EXERCISE 16.3 PAGE NO: 16.40

1. Find the angle to intersection of the following curves:
(i) $y^2 = x$ and $x^2 = y$

Solution:

Given curves $y^2 = x$ ... (1)

And $x^2 = y$ ... (2)

First curve is $y^2 = x$

Differentiating above with respect to x,

$\Rightarrow 2y.\frac{dy}{dx} = 1$

$\Rightarrow m_1 = \frac{dy}{dx} = \frac{1}{2x}$ ... (3)

The second curve is $x^2 = y$

$\Rightarrow 2x = \frac{dy}{dx}$

$\Rightarrow m_2 = \frac{dy}{dx} = 2x$ ... (4)

Substituting (1) in (2), we get

$\Rightarrow x^2 = y$

$\Rightarrow (y^2)^2 = y$

$\Rightarrow y^4 - y = 0$

$\Rightarrow y (y^3 - 1) = 0$

$\Rightarrow y = 0$ or $y = 1$

Substituting y = 0 & y = 1 in (1) in (2),

$x = y^2$

When y = 0, x = 0

When y = 1, x = 1

Substituting above values for m_1 & m_2, we get,

When x = 0,

$m_1 = \frac{dy}{dx} = \frac{1}{2\times0} = \infty$

When x = 1,

$m_1 = \frac{dy}{dx} = \frac{1}{2\times1} = \frac{1}{2}$

Values of m_1 is ∞ & $\frac{1}{2}$

When y = 0,

$m_2 = \frac{dy}{dx} = 2x = 2\times0 = 0$

When x = 1,

$m_2 = \frac{dy}{dx} = 3x = 2\times1 = 2$

Values of m_2 is 0 & 2

When $m_1 = \infty$ & $m_2 = 0$

$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

$\tan\theta = \left|\frac{0-\infty}{1+\infty\times0}\right|$

$\tan\theta = \infty$

$\theta = \tan^{-1}(\infty)$

$\therefore \tan^{-1}(\infty) = \frac{\pi}{2}$

$\theta = \frac{\pi}{2}$

When $m_1 = \frac{1}{2}$ & $m_2 = 2$

Angle of intersection of two curves is given by $\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

Tan θ $= \left|\frac{2-\frac{1}{2}}{1+\frac{1}{2}\times 2}\right|$

Tan θ $= \left|\frac{\frac{3}{2}}{2}\right|$

Tan θ $= \left|\frac{3}{4}\right|$

$\theta = \tan^{-1}\left(\frac{3}{4}\right)$

(ii) $y = x^2$ and $x^2 + y^2 = 20$

Solution:

Given curves $y = x^2$ ... (1) and $x^2 + y^2 = 20$ ... (2)

Now consider first curve $y = x^2$

$\Rightarrow m_1 = \frac{dy}{dx} = 2x$ ... (3)

Consider second curve is $x^2 + y^2 = 20$

Differentiating above with respect to x,

$\Rightarrow 2x + 2y.\frac{dy}{dx} = 0$

$\Rightarrow y.\frac{dy}{dx} = -x$

$\Rightarrow m_2 = \frac{dy}{dx} = \frac{-x}{y}$ ... (4)

Substituting (1) in (2), we get

$\Rightarrow y + y^2 = 20$

$\Rightarrow y^2 + y - 20 = 0$

We will use factorization method to solve the above Quadratic equation

$\Rightarrow y^2 + 5y - 4y - 20 = 0$

$\Rightarrow y(y + 5) - 4(y + 5) = 0$

⇒ (y + 5) (y – 4) = 0

⇒ y = – 5 & y = 4

Substituting y = – 5 & y = 4 in (1) in (2),

$y = x^2$

When y = – 5,

⇒ $-5 = x^2$

⇒ $x = \sqrt{-5}$

When y = 4,

⇒ $4 = x^2$

⇒ x = ±2

Substituting above values for m_1 & m_2, we get,

When x = 2,

$m_1 = \frac{dy}{dx} = 2 \times 2$

= 4

When x = 1,

$m_1 = \frac{dy}{dx} = 2 \times -2$

= - 4

Values of m_1 is 4 & – 4

When y = 4 & x = 2

$m_2 = \frac{dy}{dx} = \frac{-x}{y} = \frac{-2}{4} = \frac{-1}{2}$

When y = 4 & x = – 2

$m_2 = \frac{dy}{dx} = \frac{-x}{y} = \frac{2}{4} = \frac{1}{2}$

Values of m_2 is $\frac{-1}{2}$ & $\frac{1}{2}$

When $m_1 = \infty$ & $m_2 = 0$

Angle of intersection of two curves is given by $\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

$\tan\theta = \left|\frac{\frac{-1}{2} - 4}{1 + 2\times 4}\right|$

$\tan\theta = \left|\frac{\frac{-9}{2}}{1-2}\right|$

$\tan\theta = \left|\frac{9}{2}\right|$

$\theta = \tan^{-1}(\frac{9}{2})$

(iii) $2y^2 = x^3$ and $y^2 = 32x$

Solution:

Given curves $2y^2 = x^3$ ... (1) and $y^2 = 32x$ ... (2)

First curve is $2y^2 = x^3$

Differentiating above with respect to x,

$\Rightarrow 4y.\frac{dy}{dx} = 3x^2$

$\Rightarrow m_1 = \frac{dy}{dx} = \frac{3x^2}{4y}$ ... (3)

Second curve is $y^2 = 32x$

$\Rightarrow 2y.\frac{dy}{dx} = 32$

$\Rightarrow y.\frac{dy}{dx} = 16$

$\Rightarrow m_2 = \frac{dy}{dx} = \frac{16}{y}$ ... (4)

Substituting (2) in (1), we get

$\Rightarrow 2y^2 = x^3$

$\Rightarrow 2(32x) = x^3$

$\Rightarrow 64x = x^3$

$\Rightarrow x^3 - 64x = 0$

$\Rightarrow x(x^2 - 64) = 0$

$\Rightarrow x = 0$ & $(x^2 - 64) = 0$

$\Rightarrow x = 0$ & ± 8

Substituting x = 0 & x = ±8 in (1) in (2),

$y^2 = 32x$

When x = 0, y = 0

When x = 8

$\Rightarrow y^2 = 32 \times 8$

$\Rightarrow y^2 = 256$

$\Rightarrow y = \pm 16$

Substituting above values for m_1 & m_2, we get,

When x = 0, y = 16

$m_1 = \frac{dy}{dx}$

$\Rightarrow \frac{3\times 0^2}{4\times 8}$

= 0

When x = 8, y = 16

$m_1 = \frac{dy}{dx}$

$\Rightarrow \frac{3\times 8^2}{4\times 16}$

= 3

Values of m_1 is 0 & 3

When x = 0, y = 0,

$m_2 = \frac{dy}{dx}$

$\Rightarrow \frac{16}{y} = \frac{16}{0} = \infty$

When y = 16,

$m_2 = \frac{dy}{dx}$

$\Rightarrow \frac{16}{y} = \frac{16}{16}$

= 1

Values of m_2 is ∞ & 1

When $m_1 = 0$ & $m_2 = \infty$

$\Rightarrow \text{Tan } \theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

$\Rightarrow \text{Tan } \theta = \left|\frac{\infty - 0}{1 + \infty \times 0}\right|$

$\Rightarrow \text{Tan } \theta = \infty$

$\Rightarrow \theta = \tan^{-1}(\infty)$

$\therefore \text{Tan}^{-1}(\infty) = \frac{\pi}{2}$

$\Rightarrow \theta = \frac{\pi}{2}$

When $m_1 = \frac{1}{2}$ & $m_2 = 2$

Angle of intersection of two curves is given by $\tan \theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

$\Rightarrow \text{Tan } \theta = \left|\frac{3-1}{1 + 3 \times 1}\right|$

$\Rightarrow \text{Tan } \theta = \left|\frac{2}{4}\right|$

$\Rightarrow \text{Tan } \theta = \left|\frac{1}{2}\right|$

$\Rightarrow \theta = \tan^{-1}(\frac{1}{2})$

(iv) $x^2 + y^2 – 4x – 1 = 0$ and $x^2 + y^2 – 2y – 9 = 0$

Solution:

Given curves $x^2 + y^2 – 4x – 1 = 0$ ... (1) and $x^2 + y^2 – 2y – 9 = 0$ ... (2)

First curve is $x^2 + y^2 – 4x – 1 = 0$

$\Rightarrow x^2 – 4x + 4 + y^2 – 4 – 1 = 0$

$\Rightarrow (x – 2)^2 + y^2 – 5 = 0$

Now, Subtracting (2) from (1), we get

$\Rightarrow x^2 + y^2 – 4x – 1 – (x^2 + y^2 – 2y – 9) = 0$

$\Rightarrow x^2 + y^2 – 4x – 1 – x^2 – y^2 + 2y + 9 = 0$

$\Rightarrow – 4x – 1 + 2y + 9 = 0$

$\Rightarrow – 4x + 2y + 8 = 0$

$\Rightarrow 2y = 4x – 8$

$\Rightarrow y = 2x – 4$

Substituting $y = 2x – 4$ in (3), we get,

$\Rightarrow (x – 2)^2 + (2x – 4)^2 – 5 = 0$

$\Rightarrow (x – 2)^2 + 4(x – 2)^2 – 5 = 0$

$\Rightarrow (x – 2)^2(1 + 4) – 5 = 0$

$\Rightarrow 5(x – 2)^2 – 5 = 0$

$\Rightarrow (x – 2)^2 – 1 = 0$

$\Rightarrow (x – 2)^2 = 1$

$\Rightarrow (x – 2) = \pm 1$

$\Rightarrow x = 1 + 2$ or $x = – 1 + 2$

$\Rightarrow x = 3$ or $x = 1$

So, when $x = 3$

$y = 2\times 3 – 4$

$\Rightarrow y = 6 – 4 = 2$

So, when $x = 1$

$y = 2 \times 1 – 4$

$\Rightarrow y = 2 – 4 = – 2$

The point of intersection of two curves are (3, 2) & (1, – 2)

Now, differentiating curves (1) & (2) with respect to x, we get

$\Rightarrow x^2 + y^2 – 4x – 1 = 0$

$\Rightarrow 2x + 2y \frac{dy}{dx} – 4 – 0 = 0$

$\Rightarrow x + y\frac{dy}{dx} - 2 = 0$

$\Rightarrow y\frac{dy}{dx} = 2 - x$

$\Rightarrow \frac{dy}{dx} = \frac{2-x}{y}$ ... (3)

$\Rightarrow x^2 + y^2 - 2y - 9 = 0$

$\Rightarrow 2x + 2y\frac{dy}{dx} - 2\frac{dy}{dx} - 0 = 0$

$\Rightarrow x + y\frac{dy}{dx} - \frac{dy}{dx} = 0$

$\Rightarrow x + (y-1)\frac{dy}{dx} = 0$

$\Rightarrow \frac{dy}{dx} = \frac{-x}{y-1}$ ... (4)

At (3, 2) in equation (3), we get

$\Rightarrow \frac{dy}{dx} = \frac{2-3}{2}$

$\Rightarrow m_1 = \frac{dy}{dx} = \frac{-1}{2}$

At (3, 2) in equation (4), we get

$\Rightarrow \frac{dy}{dx} = \frac{-3}{2-1}$

$\Rightarrow \frac{dy}{dx} = -3$

$\Rightarrow m_2 = \frac{dy}{dx} = -3$

When $m_1 = \frac{-1}{2}$ & $m_2 = 0$

Angle of intersection of two curves is given by $\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

$\Rightarrow$ Tan θ $= \left|\frac{\frac{-1}{2}+3}{1+\frac{3}{2}}\right| = 1$

$\Rightarrow \theta = \tan^{-1}(1) = \frac{\pi}{4}$

$(v)\ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\ and\ x^2 + y^2 = ab$

Solution:

Given curves $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ... (1) and $x^2 + y^2 = ab$ ... (2)

Second curve is $x^2 + y^2 = ab$

$y^2 = ab - x^2$

Substituting this in equation (1),

$\Rightarrow \frac{x^2}{a^2} + \frac{ab-x^2}{b^2} = 1$

$\Rightarrow \frac{x^2b^2 + a^2(ab-x^2)}{a^2b^2} = 1$

$\Rightarrow x^2b^2 + a^3b - a^2x^2 = a^2b^2$

$\Rightarrow x^2b^2 - a^2x^2 = a^2b^2 - a^3b$

$\Rightarrow x^2(b^2 - a^2) = a^2b(b - a)$

$\Rightarrow x^2 = \frac{a^2b(b-a)}{x^2\ (b^2-a^2)}$

$\Rightarrow x^2 = \frac{a^2b(b-a)}{x^2\ (b-a)(b+a)}$

$\Rightarrow x^2 = \frac{a^2b}{(b+a)}$

$\therefore a^2 - b^2 = (a + b)\ (a - b)$

$\Rightarrow x = \pm\sqrt{\frac{a^2b}{(b+a)}}$ ... (3)

Since, $y^2 = ab - x^2$

$\Rightarrow y^2 = ab - (\frac{a^2b}{(b+a)})$

$\Rightarrow y^2 = \frac{ab^2 + a^2b - a^2b}{(b+a)}$

$\Rightarrow y^2 = \frac{ab^2}{(b+a)}$

$\Rightarrow y = \pm\sqrt{\frac{ab^2}{(b+a)}}$ ... (4)

Since, curves are $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ & $x^2 + y^2 = ab$

Differentiating above with respect to x

$\Rightarrow \frac{2x}{a^2} + \frac{2y}{b^2}.\frac{dy}{dx} = 0$

$\Rightarrow \frac{y}{b^2}.\frac{dy}{dx} = -\frac{x}{a^2}$

$\Rightarrow \frac{dy}{dx} = \frac{-\frac{x}{a^2}}{\frac{y}{b^2}}$

$\Rightarrow \frac{dy}{dx} = \frac{-b^2x}{a^2y}$

$\Rightarrow m_1 = \frac{dy}{dx} = \frac{-b^2x}{a^2y}$ ... (5)

Second curve is $x^2 + y^2 = ab$

$\Rightarrow 2x + 2y.\frac{dy}{dx} = 0$

$\Rightarrow m_2 = \frac{dy}{dx} = \frac{-x}{y}$ ... (6)

Substituting (3) in (4), above values for m_1 & m_2, we get, At $\left(\sqrt{\frac{a^2b}{(b+a)}}, \sqrt{\frac{ab^2}{(b+a)}}\right)$ in equation (5), we get

$$\Rightarrow \frac{dy}{dx} = \frac{-b^2 \times \sqrt{\frac{a^2b}{(b+a)}}}{a^2 \times \sqrt{\frac{ab^2}{(b+a)}}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-b^2 \times a\sqrt{\frac{b}{(b+a)}}}{a^2 \times b\sqrt{\frac{a}{(b+a)}}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-b^2 a\sqrt{b}}{a^2 b\sqrt{a}}$$

$$\Rightarrow m_1 = \frac{dy}{dx} = \frac{-b\sqrt{b}}{a\sqrt{a}}$$

At $\left(\sqrt{\frac{a^2b}{(b+a)}}, \sqrt{\frac{ab^2}{(b+a)}}\right)$ in equation (6), we get

$$\Rightarrow \frac{dy}{dx} = \frac{-\sqrt{\frac{a^2b}{(b+a)}}}{\sqrt{\frac{ab^2}{(b+a)}}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-a\sqrt{\frac{b}{(b+a)}}}{b\sqrt{\frac{a}{(b+a)}}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-a\sqrt{b}}{b\sqrt{a}}$$

$$\Rightarrow m_2 = \frac{dy}{dx} = -\sqrt{\frac{a}{b}}$$

When $m_1 = \frac{-b\sqrt{b}}{a\sqrt{a}}$ & $m_2 = -\sqrt{\frac{a}{b}}$

Angle of intersection of two curves is given by $\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

$$\Rightarrow \text{Tan}\,\theta = \left|\frac{\frac{-b\sqrt{b}}{a\sqrt{a}} - -\sqrt{\frac{a}{b}}}{1 + \frac{-b\sqrt{b}}{a\sqrt{a}} \times -\sqrt{\frac{a}{b}}}\right|$$

$$\Rightarrow \text{Tan}\,\theta = \left|\frac{\frac{-b\sqrt{b}}{a\sqrt{a}} + \sqrt{\frac{a}{b}}}{1 + \frac{b}{a}}\right|$$

$$\Rightarrow \text{Tan}\,\theta = \left|\frac{\frac{-b\sqrt{b} \times \sqrt{b} + a\sqrt{a} \times \sqrt{a}}{a\sqrt{a} \times \sqrt{b}}}{1 + \frac{b}{a}}\right|$$

$$\Rightarrow \text{Tan}\,\theta = \left|\frac{\frac{-b \times b + a \times a}{a\sqrt{ab}}}{1 + \frac{b}{a}}\right|$$

$$\Rightarrow \text{Tan}\,\theta = \left|\frac{\frac{a^2 - b^2}{a\sqrt{ab}}}{\frac{a+b}{a}}\right|$$

$$\Rightarrow \text{Tan}\,\theta = \left|\frac{\frac{(a+b)(a-b)}{\sqrt{ab}}}{a+b}\right|$$

$$\Rightarrow \text{Tan}\,\theta = \left|\frac{(a-b)}{\sqrt{ab}}\right|$$

$$\Rightarrow \theta = \tan^{-1}\left(\frac{(a-b)}{\sqrt{ab}}\right)$$

2. Show that the following set of curves intersect orthogonally:
(i) $y = x^3$ and $6y = 7 - x^2$

Solution:
Given curves $y = x^3$ ... (1) and $6y = 7 - x^2$ ... (2)
Solving (1) & (2), we get
$\Rightarrow 6y = 7 - x^2$
$\Rightarrow 6(x^3) = 7 - x^2$
$\Rightarrow 6x^3 + x^2 - 7 = 0$

Since $f(x) = 6x^3 + x^2 - 7$,
We have to find f(x) = 0, so that x is a factor of f(x).
When x = 1
$f(1) = 6(1)^3 + (1)^2 - 7$
$f(1) = 6 + 1 - 7$
$f(1) = 0$
Hence, x = 1 is a factor of f(x).
Substituting x = 1 in $y = x^3$, we get
$y = 1^3$
$y = 1$
The point of intersection of two curves is (1, 1)
First curve $y = x^3$
Differentiating above with respect to x,

$\Rightarrow 6\frac{dy}{dx} = 0 - 2x$

$\Rightarrow m_2 = \frac{-2x}{6}$

$\Rightarrow m_2 = \frac{-x}{3}$ ... (4)

At (1, 1), we have,

$m_1 = 3x^2$

$\Rightarrow 3 \times (1)^2$

$m_1 = 3$

At (1, 1), we have,

$\Rightarrow m_2 = \frac{-x}{3}$

$\Rightarrow \frac{-1}{3}$

$\Rightarrow m_2 = \frac{-1}{3}$

When $m_1 = 3$ & $m_2 = \frac{-1}{3}$

Two curves intersect orthogonally if $m_1m_2 = -1$

$\Rightarrow 3x^{\frac{-1}{3}} = -1$

∴ Two curves $y = x^3$ & $6y = 7 - x^2$ intersect orthogonally.

(ii) $x^3 - 3xy^2 = -2$ and $3x^2 y - y^3 = 2$

Solution:

Given curves $x^3 - 3xy^2 = -2$ ... (1) and $3x^2y - y^3 = 2$ ... (2)

Adding (1) & (2), we get

$\Rightarrow x^3 - 3xy^2 + 3x^2y - y^3 = -2 + 2$

$\Rightarrow x^3 - 3xy^2 + 3x^2y - y^3 = -0$

$\Rightarrow (x - y)^3 = 0$

$\Rightarrow (x - y) = 0$

$\Rightarrow x = y$

Substituting x = y on $x^3 - 3xy^2 = -2$

$\Rightarrow x^3 - 3 \times x \times x^2 = -2$

$\Rightarrow x^3 - 3x^3 = -2$

$\Rightarrow -2x^3 = -2$

$\Rightarrow x^3 = 1$

$\Rightarrow x = 1$

Since x = y

y = 1

The point of intersection of two curves is (1, 1)

First curve $x^3 - 3xy^2 = -2$

Differentiating above with respect to x,

$\Rightarrow 3x^2 - 3(1 \times y^2 + x \times 2y\frac{dy}{dx}) = 0$

$\Rightarrow 3x^2 - 3y^2 - 6xy\frac{dy}{dx} = 0$

$\Rightarrow 3x^2 - 3y^2 = 6xy\frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx} = \frac{3x^2 - 3y^2}{6xy}$

$\Rightarrow \frac{dy}{dx} = \frac{3(x^2 - y^2)}{6xy}$

$\Rightarrow m_1 = \frac{(x^2-y^2)}{2xy}$ ... (3)

Second curve $3x^2y - y^3 = 2$

Differentiating above with respect to x

$\Rightarrow 3(2x \times y + x^2 \times \frac{dy}{dx}) - 3y^2\frac{dy}{dx} = 0$

$\Rightarrow 6xy + 3x^2\frac{dy}{dx} - 3y^2\frac{dy}{dx} = 0$

$\Rightarrow 6xy + (3x^2 - 3y^2)\frac{dy}{dx} = 0$

$\Rightarrow \frac{dy}{dx} = \frac{-6xy}{3x^2-3y^2}$

$\Rightarrow \frac{dy}{dx} = \frac{-2xy}{x^2-y^2}$

$\Rightarrow m_2 = \frac{-2xy}{x^2-y^2}$ ... (4)

When $m_1 = \frac{(x^2-y^2)}{2xy}$ & $m_2 = \frac{-2xy}{x^2-y^2}$

Two curves intersect orthogonally if $m_1m_2 = -1$

$\Rightarrow \frac{(x^2-y^2)}{2xy} \times \frac{-2xy}{x^2-y^2} = -1$

∴ Two curves $x^3 - 3xy^2 = -2$ & $3x^2y - y^3 = 2$ intersect orthogonally.

(iii) $x^2 + 4y^2 = 8$ and $x^2 - 2y^2 = 4$.

Solution:

Given curves $x^2 + 4y^2 = 8$ ... (1) and $x^2 - 2y^2 = 4$ ... (2)

Solving (1) & (2), we get,

From 2nd curve,

$x^2 = 4 + 2y^2$

Substituting on $x^2 + 4y^2 = 8$,

$\Rightarrow 4 + 2y^2 + 4y^2 = 8$

$\Rightarrow 6y^2 = 4$

$\Rightarrow y^2 = \frac{4}{6}$

$\Rightarrow y = \pm\sqrt{\frac{2}{3}}$

Substituting on $y = \pm\sqrt{\frac{2}{3}}$, we get,

$\Rightarrow x^2 = 4 + 2(\pm\sqrt{\frac{2}{3}})^2$

$\Rightarrow x^2 = 4 + 2(\frac{2}{3})$

$\Rightarrow x^2 = 4 + \frac{4}{3}$

$\Rightarrow x^2 = \frac{16}{3}$

$\Rightarrow x = \pm\sqrt{\frac{16}{3}}$

$\Rightarrow x = \pm\frac{4}{\sqrt{3}}$

$\therefore$ The point of intersection of two curves $(\frac{4}{\sqrt{3}}, \sqrt{\frac{2}{3}})$ & $(-\frac{4}{\sqrt{3}}, -\sqrt{\frac{2}{3}})$

Now, differentiating curves (1) & (2) with respect to x, we get

$\Rightarrow x^2 + 4y^2 = 8$

$\Rightarrow 2x + 8y.\frac{dy}{dx} = 0$

$\Rightarrow 8y.\frac{dy}{dx} = -2x$

$\Rightarrow \frac{dy}{dx} = \frac{-x}{4y}$ ... (3)

$\Rightarrow x^2 - 2y^2 = 4$

$\Rightarrow 2x - 4y.\frac{dy}{dx} = 0$

$\Rightarrow x - 2y.\frac{dy}{dx} = 0$

$\Rightarrow 4y\frac{dy}{dx} = x$

$\Rightarrow \frac{dy}{dx} = \frac{x}{2y}$ ... (4)

At $(\frac{4}{\sqrt{3}}, \sqrt{\frac{2}{3}})$ in equation (3), we get

$$\Rightarrow \frac{dy}{dx} = \frac{-\frac{4}{\sqrt{3}}}{4 \times \sqrt{\frac{2}{3}}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-\frac{1}{\sqrt{3}}}{\sqrt{\frac{2}{3}}}$$

$\Rightarrow m_1 = \frac{-1}{\sqrt{2}}$

At $(\frac{4}{\sqrt{3}}, \sqrt{\frac{2}{3}})$ in equation (4), we get

$$\Rightarrow \frac{dy}{dx} = \frac{\frac{4}{\sqrt{3}}}{2 \times \sqrt{\frac{2}{3}})}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\frac{2}{\sqrt{3}}}{\sqrt{\frac{2}{3}})}$$

$\Rightarrow \frac{dy}{dx} = \frac{2}{\sqrt{2}}$

$\Rightarrow \frac{dy}{dx} = \sqrt{2}$

$\Rightarrow m_2 = 1$

When $m_1 = \frac{-1}{\sqrt{2}}$ & $m_2 = \sqrt{2}$

Two curves intersect orthogonally if $m_1m_2 = -1$

$\Rightarrow \frac{-1}{\sqrt{2}} \times \sqrt{2} = -1$

∴ Two curves $x^2 + 4y^2 = 8$ & $x^2 - 2y^2 = 4$ intersect orthogonally.

3. $x^2 = 4y$ and $4y + x^2 = 8$ at (2, 1)

Solution:

Given curves $x^2 = 4y$ ... (1) and $4y + x^2 = 8$ ... (2)

The point of intersection of two curves (2, 1)

Solving (1) & (2), we get,

First curve is $x^2 = 4y$

Differentiating above with respect to x,

$\Rightarrow 2x = 4.\frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx} = \frac{2x}{4}$

$\Rightarrow m_1 = \frac{x}{2}$ ... (3)

Second curve is $4y + x^2 = 8$

$\Rightarrow 4.\frac{dy}{dx} + 2x = 0$

$\Rightarrow \frac{dy}{dx} = \frac{-2x}{4}$

$\Rightarrow m_2 = \frac{-x}{2}$ ... (4)

Substituting (2, 1) for m_1 & m_2, we get,

$m_1 = \frac{x}{2}$

$\Rightarrow \frac{2}{2}$

$m_1 = 1$ ... (5)

$m_2 = \frac{-x}{2}$

$\Rightarrow \frac{-2}{2}$

$m_2 = -1$ ... (6)

When $m_1 = 1$ & $m_2 = -1$

Two curves intersect orthogonally if $m_1m_2 = -1$

$\Rightarrow 1 \times -1 = -1$

∴ Two curves $x^2 = 4y$ & $4y + x^2 = 8$ intersect orthogonally.

(ii) $x^2 = y$ and $x^3 + 6y = 7$ at (1, 1)

Solution:

Given curves $x^2 = y$ ... (1) and $x^3 + 6y = 7$ ... (2)
The point of intersection of two curves (1, 1)
Solving (1) & (2), we get,
First curve is $x^2 = y$
Differentiating above with respect to x,

$\Rightarrow 2x = \frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx} = 2x$

$\Rightarrow m_1 = 2x$ ... (3)

Second curve is $x^3 + 6y = 7$

Differentiating above with respect to x,

$\Rightarrow 3x^2 + 6.\frac{dy}{dx} = 0$

$\Rightarrow \frac{dy}{dx} = \frac{-3x^2}{6}$

$\Rightarrow \frac{dy}{dx} = \frac{-x^2}{2}$

$\Rightarrow m_2 = \frac{-x^2}{2}$ ... (4)

Substituting (1, 1) for m_1 & m_2, we get,

$m_1 = 2x$

$\Rightarrow 2 \times 1$

$m_1 = 2$ ... (5)

$m_2 = \frac{-x^2}{2}$

$\Rightarrow \frac{-1^2}{2}$

$m_2 = -\frac{-1}{2}$ ... (6)

When $m_1 = 2$ & $m_2 = -\frac{-1}{2}$

Two curves intersect orthogonally if $m_1m_2 = -1$

$\Rightarrow 2\times\frac{-1}{2} = -1$

∴ Two curves $x^2 = y$ & $x^3 + 6y = 7$ intersect orthogonally.

(iii) $y^2 = 8x$ and $2x^2 + y^2 = 10$ at (1, 2√2)

Solution:

Given curves $y^2 = 8x$ ... (1) and $2x^2 + y^2 = 10$ ... (2)

The point of intersection of two curves are (0, 0) & (1, 2√

Now, differentiating curves (1) & (2) w.r.t x, we get

$\Rightarrow y^2 = 8x$

$\Rightarrow 2y.\frac{dy}{dx} = 8$

$\Rightarrow \frac{dy}{dx} = \frac{8}{2y}$

$\Rightarrow \frac{dy}{dx} = \frac{4}{y}$ ... (3)

$\Rightarrow 2x^2 + y^2 = 10$

Differentiating above with respect to x,

$\Rightarrow 4x + 2y.\frac{dy}{dx} = 0$

$\Rightarrow 2x + y.\frac{dy}{dx} = 0$

$\Rightarrow y.\frac{dy}{dx} = -2x$

$\Rightarrow \frac{dy}{dx} = \frac{-2x}{y}$ ... (4)

Substituting (1, 2√2) for m_1 & m_2, we get,

$m_1 = \frac{4}{y}$

$\Rightarrow \frac{4}{2\sqrt{2}}$

$m_1 = \sqrt{2}$ ... (5)

$m_2 = \frac{-2x}{y}$

$\Rightarrow \frac{-2\times1}{2\sqrt{2}}$

$m_2 = -\frac{-1}{\sqrt{2}}$ ... (6)

When $m_1 = \sqrt{2}$ & $m_2 = \frac{-1}{\sqrt{2}}$

Two curves intersect orthogonally if $m_1m_2 = -1$

$\Rightarrow \sqrt{2} \times \frac{-1}{\sqrt{2}} = -1$

∴ Two curves $y^2 = 8x$ & $2x^2 + y^2 = 10$ intersect orthogonally.

4. Show that the curves $4x = y^2$ and $4xy = k$ cut at right angles, if $k^2 = 512$.

Solution:

Given curves $4x = y^2$ ... (1) and $4xy = k$ ... (2)

We have to prove that two curves cut at right angles if $k^2 = 512$

Now, differentiating curves (1) & (2) w.r.t x, we get

$\Rightarrow 4x = y^2$

$\Rightarrow 4 = 2y. \frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx} = \frac{2}{y}$

$m_1 = \frac{2}{y}$ ... (3)

$\Rightarrow 4xy = k$

Differentiating above with respect to x,

$\Rightarrow 4(y + x\frac{dy}{dx}) = 0$

$\Rightarrow y + x\frac{dy}{dx} = 0$

$\Rightarrow \frac{dy}{dx} = \frac{-y}{x}$

$\Rightarrow m_2 = \frac{-y}{x}$ ... (4)

Two curves intersect orthogonally if $m_1m_2 = -1$

Since m_1 and m_2 cuts orthogonally,

$\Rightarrow \frac{2}{y} \times \frac{-y}{x} = -1$

$\Rightarrow \frac{-2}{x} = -1$

$\Rightarrow x = 2$

Now, Solving (1) & (2), we get,

$4xy = k$ & $4x = y^2$

$\Rightarrow (y^2)\, y = k$

$\Rightarrow y^3 = k$

$\Rightarrow y = k^{\frac{1}{3}}$

Substituting $y = k^{\frac{1}{3}}$ in $4x = y^2$, we get,

$\Rightarrow 4x = (k^{\frac{1}{3}})^2$

$\Rightarrow 4 \times 2 = k^{\frac{2}{3}}$

$\Rightarrow k^{\frac{2}{3}} = 8$

$\Rightarrow k^2 = 8^3$

$\Rightarrow k^2 = 512$

5. Show that the curves $2x = y^2$ and $2xy = k$ cut at right angles, if $k^2 = 8$.

Solution:

Given curves $2x = y^2$ ... (1) and $2xy = k$ ... (2)
We have to prove that two curves cut at right angles if $k^2 = 8$
Now, differentiating curves (1) & (2) with respect to x, we get
$\Rightarrow 2x = y^2$

$\Rightarrow 2 = 2y.\frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx} = \frac{1}{y}$

$m_1 = \frac{1}{y}$ ... (3)

$\Rightarrow 2xy = k$

Differentiating above with respect to x,

$\Rightarrow 2(y + x\frac{dy}{dx}) = 0$

$\Rightarrow y + x\frac{dy}{dx} = 0$

$\Rightarrow \frac{dy}{dx} = \frac{-y}{x}$

$\Rightarrow m_2 = \frac{-y}{x}$ ... (4)

Two curves intersect orthogonally if $m_1m_2 = -1$

Since m_1 and m_2 cuts orthogonally,

$\Rightarrow \frac{1}{y} \times \frac{-y}{x} = -1$

$\Rightarrow \frac{-1}{x} = -1$

$\Rightarrow x = 1$

Now, solving (1) & (2), we get,

$2xy = k$ & $2x = y^2$

$\Rightarrow (y^2)\, y = k$

$\Rightarrow y^3 = k$

$\Rightarrow y = k^{\frac{1}{3}}$

Substituting y = $k^{\frac{1}{3}}$ in 2x = y^2, we get,

$\Rightarrow 2x = (k^{\frac{1}{3}})^2$

$\Rightarrow 2 \times 1 = k^{\frac{2}{3}}$

$\Rightarrow k^{\frac{2}{3}} = 2$

$\Rightarrow k^2 = 2^3$

$\Rightarrow k^2 = 8$