EXERCISE 3.1 PAGE NO: 3.7

1. Define a function as a set of ordered pairs.
Solution:
Let A and B be two non-empty sets. A relation from A to B, i.e., a subset of A×B, is called a function (or a mapping) from A to B, if
(i) for each a ∈ A there exists b ∈ B such that (a, b) ∈ f
(ii) (a, b) ∈ f and (a, c) ∈ f ⇒ b = c

2. Define a function as a correspondence between two sets.
Solution:
Let A and B be two non-empty sets. Then a function 'f' from set A to B is a rule or method or correspondence which associates elements of set A to elements of set B such that:
(i) all elements of set A are associated to elements in set B.
(ii) an element of set A is associated to a unique element in set B.

3. What is the fundamental difference between a relation and a function? Is every relation a function?
Solution:
Let 'f' be a function and R be a relation defined from set X to set Y.

The domain of the relation R might be a subset of the set X, but the domain of the function f must be equal to X. This is because each element of the domain of a function must have an element associated with it, whereas this is not necessary for a relation.

In relation, one element of X might be associated with one or more elements of Y, while it must be associated with only one element of Y in a function.

Thus, not every relation is a function. However, every function is necessarily a relation.

4. Let A = {–2, –1, 0, 1, 2} and f: A → Z be a function defined by $f(x) = x^2 – 2x – 3$. Find:
(i) range of f i.e. f (A)
(ii) pre-images of 6, –3 and 5
Solution:
Given:
A = {–2, –1, 0, 1, 2}
f : A → Z such that $f(x) = x^2 – 2x – 3$

(i) Range of f i.e. f (A)
A is the domain of the function f. Hence, range is the set of elements f(x) for all x ∈ A.
Substituting x = –2 in f(x), we get
$f(-2) = (-2)^2 - 2(-2) - 3$
$= 4 + 4 - 3$
$= 5$
Substituting x = –1 in f(x), we get
$f(-1) = (-1)^2 - 2(-1) - 3$
$= 1 + 2 - 3$
$= 0$
Substituting x = 0 in f(x), we get
$f(0) = (0)^2 - 2(0) - 3$
$= 0 - 0 - 3$
$= -3$
Substituting x = 1 in f(x), we get
$f(1) = 1^2 - 2(1) - 3$
$= 1 - 2 - 3$
$= -4$
Substituting x = 2 in f(x), we get
$f(2) = 2^2 - 2(2) - 3$
$= 4 - 4 - 3$
$= -3$
Thus, the range of f is {-4, -3, 0, 5}.

(ii) pre-images of 6, –3 and 5
Let x be the pre-image of 6 ⇒ f(x) = 6
$x^2 - 2x - 3 = 6$
$x^2 - 2x - 9 = 0$
$x = [-(-2) \pm \sqrt{((-2)^2 - 4(1)(-9))}] / 2(1)$
$= [2 \pm \sqrt{(4+36)}] / 2$
$= [2 \pm \sqrt{40}] / 2$
$= 1 \pm \sqrt{10}$
However, $1 \pm \sqrt{10} \notin A$
Thus, there exists no pre-image of 6.

Now, let x be the pre-image of –3 ⇒ f(x) = –3
$x^2 - 2x - 3 = -3$
$x^2 - 2x = 0$
$x(x - 2) = 0$

x = 0 or 2
Clearly, both 0 and 2 are elements of A.
Thus, 0 and 2 are the pre-images of –3.

Now, let x be the pre-image of 5 ⇒ f(x) = 5
$x^2 - 2x - 3 = 5$
$x^2 - 2x - 8 = 0$
$x^2 - 4x + 2x - 8 = 0$
x(x – 4) + 2(x – 4) = 0
(x + 2)(x – 4) = 0
x = –2 or 4
However, 4 ∉ A but –2 ∈ A
Thus, –2 is the pre-images of 5.
∴ Ø, {0, 2}, -2 are the pre-images of 6, -3, 5

5. If a function f: R → R be defined by

$$f(x) = \begin{cases} 3x - 2, x < 0 \\ 1, x = 0 \\ 4x + 1, x > 0 \end{cases}$$

Find: f (1), f (–1), f (0), f (2).
Solution:
Given:
Let us find f (1), f (–1), f (0) and f (2).
When x > 0, f (x) = 4x + 1
Substituting x = 1 in the above equation, we get
f (1) = 4(1) + 1
= 4 + 1
= 5

When x < 0, f(x) = 3x – 2
Substituting x = –1 in the above equation, we get
f (–1) = 3(–1) – 2
= –3 – 2
= –5

When x = 0, f(x) = 1
Substituting x = 0 in the above equation, we get
f (0) = 1

When $x > 0$, $f(x) = 4x + 1$
Substituting $x = 2$ in the above equation, we get
$f(2) = 4(2) + 1$
$= 8 + 1$
$= 9$
$\therefore f(1) = 5, f(-1) = -5, f(0) = 1$ and $f(2) = 9$.

6. A function f: R → R is defined by $f(x) = x^2$. Determine
(i) range of f
(ii) {x: f(x) = 4}
(iii) {y: f(y) = –1}
Solution:
Given:
$f : R \rightarrow R$ and $f(x) = x^2$.
(i) range of f
Domain of f = R (set of real numbers)
We know that the square of a real number is always positive or equal to zero.
$\therefore$ range of f = $R^+ \cup \{0\}$

(ii) {x: f(x) = 4}
Given:
$f(x) = 4$
we know, $x^2 = 4$
$x^2 - 4 = 0$
$(x - 2)(x + 2) = 0$
$\therefore x = \pm 2$
$\therefore \{x: f(x) = 4\} = \{-2, 2\}$

(iii) {y: f(y) = –1}
Given:
$f(y) = -1$
$y^2 = -1$
However, the domain of f is R, and for every real number y, the value of y^2 is non-negative.
Hence, there exists no real y for which $y^2 = -1$.
$\therefore \{y: f(y) = -1\} = \emptyset$

7. Let f: $R^+ \rightarrow R$, where R^+ is the set of all positive real numbers, be such that f(x) =

$\log_e x$. Determine
(i) the image set of the domain of f
(ii) {x: f (x) = –2}
(iii) whether f (xy) = f (x) + f (y) holds.
Solution:
Given f: $R^+ \rightarrow R$ and $f(x) = \log_e x$.
(i) the image set of the domain of f
Domain of f = R^+ (set of positive real numbers)
We know the value of logarithm to the base e (natural logarithm) can take all possible real values.
∴ The image set of f = R

(ii) {x: f(x) = –2}
Given f(x) = –2
$\log_e x = -2$
∴ $x = e^{-2}$ [since, $\log_b a = c \Rightarrow a = b^c$]
∴ {x: f(x) = –2} = $\{e^{-2}\}$

(iii) Whether f (xy) = f (x) + f (y) holds.
We have $f(x) = \log_e x \Rightarrow f(y) = \log_e y$
Now, let us consider f (xy)
F (xy) = $\log_e (xy)$
f (xy) = $\log_e (x \times y)$ [since, $\log_b (a \times c) = \log_b a + \log_b c$]
f (xy) = $\log_e x + \log_e y$
f (xy) = f (x) + f (y)
∴ the equation f (xy) = f (x) + f (y) holds.

8. Write the following relations as sets of ordered pairs and find which of them are functions:
(i) {(x, y): y = 3x, x ∈ {1, 2, 3}, y ∈ {3, 6, 9, 12}}
(ii) {(x, y): y > x + 1, x = 1, 2 and y = 2, 4, 6}
(iii) {(x, y): x + y = 3, x, y ∈ {0, 1, 2, 3}}
Solution:
(i) {(x, y): y = 3x, x ∈ {1, 2, 3}, y ∈ {3, 6, 9, 12}}
When x = 1, y = 3(1) = 3
When x = 2, y = 3(2) = 6
When x = 3, y = 3(3) = 9
∴ R = {(1, 3), (2, 6), (3, 9)}
Hence, the given relation R is a function.

(ii) {(x, y): y > x + 1, x = 1, 2 and y = 2, 4, 6}
When x = 1, y > 1 + 1 or y > 2 ⇒ y = {4, 6}
When x = 2, y > 2 + 1 or y > 3 ⇒ y = {4, 6}
∴ R = {(1, 4), (1, 6), (2, 4), (2, 6)}
Hence, the given relation R is not a function.

(iii) {(x, y): x + y = 3, x, y ∈ {0, 1, 2, 3}}
When x = 0, 0 + y = 3 ⇒ y = 3
When x = 1, 1 + y = 3 ⇒ y = 2
When x = 2, 2 + y = 3 ⇒ y = 1
When x = 3, 3 + y = 3 ⇒ y = 0
∴ R = {(0, 3), (1, 2), (2, 1), (3, 0)}
Hence, the given relation R is a function.

9. Let f: R → R and g: C → C be two functions defined as f(x) = x^2 and g(x) = x^2. Are they equal functions?
Solution:
Given:
f: R → R ∈ f(x) = x^2 and g : R → R ∈ g(x) = x^2
f is defined from R to R, the domain of f = R.
g is defined from C to C, the domain of g = C.
Two functions are equal only when the domain and codomain of both the functions are equal.
In this case, the domain of f ≠ domain of g.
∴ f and g are not equal functions.

EXERCISE 3.2 PAGE NO: 3.11

1. If f (x) = x^2 – 3x + 4, then find the values of x satisfying the equation f (x) = f (2x + 1).

Solution:

Given:

f(x) = $x^2 - 3x + 4$.

Let us find x satisfying f (x) = f (2x + 1).

We have,

$f(2x + 1) = (2x + 1)^2 - 3(2x + 1) + 4$

$= (2x)^2 + 2(2x)(1) + 1^2 - 6x - 3 + 4$

$= 4x^2 + 4x + 1 - 6x + 1$

$= 4x^2 - 2x + 2$

Now, f (x) = f (2x + 1)

$x^2 - 3x + 4 = 4x^2 - 2x + 2$

$4x^2 - 2x + 2 - x^2 + 3x - 4 = 0$

$3x^2 + x - 2 = 0$

$3x^2 + 3x - 2x - 2 = 0$

$3x(x + 1) - 2(x + 1) = 0$

$(x + 1)(3x - 2) = 0$

x + 1 = 0 or 3x – 2 = 0

x = –1 or 3x = 2

x = –1 or 2/3

∴ The values of x are –1 and 2/3.

2. If f (x) = $(x - a)^2 (x - b)^2$, find f (a + b).

Solution:

Given:

F (x) = $(x - a)^2(x - b)^2$

Let us find f (a + b).

We have,

$f(a + b) = (a + b - a)^2 (a + b - b)^2$

$f(a + b) = (b)^2 (a)^2$

∴ $f(a + b) = a^2b^2$

3. If y = f (x) = (ax - b) / (bx - a), show that x = f (y).

Solution:

Given:

y = f (x) = (ax - b) / (bx - a) ⇒ f (y) = (ay - b) / (by - a)
Let us prove that x = f (y).
We have,
y = (ax - b) / (bx - a)
By cross-multiplying,
y(bx – a) = ax – b
bxy – ay = ax – b
bxy – ax = ay – b
x(by – a) = ay – b
x = (ay - b) / (by - a) = f (y)
∴ x = f (y)
Hence proved.

4. If f (x) = 1 / (1 - x), show that f [f {f (x)}] = x.
Solution:
Given:
f (x) = 1 / (1 - x)
Let us prove that f [f {f (x)}] = x.
Firstly, let us solve for f {f (x)}.
f {f (x)} = f {1/(1 - x)}
= 1 / 1 - (1/(1 - x))
= 1 / [(1 – x - 1)/(1 - x)]
= 1 / (-x/(1 - x))
= (1 - x) / -x
= (x – 1) / x
∴ f {f (x)} = (x – 1) / x

Now, we shall solve for f [f {f (x)}]
f [f {f (x)}] = f [(x-1)/x]
= 1 / [1 – (x-1)/x]
= 1 / [(x – (x-1))/x]
= 1 / [(x - x + 1)/x]
= 1 / (1/x)
∴ f [f {f (x)}] = x
Hence proved.

5. If f (x) = (x + 1) / (x - 1), show that f [f (x)] = x.
Solution:
Given:

f (x) = (x + 1) / (x - 1)
Let us prove that f [f (x)] = x.
f [f (x)] = f [(x+1)/(x-1)]
= [(x+1)/(x-1) + 1] / [(x+1)/(x-1) - 1]
= [[(x+1) + (x-1)]/(x-1)] / [[(x+1) – (x-1)]/(x-1)]
= [(x+1) + (x-1)] / [(x+1) – (x-1)]
= (x+1+x-1)/(x+1-x+1)
= 2x/2
= x
∴ f [f (x)] = x
Hence proved.

6. If

$$f(x) = \begin{cases} x^2, \text{when } x < 0 \\ x, \text{when } 0 \le x < 1 \\ \frac{1}{x}, \text{when } x \ge 1 \end{cases}$$

Find:
(i) f (1/2)
(ii) f (-2)
(iii) f (1)
(iv) f (√3)
(v) f (√-3)
Solution:
(i) f (1/2)
When, 0 ≤ x ≤ 1, f(x) = x
∴ f (1/2) = ½

(ii) f (-2)
When, x < 0, f(x) = x^2
f (–2) = $(–2)^2$
= 4
∴ f (–2) = 4

(iii) f (1)
When, x ≥ 1, f (x) = 1/x
f (1) = 1/1
∴ f(1) = 1

(iv) f (√3)
We have √3 = 1.732 > 1
When, x ≥ 1, f (x) = 1/x
∴ f (√3) = 1/√3

(v) f (√-3)
We know √-3 is not a real number and the function f(x) is defined only when x ∈ R.
∴ f (√-3) does not exist.

EXERCISE 3.3 PAGE NO: 3.18

1. Find the domain of each of the following real valued functions of real variable:
(i) f (x) = 1/x
(ii) f (x) = 1/(x-7)
(iii) f (x) = (3x-2)/(x+1)
(iv) f (x) = (2x+1)/(x^2-9)
(v) f (x) = (x^2+2x+1)/(x^2-8x+12)
Solution:
(i) f (x) = 1/x
We know, f (x) is defined for all real values of x, except for the case when x = 0.
∴ Domain of f = R – {0}

(ii) f (x) = 1/(x-7)
We know, f (x) is defined for all real values of x, except for the case when x – 7 = 0 or x = 7.
∴ Domain of f = R – {7}

(iii) f (x) = (3x-2)/(x+1)
We know, f(x) is defined for all real values of x, except for the case when x + 1 = 0 or x = –1.
∴ Domain of f = R – {–1}

(iv) f (x) = (2x+1)/(x^2-9)
We know, f (x) is defined for all real values of x, except for the case when $x^2 – 9 = 0$.
$x^2 – 9 = 0$
$x^2 – 3^2 = 0$
(x + 3)(x – 3) = 0
x + 3 = 0 or x – 3 = 0
x = ± 3
∴ Domain of f = R – {–3, 3}

(v) f (x) = (x^2+2x+1)/(x^2-8x+12)
We know, f(x) is defined for all real values of x, except for the case when $x^2 – 8x + 12 = 0$.
$x^2 – 8x + 12 = 0$
$x^2 – 2x – 6x + 12 = 0$
x(x – 2) – 6(x – 2) = 0
(x – 2)(x – 6) = 0

x – 2 = 0 or x – 6 = 0
x = 2 or 6
∴ Domain of f = R – {2, 6}

2. Find the domain of each of the following real valued functions of real variable:
(i) f (x) = √(x-2)
(ii) f (x) = 1/(√(x^2-1))
(iii) f (x) = √(9-x^2)
(iv) f (x) = √(x-2)/(3-x)
Solution:
(i) f (x) = √(x-2)
We know the square of a real number is never negative.
f (x) takes real values only when x – 2 ≥ 0
x ≥ 2
∴ x ∈ [2, ∞)
∴ Domain (f) = [2, ∞)

(ii) f (x) = 1/(√(x^2-1))
We know the square of a real number is never negative.
f (x) takes real values only when x^2 – 1 ≥ 0
$x^2 - 1^2 \geq 0$
(x + 1) (x – 1) ≥ 0
x ≤ –1 or x ≥ 1
∴ x ∈ (–∞, –1] ∪ [1, ∞)
In addition, f (x) is also undefined when x^2 – 1 = 0 because denominator will be zero and the result will be indeterminate.
$x^2 - 1 = 0 \Rightarrow x = \pm 1$
So, x ∈ (–∞, –1] ∪ [1, ∞) – {–1, 1}
x ∈ (–∞, –1) ∪ (1, ∞)
∴ Domain (f) = (–∞, –1) ∪ (1, ∞)

(iii) f (x) = √(9-x^2)
We know the square of a real number is never negative.
f (x) takes real values only when 9 – x^2 ≥ 0
$9 \geq x^2$
$x^2 \leq 9$
$x^2 - 9 \leq 0$
$x^2 - 3^2 \leq 0$
(x + 3)(x – 3) ≤ 0

$x \geq -3$ and $x \leq 3$
$x \in [-3, 3]$
∴ Domain (f) = [–3, 3]

(iv) f (x) = $\sqrt{(x-2)/(3-x)}$
We know the square root of a real number is never negative.
f (x) takes real values only when x – 2 and 3 – x are both positive and negative.
(a) Both x – 2 and 3 – x are positive
$x - 2 \geq 0$
$x \geq 2$

$3 - x \geq 0$
$x \leq 3$
Hence, $x \geq 2$ and $x \leq 3$
∴ $x \in [2, 3]$

(b) Both x – 2 and 3 – x are negative
$x - 2 \leq 0$
$x \leq 2$

$3 - x \leq 0$
$x \geq 3$
Hence, $x \leq 2$ and $x \geq 3$
However, the intersection of these sets is null set. Thus, this case is not possible.
Hence, $x \in [2, 3] - [3]$
$x \in [2, 3]$
∴ Domain (f) = [2, 3]

3. Find the domain and range of each of the following real valued functions:
(i) f (x) = (ax+b)/(bx-a)
(ii) f (x) = (ax-b)/(cx-d)
(iii) f (x) = $\sqrt{(x-1)}$
(iv) f (x) = $\sqrt{(x-3)}$
(v) f (x) = (x-2)/(2-x)
(vi) f (x) = |x-1|
(vii) f (x) = -|x|
(viii) f (x) = $\sqrt{(9-x^2)}$
Solution:
(i) f (x) = (ax+b)/(bx-a)

f(x) is defined for all real values of x, except for the case when bx – a = 0 or x = a/b.
Domain (f) = R – (a/b)
Let f (x) = y
(ax+b)/(bx-a) = y
ax + b = y(bx – a)
ax + b = bxy – ay
ax – bxy = –ay – b
x(a – by) = –(ay + b)
∴ x = - (ay+b)/(a-by)
When a – by = 0 or y = a/b
Hence, f(x) cannot take the value a/b.
∴ Range (f) = R – (a/b)

(ii) f (x) = (ax-b)/(cx-d)
f(x) is defined for all real values of x, except for the case when cx – d = 0 or x = d/c.
Domain (f) = R – (d/c)
Let f (x) = y
(ax-b)/(cx-d) = y
ax – b = y(cx – d)
ax – b = cxy – dy
ax – cxy = b – dy
x(a – cy) = b – dy
∴ x = (b-dy)/(a-cy)
When a – cy = 0 or y = a/c,
Hence, f(x) cannot take the value a/c.
∴ Range (f) = R – (a/c)

(iii) f (x) = √(x-1)
We know the square of a real number is never negative.
f(x) takes real values only when x – 1 ≥ 0
x ≥ 1
∴ x ∈ [1, ∞)
Thus, domain (f) = [1, ∞)
When x ≥ 1, we have x – 1 ≥ 0
Hence, √(x-1) ≥ 0 ⇒ f (x) ≥ 0
f(x) ∈ [0, ∞)
∴ Range (f) = [0, ∞)

(iv) f (x) = √(x-3)

We know the square of a real number is never negative.
f (x) takes real values only when x – 3 ≥ 0
x ≥ 3
∴ x ∈ [3, ∞)
Domain (f) = [3, ∞)
When x ≥ 3, we have x – 3 ≥ 0
Hence, √(x-3) ≥ 0 ⇒ f (x) ≥ 0
f(x) ∈ [0, ∞)
∴ Range (f) = [0, ∞)

(v) f (x) = (x-2)/(2-x)
f(x) is defined for all real values of x, except for the case when 2 – x = 0 or x = 2.
Domain (f) = R – {2}
We have, f (x) = (x-2)/(2-x)
f (x) = -(2-x)/(2-x)
= –1
When x ≠ 2, f(x) = –1
∴ Range (f) = {–1}

(vi) f (x) = |x-1|

$$we\ know\ |x| = \begin{cases} -x, x < 0 \\ x, x \geq 0 \end{cases}$$

Now we have,

$$|x-1| = \begin{cases} -(x-1), x-1 < 0 \\ x-1, x-1 \geq 0 \end{cases}$$

$$\therefore f(x) = |x-1| = \begin{cases} 1-x, x < 1 \\ x-1, x \geq 1 \end{cases}$$

Hence, f(x) is defined for all real numbers x.
Domain (f) = R
When, x < 1, we have x – 1 < 0 or 1 – x > 0.
|x – 1| > 0 ⇒ f(x) > 0

When, x ≥ 1, we have x – 1 ≥ 0.
|x – 1| ≥ 0 ⇒ f(x) ≥ 0

∴ f(x) ≥ 0 or f(x) ∈ [0, ∞)
Range (f) = [0, ∞)

(vii) f (x) = -|x|

$$we\ know\ |x| = \begin{cases} -x, x < 0 \\ x, x \geq 0 \end{cases}$$

Now we have,

$$-|x| = \begin{cases} -(-x), x < 0 \\ -x, x \geq 0 \end{cases}$$

$$\therefore f(x) = -|x| = \begin{cases} x, x < 0 \\ -x, x \geq 0 \end{cases}$$

Hence, f(x) is defined for all real numbers x.
Domain (f) = R
When, $x < 0$, we have $-|x| < 0$
$f(x) < 0$

When, $x \geq 0$, we have $-x \leq 0$.
$-|x| \leq 0 \Rightarrow f(x) \leq 0$

$\therefore f(x) \leq 0$ or $f(x) \in (-\infty, 0]$
Range (f) = $(-\infty, 0]$

(viii) $f(x) = \sqrt{(9-x^2)}$
We know the square of a real number is never negative.
f(x) takes real values only when $9 - x^2 \geq 0$
$9 \geq x^2$
$x^2 \leq 9$
$x^2 - 9 \leq 0$
$x^2 - 3^2 \leq 0$
$(x + 3)(x - 3) \leq 0$
$x \geq -3$ and $x \leq 3$
$\therefore x \in [-3, 3]$
Domain (f) = $[-3, 3]$
When, $x \in [-3, 3]$, we have $0 \leq 9 - x^2 \leq 9$
$0 \leq \sqrt{(9-x^2)} \leq 3 \Rightarrow 0 \leq f(x) \leq 3$
$\therefore f(x) \in [0, 3]$
Range (f) = $[0, 3]$

EXERCISE 3.4 PAGE NO: 3.38

1. Find f + g, f – g, cf (c ∈ R, c ≠ 0), fg, 1/f and f/g in each of the following:
(i) f (x) = x^3 + 1 and g (x) = x + 1
(ii) f (x) = $\sqrt{(x-1)}$ and g (x) = $\sqrt{(x+1)}$
Solution:
(i) f (x) = x^3 + 1 and g(x) = x + 1
We have f(x): R → R and g(x): R → R

(a) f + g
We know, (f + g) (x) = f(x) + g(x)
(f + g) (x) = x^3 + 1 + x + 1
= x^3 + x + 2
So, (f + g) (x): R → R
∴ f + g: R → R is given by (f + g) (x) = x^3 + x + 2

(b) f – g
We know, (f – g) (x) = f(x) – g(x)
(f – g) (x) = x^3 + 1 – (x + 1)
= x^3 + 1 – x – 1
= x^3 – x
So, (f – g) (x): R → R
∴ f – g: R → R is given by (f – g) (x) = x^3 – x

(c) cf (c ∈ R, c ≠ 0)
We know, (cf) (x) = c × f(x)
(cf)(x) = c(x^3 + 1)
= cx^3 + c
So, (cf) (x) : R → R
∴ cf: R → R is given by (cf) (x) = cx^3 + c

(d) fg
We know, (fg) (x) = f(x) g(x)
(fg) (x) = (x^3 + 1) (x + 1)
= (x + 1) (x^2 – x + 1) (x + 1)
= $(x + 1)^2$ (x^2 – x + 1)
So, (fg) (x): R → R
∴ fg: R → R is given by (fg) (x) = $(x + 1)^2$(x^2 – x + 1)

(e) 1/f
We know, (1/f) (x) = 1/f (x)
$1/f(x) = 1/(x^3 + 1)$
Observe that 1/f(x) is undefined when f(x) = 0 or when x = – 1.
So, 1/f: R – {–1} → R is given by $1/f(x) = 1/(x^3 + 1)$

(f) f/g
We know, (f/g) (x) = f(x)/g(x)
$(f/g)(x) = (x^3 + 1)/(x + 1)$
Observe that $(x^3 + 1)/(x + 1)$ is undefined when g(x) = 0 or when x = –1.
Using $x^3 + 1 = (x + 1)(x^2 – x + 1)$, we have
$(f/g)(x) = [(x+1)(x^2 - x+1)/(x+1)]$
$= x^2 – x + 1$
∴ f/g: R – {–1} → R is given by $(f/g)(x) = x^2 – x + 1$

(ii) f (x) = √(x-1) and g (x) = √(x+1)
We have f(x): [1, ∞) → R^+ and g(x): [–1, ∞) → R^+ as real square root is defined only for non-negative numbers.

(a) f + g
We know, (f + g) (x) = f(x) + g(x)
(f+g) (x) = √(x-1) + √(x+1)
Domain of (f + g) = Domain of f ∩ Domain of g
Domain of (f + g) = [1, ∞) ∩ [–1, ∞)
Domain of (f + g) = [1, ∞)
∴ f + g: [1, ∞) → R is given by (f+g) (x) = √(x-1) + √(x+1)

(b) f – g
We know, (f – g) (x) = f(x) – g(x)
(f-g) (x) = √(x-1) - √(x+1)
Domain of (f – g) = Domain of f ∩ Domain of g
Domain of (f – g) = [1, ∞) ∩ [–1, ∞)
Domain of (f – g) = [1, ∞)
∴ f – g: [1, ∞) → R is given by (f-g) (x) = √(x-1) - √(x+1)

(c) cf (c ∈ R, c ≠ 0)
We know, (cf) (x) = c × f(x)
(cf) (x) = c√(x-1)
Domain of (cf) = Domain of f

Domain of (cf) = [1, ∞)
∴ cf: [1, ∞) → R is given by (cf) (x) = c√(x-1)

(d) fg
We know, (fg) (x) = f(x) g(x)
(fg) (x) = √(x-1) √(x+1)
= $\sqrt{(x^2 - 1)}$
Domain of (fg) = Domain of f ∩ Domain of g
Domain of (fg) = [1, ∞) ∩ [–1, ∞)
Domain of (fg) = [1, ∞)
∴ fg: [1, ∞) → R is given by (fg) (x) = $\sqrt{(x^2 - 1)}$

(e) 1/f
We know, (1/f) (x) = 1/f(x)
(1/f) (x) = 1/√(x-1)
Domain of (1/f) = Domain of f
Domain of (1/f) = [1, ∞)
Observe that 1/√(x-1) is also undefined when x – 1 = 0 or x = 1.
∴ 1/f: (1, ∞) → R is given by (1/f) (x) = 1/√(x-1)

(f) f/g
We know, (f/g) (x) = f(x)/g(x)
(f/g) (x) = √(x-1)/√(x+1)
(f/g) (x) = √[(x-1)/(x+1)]
Domain of (f/g) = Domain of f ∩ Domain of g
Domain of (f/g) = [1, ∞) ∩ [–1, ∞)
Domain of (f/g) = [1, ∞)
∴ f/g: [1, ∞) → R is given by (f/g) (x) = √[(x-1)/(x+1)]

2. Let f(x) = 2x + 5 and g(x) = x^2 + x. Describe
(i) f + g
(ii) f – g
(iii) fg
(iv) f/g
Find the domain in each case.
Solution:
Given:
f(x) = 2x + 5 and g(x) = x^2 + x
Both f(x) and g(x) are defined for all x ∈ R.

So, domain of f = domain of g = R

(i) f + g

We know, (f + g)(x) = f(x) + g(x)

(f + g)(x) = 2x + 5 + x^2 + x

= x^2 + 3x + 5

(f + g)(x) Is defined for all real numbers x.

∴ The domain of (f + g) is R

(ii) f – g

We know, (f – g)(x) = f(x) – g(x)

(f – g)(x) = 2x + 5 – (x^2 + x)

= 2x + 5 – x^2 – x

= 5 + x – x^2

(f – g)(x) is defined for all real numbers x.

∴ The domain of (f – g) is R

(iii) fg

We know, (fg)(x) = f(x)g(x)

(fg)(x) = (2x + 5)(x^2 + x)

= 2x(x^2 + x) + 5(x^2 + x)

= 2x^3 + 2x^2 + 5x^2 + 5x

= 2x^3 + 7x^2 + 5x

(fg)(x) is defined for all real numbers x.

∴ The domain of fg is R

(iv) f/g

We know, (f/g) (x) = f(x)/g(x)

(f/g) (x) = (2x+5)/(x^2+x)

(f/g) (x) is defined for all real values of x, except for the case when x^2 + x = 0.

x^2 + x = 0

x(x + 1) = 0

x = 0 or x + 1 = 0

x = 0 or –1

When x = 0 or –1, (f/g) (x) will be undefined as the division result will be indeterminate.

∴ The domain of f/g = R – {–1, 0}

3. If f(x) be defined on [–2, 2] and is given by $f(x)=\begin{cases}-1, -2\le x\le 0\\ x-1, 0<x\le 2\end{cases}$ **and g(x) = f(|x|) + |f(x)|. Find g(x).**

Solution:

Given:

$$f(x) = \begin{cases} -1, -2 \le x \le 0 \\ x-1, 0 < x \le 2 \end{cases}$$ and

g(x) = f(|x|) + |f(x)|

Now we have,

$$f(|x|) = \begin{cases} -1, -2 \le |x| \le 0 \\ |x|-1, 0 < |x| \le 2 \end{cases}$$

However, |x| ≥ 0 ⇒ f(|x|) = |x| – 1 when 0 < |x| ≤ 2

We also have,

$$|f(x)| = \begin{cases} |-1|, -2 \le x \le 0 \\ |x-1|, 0 < x \le 2 \end{cases}$$
$$= \begin{cases} 1, -2 \le x \le 0 \\ |x-1|, 0 < x \le 2 \end{cases}$$

We also know,

$$|x-1| = \begin{cases} -(x-1), x-1 < 0 \\ x-1, x-1 \ge 0 \end{cases}$$
$$= \begin{cases} -(x-1), x < 1 \\ x-1, x \ge 1 \end{cases}$$

Here, we shall only the range between [0, 2].

$$|x-1| = \begin{cases} -(x-1), 0 < x < 1 \\ x-1, 1 \le x \le 2 \end{cases}$$

Substituting this value of |x – 1| in |f(x)|, we get

$$|f(x)| = \begin{cases} 1, -2 \le x \le 0 \\ -(x-1), 0 < x < 1 \\ x-1, 1 \le x \le 2 \end{cases}$$
$$= \begin{cases} 1, -2 \le x \le 0 \\ 1-x, 0 < x < 1 \\ x-1, 1 \le x \le 2 \end{cases}$$

Now, we need to find g(x)

g(x) = f(|x|) + |f(x)|

$$= |x| - 1 \text{ when } 0 < |x| \le 2 + \begin{cases} 1, -2 \le x \le 0 \\ 1-x, 0 < x < 1 \\ x-1, 1 \le x \le 2 \end{cases}$$

$$g(x)=\begin{cases}-x-1,-2\le x\le 0\\ x-1,0<x<1\\ x-1,1\le x\le 2\end{cases}+\begin{cases}1,-2\le x\le 0\\ 1-x,0<x<1\\ x-1,1\le x\le 2\end{cases}$$
$$=\begin{cases}-x-1+1,-2\le x\le 0\\ x-1+1-x,0<x<1\\ x-1+x-1,1\le x\le 2\end{cases}$$
$$=\begin{cases}-x,-2\le x\le 0\\ 0,0<x<1\\ 2(x-1),1\le x\le 2\end{cases}$$

∴ g(x) = f(|x|) + |f(x)| |
$$=\begin{cases}-x,-2\le x\le 0\\ 0,0<x<1\\ 2(x-1),1\le x\le 2\end{cases}$$

4. Let f, g be two real functions defined by f(x) = √(x+1) and g(x) = √(9-x^2). Then, describe each of the following functions.
(i) f + g
(ii) g – f
(iii) fg
(iv) f/g
(v) g/f
(vi) 2f - √5g
(vii) f^2 + 7f
(viii) 5/g
Solution:
Given:
f(x) = √(x+1) and g(x) = √(9-x^2)
We know the square of a real number is never negative.
So, f(x) takes real values only when x + 1 ≥ 0
x ≥ –1, x ∈ [–1, ∞)
Domain of f = [–1, ∞)

Similarly, g(x) takes real values only when 9 – x^2 ≥ 0
9 ≥ x^2
x^2 ≤ 9
x^2 – 9 ≤ 0
x^2 – 3^2 ≤ 0
(x + 3)(x – 3) ≤ 0

$x \geq -3$ and $x \leq 3$
$\therefore x \in [-3, 3]$
Domain of g = [–3, 3]

(i) f + g
We know, (f + g)(x) = f(x) + g(x)
$(f + g)(x) = \sqrt{(x+1)} + \sqrt{(9-x^2)}$
Domain of f + g = Domain of f ∩ Domain of g
= [–1, ∞) ∩ [–3, 3]
= [–1, 3]
$\therefore f + g: [-1, 3] \to R$ is given by $(f + g)(x) = f(x) + g(x) = \sqrt{(x+1)} + \sqrt{(9-x^2)}$

(ii) g – f
We know, (g – f)(x) = g(x) – f(x)
$(g - f)(x) = \sqrt{(9-x^2)} - \sqrt{(x+1)}$
Domain of g – f = Domain of g ∩ Domain of f
= [–3, 3] ∩ [–1, ∞)
= [–1, 3]
$\therefore g - f: [-1, 3] \to R$ is given by $(g - f)(x) = g(x) - f(x) = \sqrt{(9-x^2)} - \sqrt{(x+1)}$

(iii) fg
We know, (fg) (x) = f(x)g(x)
$(fg)(x) = \sqrt{(x+1)}\ \sqrt{(9-x^2)}$
$= \sqrt{[(x+1)(9-x^2)]}$
$= \sqrt{[x(9-x^2) + (9-x^2)]}$
$= \sqrt{(9x-x^3+9-x^2)}$
$= \sqrt{(9+9x-x^2-x^3)}$
Domain of fg = Domain of f ∩ Domain of g
= [–1, ∞) ∩ [–3, 3]
= [–1, 3]
$\therefore fg: [-1, 3] \to R$ is given by $(fg)(x) = f(x)\,g(x) = \sqrt{(x+1)}\ \sqrt{(9-x^2)} = \sqrt{(9+9x-x^2-x^3)}$

(iv) f/g
We know, (f/g) (x) = f(x)/g(x)
$(f/g)(x) = \sqrt{(x+1)} / \sqrt{(9-x^2)}$
$= \sqrt{[(x+1) / (9-x^2)]}$
Domain of f/g = Domain of f ∩ Domain of g
= [–1, ∞) ∩ [–3, 3]
= [–1, 3]

However, (f/g) (x) is defined for all real values of $x \in [-1, 3]$, except for the case when $9 - x^2 = 0$ or $x = \pm 3$
When $x = \pm 3$, (f/g) (x) will be undefined as the division result will be indeterminate.
Domain of f/g = [–1, 3] – {–3, 3}
Domain of f/g = [–1, 3)
∴ f/g: [–1, 3) → R is given by (f/g) (x) = f(x)/g(x) = $\sqrt{(x+1)} / \sqrt{(9-x^2)}$

(v) g/f
We know, (g/f) (x) = g(x)/f(x)
(g/f) (x) = $\sqrt{(9-x^2)} / \sqrt{(x+1)}$
$= \sqrt{[(9-x^2) / (x+1)]}$
Domain of g/f = Domain of f ∩ Domain of g
= [–1, ∞) ∩ [–3, 3]
= [–1, 3]
However, (g/f) (x) is defined for all real values of $x \in [-1, 3]$, except for the case when $x + 1 = 0$ or $x = -1$
When x = –1, (g/f) (x) will be undefined as the division result will be indeterminate.
Domain of g/f = [–1, 3] – {–1}
Domain of g/f = (–1, 3]
∴ g/f: (–1, 3] → R is given by (g/f) (x) = g(x)/f(x) = $\sqrt{(9-x^2)} / \sqrt{(x+1)}$

(vi) 2f - √5g
We know, (2f – √5g) (x) = 2f(x) – √5g(x)
(2f - √5g) (x) = 2f (x) - √5g (x)
$= 2\sqrt{(x+1)} - \sqrt{5}\sqrt{(9-x^2)}$
$= 2\sqrt{(x+1)} - \sqrt{(45- 5x^2)}$
Domain of 2f - √5g = Domain of f ∩ Domain of g
= [–1, ∞) ∩ [–3, 3]
= [–1, 3]
∴ 2f - √5g: [–1, 3] → R is given by (2f - √5g) (x) = 2f (x) - √5g (x) = $2\sqrt{(x+1)} - \sqrt{(45- 5x^2)}$

(vii) $f^2 + 7f$
We know, $(f^2 + 7f)(x) = f^2(x) + (7f)(x)$
$(f^2 + 7f)(x)$ = f(x) f(x) + 7f(x)
$= \sqrt{(x+1)}\sqrt{(x+1)} + 7\sqrt{(x+1)}$
$= x + 1 + 7\sqrt{(x+1)}$
Domain of $f^2 + 7f$ is same as domain of f.
Domain of $f^2 + 7f$ = [–1, ∞)
∴ $f^2 + 7f$: [–1, ∞) → R is given by $(f^2 + 7f)(x)$ = f(x) f(x) + 7f(x) = $x + 1 + 7\sqrt{(x+1)}$

(viii) 5/g
We know, (5/g) (x) = 5/g(x)
(5/g) (x) = $5/\sqrt{(9-x^2)}$
Domain of 5/g = Domain of g = [–3, 3]
However, (5/g) (x) is defined for all real values of x ∈ [–3, 3], except for the case when $9 - x^2 = 0$ or x = ± 3
When x = ±3, (5/g) (x) will be undefined as the division result will be indeterminate.
Domain of 5/g = [–3, 3] – {–3, 3}
= (–3, 3)
∴ 5/g: (–3, 3) → R is given by (5/g) (x) = 5/g(x) = $5/\sqrt{(9-x^2)}$

5. If f(x) = $\log_e$ (1 – x) and g(x) = [x], then determine each of the following functions:
(i) f + g
(ii) fg
(iii) f/g
(iv) g/f
Also, find (f + g) (–1), (fg) (0), (f/g) (1/2) and (g/f) (1/2).
Solution:
Given:
f(x) = $\log_e$ (1 – x) and g(x) = [x]
We know, f(x) takes real values only when 1 – x > 0
1 > x
x < 1, ∴ x ∈ (–∞, 1)
Domain of f = (–∞, 1)

Similarly, g(x) is defined for all real numbers x.
Domain of g = [x], x ∈ R
= R

(i) f + g
We know, (f + g) (x) = f(x) + g(x)
(f + g) (x) = $\log_e$ (1 – x) + [x]
Domain of f + g = Domain of f ∩ Domain of g
Domain of f + g = (–∞, 1) ∩ R
= (–∞, 1)
∴ f + g: (–∞, 1) → R is given by (f + g) (x) = $\log_e$ (1 – x) + [x]

(ii) fg

We know, (fg) (x) = f(x) g(x)
(fg) (x) = $\log_e (1 - x) \times [x]$
= $[x] \log_e (1 - x)$
Domain of fg = Domain of f ∩ Domain of g
= (–∞, 1) ∩ R
= (–∞, 1)
∴ fg: (–∞, 1) → R is given by (fg) (x) = $[x] \log_e (1 - x)$

(iii) f/g
We know, (f/g) (x) = f(x)/g(x)
(f/g) (x) = $\log_e (1 - x) / [x]$
Domain of f/g = Domain of f ∩ Domain of g
= (–∞, 1) ∩ R
= (–∞, 1)
However, (f/g) (x) is defined for all real values of x ∈ (–∞, 1), except for the case when [x] = 0.
We have, [x] = 0 when $0 \leq x < 1$ or x ∈ [0, 1)
When $0 \leq x < 1$, (f/g) (x) will be undefined as the division result will be indeterminate.
Domain of f/g = (–∞, 1) – [0, 1)
= (–∞, 0)
∴ f/g: (–∞, 0) → R is given by (f/g) (x) = $\log_e (1 - x) / [x]$

(iv) g/f
We know, (g/f) (x) = g(x)/f(x)
(g/f) (x) = $[x] / \log_e (1 - x)$
However, (g/f) (x) is defined for all real values of x ∈ (–∞, 1), except for the case when $\log_e (1 - x) = 0$.
$\log_e (1 - x) = 0 \Rightarrow 1 - x = 1$ or x = 0
When x = 0, (g/f) (x) will be undefined as the division result will be indeterminate.
Domain of g/f = (–∞, 1) – {0}
= (–∞, 0) ∪ (0, 1)
∴ g/f: (–∞, 0) ∪ (0, 1) → R is given by (g/f) (x) = $[x] / \log_e (1 - x)$

(a) We need to find (f + g) (–1).
We have, (f + g) (x) = $\log_e (1 - x) + [x]$, x ∈ (–∞, 1)
Substituting x = –1 in the above equation, we get
(f + g)(–1) = $\log_e (1 - (-1)) + [-1]$
= $\log_e (1 + 1) + (-1)$
= $\log_e 2 - 1$

∴ $(f + g)(-1) = \log_e 2 - 1$

(b) We need to find (fg) (0).
We have, $(fg)(x) = [x] \log_e (1 - x), x \in (-\infty, 1)$
Substituting x = 0 in the above equation, we get
$(fg)(0) = [0] \log_e (1 - 0)$
$= 0 \times \log_e 1$
∴ $(fg)(0) = 0$

(c) We need to find (f/g) (1/2)
We have, $(f/g)(x) = \log_e (1 - x) / [x], x \in (-\infty, 0)$
However, 1/2 is not in the domain of f/g.
∴ (f/g) (1/2) does not exist.

(d) We need to find (g/f) (1/2)
We have, $(g/f)(x) = [x] / \log_e (1 - x), x \in (-\infty, 0) \cup (0, \infty)$
Substituting x=1/2 in the above equation, we get
$(g/f)(1/2) = [x] / \log_e (1 - x)$
$= (1/2)/ \log_e (1 - 1/2)$
$= 0.5/ \log_e (1/2)$
$= 0 / \log_e (1/2)$
$= 0$
∴ $(g/f)(1/2) = 0$