}-x\right)\tan x$

Solution:

Given:

The limit $\lim_{x\to\pi/2}\left(\frac{\pi}{2}-x\right)\tan x$

Let us assume, $y = \frac{\pi}{2} - x$

So,

$x \to \frac{\pi}{2}, y \to 0$

$$\lim_{x\to\pi/2}\left(\frac{\pi}{2}-x\right)\tan x = \lim_{y\to 0} y\tan\left(\frac{\pi}{2}-y\right)$$

$$= \lim_{y\to 0} y\frac{\sin\left(\frac{\pi}{2}-y\right)}{\cos\left(\frac{\pi}{2}-y\right)}$$ [We know that, tan = sin/cos]

$$= \lim_{y\to 0} y\frac{\cos y}{\sin y}$$

Upon simplification, we get

$$= \lim_{y\to 0}\cos y - \lim_{y\to 0}\frac{y}{\sin y}$$

Substituting the value of $y = 0$, then

$$= \cos 0 - \frac{0}{\sin 0}$$

$$= 1 - 0$$

$$= 1$$

∴ The value of $\lim_{x\to\frac{\pi}{2}}\left(\frac{\pi}{2}-x\right) = 1$

2. $\lim_{x\to\pi/2}\frac{\sin 2x}{\cos x}$

Solution:

Given:

The limit $\lim_{x\to\pi/2}\frac{\sin 2x}{\cos x}$

We know, $\sin 2x = 2\sin x.\cos x$

So,

$$\lim_{x\to\pi/2} \frac{\sin 2x}{\cos x} = \lim_{x\to\pi/2} \frac{2\sin x\cos x}{\cos x}$$

Upon simplification, we get

$$= \lim_{x\to\pi/2} 2\sin x$$

Substitute the value of x, we get

$$= 2\sin\frac{\pi}{2}$$

$$= 2 \times 1$$

$$= 2$$

∴ The value of $\lim_{x\to\pi/2} \frac{\sin 2x}{\cos x} = 2$

3. $\lim_{x\to\pi/2} \frac{\cos^2 x}{1-\sin x}$

Solution:

Given: The limit $\lim_{x\to\pi/2} \frac{\cos^2 x}{1-\sin x}$

We know that, $\cos^2 x = 1-\sin^2 x$

So, by substituting this value we get,

$$\lim_{x\to\pi/2} \frac{\cos^2 x}{1-\sin x} = \lim_{x\to\pi/2} \frac{1-\sin^2 x}{1-\sin x}$$

Upon expansion,

$$= \lim_{x\to\pi/2} \frac{(1-\sin x)(1+\sin x)}{1-\sin x}$$

When simplified, we get

$$= \lim_{x\to\pi/2} 1+\sin x$$

Now, substitute the value of x, we get

$$= 1+\sin\frac{\pi}{2}$$

$$= 1+1$$

$$= 2$$

∴ The value of $\lim_{x\to\pi/2} \frac{\cos^2 x}{1-\sin x} = 2$

4. $\lim_{x \to \pi/3} \frac{\sqrt{1-\cos 6x}}{\sqrt{2}(\pi/3 - x)}$

Solution:

Given:

The limit $\lim_{x \to \pi/3} \frac{\sqrt{1-\cos 6x}}{\sqrt{2}(\pi/3 - x)}$

We know that, $1 - \cos 2x = 2\sin^2 x$

So,

$$\lim_{x \to \frac{\pi}{3}} \frac{\sqrt{1-\cos 6x}}{\sqrt{2}\left(\frac{\pi}{3} - x\right)} = \lim_{x \to \frac{\pi}{3}} \frac{\sqrt{2\sin^2 3x}}{\sqrt{2}\left(\frac{\pi}{3} - x\right)}$$

$$= \lim_{x \to \frac{\pi}{3}} \frac{\sqrt{2}\sin 3x}{\sqrt{2}\left(\frac{\pi}{3} - x\right)}$$

$$= \lim_{x \to \frac{\pi}{3}} \frac{\sin 3x}{\left(\frac{\pi}{3} - x\right)}$$

$$= \lim_{x \to \frac{\pi}{3}} \frac{3\sin 3x}{\pi - 3x}$$

We know that, $\sin x = \sin(\pi - x)$

So,

$$\lim_{x \to \frac{\pi}{3}} \frac{\sqrt{1-\cos 6x}}{\sqrt{2}\left(\frac{\pi}{3} - x\right)} = \lim_{x \to \frac{\pi}{3}} \frac{3\sin(\pi - 3x)}{\pi - 3x}$$ [We know that, $\lim_{x \to 0} \frac{\sin x}{x} = 1$]

$= 3$

∴ The value of $\lim_{x \to \frac{\pi}{3}} \frac{\sqrt{1-\cos 6x}}{\sqrt{2}\left(\frac{\pi}{3} - x\right)} = 3$

5. $\lim_{x \to a} \frac{\cos x - \cos a}{x - a}$

Solution:

Given:

The limit $\lim_{x \to a} \frac{\cos x - \cos a}{x - a}$

We know that,

$$\left[\cos A - \cos B = 2\sin\left(\frac{A-B}{2}\right)\sin\left(\frac{A+B}{2}\right)\right]$$

By substituting in the formula, we get

$$\lim_{x\to a}\frac{\cos x-\cos a}{x-a}=\lim_{x\to a}\frac{\left(-2\sin\left(\frac{x+a}{2}\right)\sin\left(\frac{x-a}{2}\right)\right)}{x-a}$$

$$=-2\lim_{x\to a}\sin\left(\frac{x+a}{2}\right)\lim_{x\to a}\sin\frac{\left(\frac{x-a}{2}\right)}{x-a}$$

Upon simplification, we get

$$=-2\sin\left(\frac{a+a}{2}\right)\left(\lim_{x\to a}\sin\frac{\left(\frac{x-a}{2}\right)}{\frac{x-a}{2}}\right)\times\frac{1}{2}$$

$$=-2\sin a\times 1\times\frac{1}{2}$$

$$=-\sin a$$

∴ The value of $\lim_{x\to a}\frac{\cos x-\cos a}{x-a}=-\sin a$

EXERCISE 29.9

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Evaluate the following limits:

1. $\lim_{x\to\pi} \frac{1+\cos x}{\tan^2 x}$

Solution:

Given: $\lim_{x\to\pi} \frac{1+\cos x}{\tan^2 x}$

The limit $\lim_{x\to\pi} \frac{1+\cos x}{\tan^2 x}$

When $x = \pi$, the expression $\lim_{x\to\pi} \frac{1+\cos x}{\tan^2 x}$ assumes the form (0/0).

So, let us multiply the expression by $\cos^2 x$

$$\lim_{x\to\pi} \frac{1+\cos x}{\tan^2 x} = \lim_{x\to\pi}\left[\frac{(1+\cos x)}{\sin^2 x}\times\cos^2 x\right]$$

$$= \lim_{x\to\pi}\left[\frac{(1+\cos x)}{1-\cos^2 x}\times\cos^2 x\right]$$

Upon expansion, we get

$$= \lim_{x\to\pi}\left[\frac{(1+\cos x)}{(1-\cos x)(1+\cos x)}\times\cos^2 x\right]$$

$$= \lim_{x\to\pi}\left[\frac{\cos^2 x}{(1-\cos x)}\right]$$

Now, substitute the value of x, we get

$$= \frac{\cos^2 \pi}{1-\cos \pi}$$

$$= \frac{(-1)^2}{1-(-1)}$$

$$= \frac{1}{2}$$

$\therefore$ The value of $\lim_{x\to\pi} \frac{1+\cos x}{\tan^2 x} = \frac{1}{2}$

2. $\lim_{x \to \frac{\pi}{4}} \frac{\operatorname{cosec}^2 x - 2}{\cot x - 1}$

Solution:

Given: $\lim_{x \to \frac{\pi}{4}} \frac{\operatorname{cosec}^2 x - 2}{\cot x - 1}$

The limit $\lim_{x \to \frac{\pi}{4}} \frac{\operatorname{cosec}^2 x - 2}{\cot x - 1}$

When x = π/4, the expression $\lim_{x \to \frac{\pi}{4}} \frac{\operatorname{cosec}^2 x - 2}{\cot x - 1}$ assumes the form (0/0).

So,

$$\lim_{x \to \frac{\pi}{4}} \frac{\operatorname{cosec}^2 x - 2}{\cot x - 1} = \lim_{x \to \frac{\pi}{4}} \left[\frac{1 + \cot^2 x - 2}{\cot x - 1}\right]$$ [Since, $\operatorname{cosec}^2 x = 1 + \cot^2 x$]

$$= \lim_{x \to \frac{\pi}{4}} \left[\frac{\cot^2 x - 1}{\cot x - 1}\right]$$

Upon expansion, we get

$$= \lim_{x \to \frac{\pi}{4}} \left[\frac{(\cot x - 1)(\cot x + 1)}{(\cot x - 1)}\right]$$

Now, substitute the value of x, we get

$$= \cot \frac{\pi}{4} + 1$$

$$= 2$$

∴ The value of $\lim_{x \to \frac{\pi}{4}} \frac{\operatorname{cosec}^2 x - 2}{\cot x - 1} = 2$

3. $\lim_{x \to \frac{\pi}{6}} \frac{\cot^2 x - 3}{\operatorname{cosec} x - 2}$

Solution:

Given: $\lim_{x \to \frac{\pi}{6}} \frac{\cot^2 x - 3}{\operatorname{cosec} x - 2}$

The limit $\lim_{x \to \frac{\pi}{6}} \frac{\cot^2 x - 3}{\operatorname{cosec} x - 2}$

When x = π/6, the expression $\lim_{x \to \frac{\pi}{6}} \frac{\cot^2 x - 3}{\operatorname{cosec} x - 2}$ assumes the form (0/0).

$$\lim_{x \to \frac{\pi}{6}} \frac{\cot^2 x - 3}{\operatorname{cosec} x - 2} = \lim_{x \to \frac{\pi}{6}} \left[\frac{cosec^2 x - 1 - 3}{cosec x - 2}\right]$$ [Since, $\cot^2 x = \operatorname{cosec}^2 x - 1$]

$$= \lim_{x \to \frac{\pi}{6}} \left[\frac{cosec^2 x - 4}{cosec x - 2} \right]$$

Upon expansion, we get

$$= \lim_{x \to \frac{\pi}{6}} \left[\frac{(cosec x - 2)(cosec x + 2)}{(cosec x - 2)} \right]$$

Now, substitute the value of x, we get

$$= cosec \frac{\pi}{6} + 2$$

$$= 2 + 2$$

$$= 4$$

∴ The value of $\lim_{x \to \frac{\pi}{6}} \frac{\cot^2 x - 3}{cosec x - 2} = 4$

4. $\lim_{x \to \frac{\pi}{4}} \frac{2 - cosec^2 x}{1 - \cot x}$

Solution:

Given: $\lim_{x \to \frac{\pi}{4}} \frac{2 - cosec^2 x}{1 - \cot x}$

The limit $\lim_{x \to \frac{\pi}{4}} \frac{2 - cosec^2 x}{1 - \cot x}$

When x = π/4, the expression assumes the form (0/0).

So,

$$\lim_{x \to \frac{\pi}{4}} \frac{2 - cosec^2 x}{1 - \cot x} = \lim_{x \to \frac{\pi}{4}} \left[\frac{2 - (1 + \cot^2 x)}{1 - \cot x} \right]$$ [Since, $cosec^2 x = 1 + \cot^2 x$]

$$= \lim_{x \to \frac{\pi}{4}} \left[\frac{1 - \cot^2 x}{1 - \cot x} \right]$$

Upon expansion, we get

$$= \lim_{x \to \frac{\pi}{4}} \left[\frac{(1 - \cot x)(1 + \cot x)}{(1 - \cot x)} \right]$$

Now, substitute the value of x, we get

$$= 1 + \cot\left(\frac{\pi}{4}\right)$$

$$= 1 + 1$$

$$= 2$$

∴ The value of $\lim_{x \to \frac{\pi}{4}} \frac{2 - \operatorname{cosec}^2 x}{1 - \cot x} = 2$

5. $\lim_{x \to \pi} \frac{\sqrt{2 + \cos x} - 1}{(\pi - x)^2}$

Solution:

Given: $\lim_{x \to \pi} \frac{\sqrt{2 + \cos x} - 1}{(\pi - x)^2}$

The limit

When $x = \pi$, the expression $\lim_{x \to \pi} \frac{\sqrt{2 + \cos x} - 1}{(\pi - x)^2}$ assumes the form (0/0).

So, let us rationalize the numerator, we get

$$\lim_{x \to \pi} \frac{\sqrt{2 + \cos x} - 1}{(\pi - x)^2} = \lim_{x \to \pi} \left[\frac{(\sqrt{2 + \cos x} - 1) \times (\sqrt{2 + \cos x} + 1)}{(\pi - x)^2 (\sqrt{2 + \cos x} + 1)} \right]$$

Let us simplify the above expression, we get

$$= \lim_{x \to \pi} \left[\frac{2 + \cos x - 1}{(\pi - x)^2 (\sqrt{2 + \cos x} + 1)} \right]$$

$$= \lim_{x \to \pi} \left[\frac{1 + \cos x}{(\pi - x)^2 [\sqrt{2 + \cos x} + 1]} \right]$$

Now, let $x = \pi - h$

When $x = \pi$, then $h = 0$

So,

$$= \lim_{h \to 0} \left[\frac{1 + \cos(\pi - h)}{[\pi - (\pi - h)]^2 \left[\sqrt{2 + \cos(\pi - h)} + 1\right]} \right]$$

$$= \lim_{h \to 0} \left[\frac{1 - \cos h}{h^2 [\sqrt{2 - \cos h} + 1]} \right] \{\because \cos(\pi - \theta) = -\cos\theta\}$$

Let us simplify further,

$$= \lim_{h \to 0} \left[\frac{2 \sin^2\left(\frac{h}{2}\right)}{4 \times \frac{h^2}{4} [\sqrt{2 - \cos h} + 1]} \right]$$

$$= \frac{1}{2} \lim_{h \to 0} \left[\left(\frac{\sin \frac{h}{2}}{\frac{h}{2}} \right)^2 \times \frac{1}{[\sqrt{2 - \cos h} + 1]} \right]$$

Now, substitute the value of h, we get

$$= \frac{1}{2} \times 1 \times \frac{1}{(\sqrt{2 - \cos 0} + 1)}$$

$$= \frac{1}{2} \times \frac{1}{(\sqrt{1} + 1)}$$

$$= \frac{1}{2 \times 2}$$

$$= \frac{1}{4}$$

∴ The value of $\lim_{x \to \pi} \frac{\sqrt{2 + \cos x} - 1}{(\pi - x)^2} = \frac{1}{4}$

EXERCISE 29.10 PAGE NO: 29.71

Evaluate the following limits:

1. $\lim_{x\to 0} \frac{5^x - 1}{\sqrt{4+x} - 2}$

Solution:

Given:

The limit $\lim_{x\to 0} \frac{5^x - 1}{\sqrt{4+x} - 2}$

When x = 0, the expression $\lim_{x\to 0} \frac{5^x - 1}{\sqrt{4+x} - 2}$ assumes the form (0/0).

So,

As $Z = \lim_{x\to 0} \frac{5^x - 1}{\sqrt{4+x} - 2}$

Now, multiply both numerator and denominator by √(4+x) +2 so that we can remove the indeterminate form.

$$Z = \lim_{x\to 0} \frac{(5^x - 1)\sqrt{4+x}+2}{(\sqrt{4+x})^2 - 2^2}$$

$$= \lim_{x\to 0} \frac{5^x - 1}{\sqrt{4+x}-2} \times \frac{\sqrt{4+x}+2}{\sqrt{4+x}+2}$$

{By using $a^2 - b^2 = (a + b)(a - b)$}

$$Z = \lim_{x\to 0} \frac{(5^x - 1)\sqrt{4+x}+2}{4+x-4}$$

$$= \lim_{x\to 0} \frac{(5^x - 1)\sqrt{4+x}+2}{x}$$

By using basic algebra of limits, we get

$$Z = \lim_{x\to 0} \frac{(5^x - 1)}{x} \times \lim_{x\to 0} \sqrt{4+x} + 2 = \{\sqrt{4+0} + 2\}\lim_{x\to 0} \frac{(5^x - 1)}{x}$$

$= 4 \lim_{x\to 0} \frac{(5^x - 1)}{x}$ [By using the formula: $\lim_{x\to 0} \frac{(a^x - 1)}{x} = \log a$]

$Z = 4\log 5$

∴ The value of $\lim_{x\to 0} \frac{5^x - 1}{\sqrt{4+x}-2} = 4\log 5$

2. $\lim_{x\to 0} \frac{\log(1+x)}{3^x - 1}$

Solution:

Given: $\lim_{x\to 0} \frac{\log(1+x)}{3^x - 1}$

The limit

When x = 0, the expression $\lim_{x\to 0} \frac{\log(1+x)}{3^x - 1}$ assumes the form (0/0).

So,

As $Z = \lim_{x\to 0} \frac{\log(1+x)}{3^x-1}$

Let us divide numerator and denominator by x, we get

$Z = \lim_{x\to 0} \frac{\frac{\log(1+x)}{x}}{\frac{3^x-1}{x}} = \frac{\lim_{x\to 0}\frac{\log(1+x)}{x}}{\lim_{x\to 0}\frac{3^x-1}{x}}$ {by using basic limit algebra}

[By using the formula: $\lim_{x\to 0} \frac{(a^x-1)}{x} = \log a$]

$= \frac{1}{\log 3}$

∴ The value of $\lim_{x\to 0} \frac{\log(1+x)}{3^x - 1} = \frac{1}{\log 3}$

3. $\lim_{x\to 0} \frac{a^x + a^{-x} - 2}{x^2}$

Solution:

Given: $\lim_{x\to 0} \frac{a^x + a^{-x} - 2}{x^2}$

The limit

When x = 0, the expression $\lim_{x\to 0} \frac{a^x + a^{-x} - 2}{x^2}$ assumes the form (0/0).

So,

As $Z = \lim_{x\to 0} \frac{a^x+a^{-x}-2}{x^2}$

$= \lim_{x\to 0} \frac{a^{-x}(a^{2x}-2a^x+1)}{x^2}$

$= \lim_{x\to 0} \frac{(a^{2x}-2a^x+1)}{a^x x^2}$

$= \lim_{x\to 0} \frac{(a^x-1)^2}{a^x x^2}$ {By using $(a + b)^2 = a^2 + b^2 + 2ab$}

Let us use algebra of limit, we get

$Z = \lim_{x\to 0}\left(\frac{a^x-1}{x}\right)^2 \times \lim_{x\to 0}\frac{1}{a^x}$

[By using the formula: $\lim_{x\to 0}\frac{(a^x-1)}{x} = \log a$]

$Z = (\log a)^2 \frac{1}{a^0} = (\log a)^2$

∴ The value of $\lim_{x\to 0}\frac{a^x + a^{-x} - 2}{x^2} = (\log a)^2$

4. $\lim_{x\to 0}\frac{a^{mx}-1}{b^{nx}-1}, n \neq 0$

Solution:

Given: $\lim_{x\to 0}\frac{a^{mx}-1}{b^{nx}-1}, n \neq 0$

The limit $\lim_{x\to 0}\frac{a^{mx}-1}{b^{nx}-1}, n \neq 0$

When x = 0, the expression assumes the form (0/0).

So, let us include mx and nx as follows:

$Z = \lim_{x\to 0}\frac{a^{mx}-1}{b^{nx}-1} = \lim_{x\to 0}\frac{\frac{a^{mx}-1}{mx}\times mx}{\frac{b^{nx}-1}{nx}\times nx}$

$= \frac{m}{n}\lim_{x\to 0}\frac{\frac{a^{mx}-1}{mx}}{\frac{b^{nx}-1}{nx}}$

By using algebra of limits, we get

$Z = \frac{m}{n}\frac{\lim_{x\to 0}\frac{a^{mx}-1}{mx}}{\lim_{x\to 0}\frac{b^{nx}-1}{nx}}$

[By using the formula: $\lim_{x\to 0}\frac{(a^x-1)}{x} = \log a$]

$Z = \frac{m}{n}\frac{\log a}{\log b}, n \neq 0$

∴ The value of $\lim_{x\to 0}\frac{a^{mx}-1}{b^{nx}-1} = \frac{m}{n}\frac{\log a}{\log b}, n \neq 0$

5. $\lim_{x \to 0} \frac{a^x + b^x - 2}{x}$

Solution:

Given: $\lim_{x \to 0} \frac{a^x + b^x - 2}{x}$

The limit $\lim_{x \to 0} \frac{a^x + b^x - 2}{x}$

When x = 0, the expression $\lim_{x \to 0} \frac{a^x + b^x - 2}{x}$ assumes the form (0/0).

So,

As $Z = \lim_{x \to 0} \frac{a^x + b^x - 2}{x}$

$= \lim_{x \to 0} \frac{a^x - 1 + b^x - 1}{x}$

By using algebra of limits, we get

$Z = \lim_{x \to 0} \frac{a^x - 1}{x} + \lim_{x \to 0} \frac{b^x - 1}{x}$

[By using the formula: $\lim_{x \to 0} \frac{(a^x - 1)}{x} = \log a$]

$Z = \log a + \log b = \log ab$

∴ The value of $\lim_{x \to 0} \frac{a^x + b^x - 2}{x} = \log ab$

EXERCISE 29.11

Evaluate the following limits:

1. $\lim_{x \to \pi} \left(1 - \frac{x}{\pi}\right)^{\pi}$

Solution:

Given:

The limit $\lim_{x \to \pi} \left(1 - \frac{x}{\pi}\right)^{\pi}$

Let us substitute the value of x = π directly, we get

$Z = \lim_{x \to \pi} \left(1 - \frac{x}{\pi}\right)^{\pi} = \left(1 - \frac{\pi}{\pi}\right)^{\pi} = (1 - 1)^{\pi} = 0^{\pi} = 0$

Since, it is not of indeterminate form.

Z = 0

∴ The value of $\lim_{x \to \pi} \left(1 - \frac{x}{\pi}\right)^{\pi} = 0$

2. $\lim_{x \to 0^+} \left\{1 + \tan^2\sqrt{x}\right\}^{1/2x}$

Solution:

Given:

The limit $\lim_{x \to 0^+} \left\{1 + \tan^2\sqrt{x}\right\}^{1/2x}$

Let us use the theorem given below

If $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$ such that $\lim_{x \to a} \frac{f(x)}{g(x)}$ exists, then $\lim_{x \to a} [1 + f(x)]^{\frac{1}{g(x)}} = e^{\lim_{x \to a} \frac{f(x)}{g(x)}}$.

So here,

$f(x) = \tan^2 \sqrt{x}$

$g(x) = 2x$

Then,

$$\lim_{x \to 0^+} \left\{1 + \tan^2\sqrt{x}\right\}^{1/2x} = e^{\lim_{x \to 0^+} \left(\frac{\tan^2 \sqrt{x}}{2x}\right)}$$

$$= e^{\lim_{x \to 0^+} \left(\frac{\tan \sqrt{x}}{\sqrt{x}}\right) \times \left(\frac{\tan \sqrt{x}}{\sqrt{x}}\right) \times \frac{1}{2}}$$

$$= e^{1 \times 1 \times \frac{1}{2}}$$

$$= \sqrt{e}$$

∴ The value of $\lim_{x \to 0^+} \left\{1 + \tan^2\sqrt{x}\right\}^{1/2x} = \sqrt{e}$

3. $\lim_{x\to 0} (\cos x)^{1/\sin x}$

Solution:

Given:

The limit $\lim_{x\to 0} (\cos x)^{1/\sin x}$

Let us use the theorem given below

If $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$ such that $\lim_{x\to a} \frac{f(x)}{g(x)}$ exists, then $\lim_{x\to a} [1 + f(x)]^{\frac{1}{g(x)}} = e^{\lim_{x\to a} \frac{f(x)}{g(x)}}$.

So here,

f(x) = cos x - 1

g (x) = sin x

Then,

$$\lim_{x\to 0} (\cos x)^{1/\sin x} = e^{\lim_{x\to 0}\left(\frac{\cos x - 1}{\sin x}\right)}$$

$$= e^{\lim_{x\to 0}\left(\frac{-2\sin^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}}\right)}$$

$$= e^{\lim_{x\to 0}\left(-\tan\frac{x}{2}\right)}$$

$$= e^0$$

$$= 1$$

∴ The value of $\lim_{x\to 0} (\cos x)^{1/\sin x} = 1$

4. $\lim_{x\to 0} (\cos x + \sin x)^{1/x}$

Solution:

Given:

The limit $\lim_{x\to 0} (\cos x + \sin x)^{1/x}$

Let us add and subtract '1' to the given expression, we get

$$\lim_{x\to 0} [1 + \cos x + \sin x - 1]^{\frac{1}{x}}$$

Let us use the theorem given below

If $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$ such that $\lim_{x\to a} \frac{f(x)}{g(x)}$ exists, then $\lim_{x\to a} [1 + f(x)]^{\frac{1}{g(x)}} = e^{\lim_{x\to a} \frac{f(x)}{g(x)}}$.

So here,

f (x) = cos x + sin x - 1

g (x) = x

Then,

$$\lim_{x\to 0} (\cos x + \sin x)^{1/x} = e^{\lim_{x\to 0}}\left(\frac{\cos x + \sin x - 1}{x}\right)$$

Upon computing, we get

$$= e^{\lim_{x\to 0}}\left[\frac{\sin x}{x} - \frac{(1-\cos x)}{x}\right]$$

$$= e^{\lim_{x\to 0}}\left(\frac{\sin x}{x} - \frac{2\sin^2\frac{x}{2}}{x}\right)$$

$$= e^{\lim_{x\to 0}}\left(\frac{\sin x}{x} - \frac{2\sin\left(\frac{x}{2}\right)\times\sin\left(\frac{x}{2}\right)}{2\times\frac{x}{2}}\right)$$

Now, substitute the value of x, we get

$= e^{1-0}$

$= e^1$

$= e$

∴ The value of $\lim_{x\to 0} (\cos x + \sin x)^{1/x} = e$

5. $\lim_{x\to 0} (\cos x + a\sin bx)^{1/x}$

Solution:

Given:

The limit $\lim_{x\to 0} (\cos x + a\sin bx)^{1/x}$

Let us add and subtract '1' to the given expression, we get

$$\lim_{x\to 0} [1 + \cos x + a\sin bx - 1]^{\frac{1}{x}}$$

Let us use the theorem given below

If $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$ such that $\lim_{x\to a}\frac{f(x)}{g(x)}$ exists, then $\lim_{x\to a}[1+f(x)]^{\frac{1}{g(x)}} = e^{\lim_{x\to a}}\frac{f(x)}{g(x)}$.

So here,

$f(x) = \cos x + a\sin bx - 1$

$g(x) = x$

Then,

$$\lim_{x\to 0} (\cos x + a\sin bx)^{1/x} = e^{\lim_{x\to 0}}\left[\frac{\cos x + a\sin bx - 1}{x}\right]$$

Let us compute now, we get

$$= e^{\lim_{x \to 0}\left[\frac{b \times a \sin bx}{bx} - \frac{(1-\cos x)}{x}\right]}$$

$$= e^{\lim_{x \to 0}\left(\frac{ab \sin bx}{bx} - \frac{2\sin^2 \frac{x}{2}}{x}\right)}$$

Now, substitute the value of x, we get

$$= e^{ab}$$

$\therefore$ The value of $\lim_{x \to 0} (\cos x + a \sin bx)^{1/x} = e^{ab}$