EXERCISE 18.5 PAGE NO: 18.72

1. Determine two positive numbers whose sum is 15 and the sum of whose squares is minimum.

Solution:

Let the two positive numbers be a and b.

So we have a + b = 15 … (1)

Also, $a^2 + b^2$ is minima

Assume, $S = a^2 + b^2$

(From equation 1)

$\Rightarrow S = a^2 + (15 - a)^2$

$\Rightarrow S = a^2 + 225 + a^2 - 30a = 2a^2 - 30a + 225$

$\Rightarrow \frac{dS}{da} = 4a - 30$

$\Rightarrow \frac{d^2S}{da^2} = 4$

Since, $\frac{d^2S}{da^2} > 0 \Rightarrow \frac{dS}{da} = 0$ will give minimum value of S.

$4a - 30 = 0$

$a = 7.5$

Hence, two numbers will be 7.5 and 7.5.

Which implies S is minimum when a = 15/2 and b = 15/2.

2. Divide 64 into two parts such that the sum of the cubes of two parts is minimum.

Solution:

Let the two positive numbers be a and b.

Given a + b = 64 … (1)

We have, $a^3 + b^3$ is minima

Assume, $S = a^3 + b^3$
(From equation 1)
$S = a^3 + (64 - a)^3$

$\frac{dS}{da} = 3a^2 + 3(64 - a)^2 \times (-1)$

$\frac{dS}{da} = 0$ (Condition for maxima and minima)

$\Rightarrow 3a^2 + 3(64 - a)^2 \times (-1) = 0$

$\Rightarrow 3a^2 + 3(4096 + a^2 - 128a) \times (-1) = 0$

$\Rightarrow 3a^2 - 3 \times 4096 - 3a^2 + 384a = 0$

$\Rightarrow a = 32$

$\frac{d^2S}{da^2} = 6a + 6(64 - a) = 384$

Since, $\frac{d^2S}{da^2} > 0 \Rightarrow a = 32$ will give minimum value of S

Hence, the two number will be 32 and 32.

3. How should we choose two numbers, each greater than or equal to –2, whose sum is ½ so that the sum of the first and the cube of the second is minimum?

Solution:

Let a and b be two numbers such that $a, b \geq -2$

Given $a + b = \frac{1}{2}$

Assume, $S = a + b^3$

$\Rightarrow S = a + (\frac{1}{2} - a)^3$

$\frac{dS}{da} = 1 + 3(\frac{1}{2} - a)^2(-1)$

Condition maxima and minima is

$dS/da = 0$

$1 + 3(½ - a)^2 (-1) = 0$

$1 - 3(½ - a)^2 = 0$

$$\Rightarrow 3\left(\frac{1}{2} - a\right)^2 = 1$$

$$\Rightarrow \left(\frac{1}{2} - a\right)^2 = \frac{1}{3}$$

$$\Rightarrow \left(\frac{1}{2} - a\right) = \pm\frac{1}{\sqrt{3}}$$

$$\Rightarrow a = \frac{1}{2} \pm \frac{1}{\sqrt{3}}$$

$$\frac{d^2S}{da^2} = 6\left(\frac{1}{2} - a\right)$$

For S to minimum, $\frac{d^2S}{da^2} > 0$

$$\Rightarrow a = \frac{1}{2} - \frac{1}{\sqrt{3}}$$

Hence, $a = \frac{1}{2} - \frac{1}{\sqrt{3}}$ and $b = \frac{1}{\sqrt{3}}$

4. Divide 15 into two parts such that the square of one multiplied with the cube of the other is minimum.

Solution:

Let the given two numbers be x and y. Then,

x + y = 15 ….. (1)

y = (15 - x)

Now we have, $z = x^2 y^3$

$z = x^2 (15 – x)^3$ (from equation 1)

$$\Rightarrow \frac{dz}{dx} = 2x(15-x)^3 - 3x^2(15-x)^2$$

For maximum or minimum values of z, we must have

$$\frac{dz}{dx} = 0$$

$2x(15-x)^3 - 3x^2(15-x)^2 = 0$

$2x(15-x) = 3x^2$

$30x - 2x^2 = 3x^2$

$30x = 5x^2$

x = 6 and y = 9

$$\frac{d^2z}{dx^2} = 2(15-x)^3 - 6x(15-x)^2 - 6x(15-x)^2 + 6x^2(15-x)$$

At x = 6:

$$\frac{d^2z}{dx^2} = 2(9)^3 - 36(9)^2 - 36(9)^2 + 6(36)(9)$$

$$\Rightarrow \frac{d^2z}{dx^2} = -2430 < 0$$

Thus, z is maximum when x = 6 and y = 9

So, the required two parts into which 15 should be divided are 6 and 9.

5. Of all the closed cylindrical cans (right circular), which enclose a given volume of 100 cm^3, which has the minimum surface area?

Solution:

Let r and h be the radius and height of the cylinder, respectively. Then,
Volume (V) of the cylinder = $\pi r^2 h$
$\Rightarrow 100 = \pi r^2 h$
$\Rightarrow h = 100/\pi r^2$
Surface area (S) of the cylinder = $2\pi r^2 + 2\pi r h = 2\pi r^2 + 2\pi r \times 100/\pi r^2$

$$\Rightarrow S = 2\pi r^2 + \frac{200}{r}$$

On differentiating we get

$$\therefore \frac{dS}{dr} = 4\pi r - \frac{200}{r^2}$$

For the maximum or minimum, we must have,

$$\frac{dS}{dr} = 0$$

$$\Rightarrow 4\pi r - \frac{200}{r^2} = 0$$

$$\Rightarrow 4\pi r^3 = 200$$

$$\Rightarrow r = \left(\frac{50}{\pi}\right)^{\frac{1}{3}}$$

Now,

$$\frac{d^2S}{dr^2} = 4\pi + \frac{400}{r^3}$$

$$\Rightarrow \frac{d^2S}{dr^2} > 0 \text{ when r } = \left(\frac{50}{\pi}\right)^{\frac{1}{3}}$$

Thus, the surface area is minimum when $r = \left(\frac{50}{\pi}\right)^{\frac{1}{3}}$.

$$\text{At r } = \left(\frac{50}{\pi}\right)^{\frac{1}{3}}$$

$$h = \frac{100}{\pi\left(\frac{50}{\pi}\right)^{\frac{2}{3}}} = 2\left(\frac{50}{\pi}\right)^{\frac{1}{3}}$$

6. A beam is supported at the two ends and is uniformly loaded. The bending moment M at a distance x from one end is given by

(i) $M = \frac{WL}{2}x - \frac{W}{2}x^2$

(ii) $M = \frac{Wx}{3} - \frac{W}{3}\frac{x^3}{L^2}$

Find the point at which M is maximum in each case.

Solution:

Condition for maxima and minima is $\frac{dM}{dx} = 0$

And for M to maximum $\frac{d^2M}{dx^2} < 0$.

(i) $M = \frac{WL}{2}x - \frac{W}{2}x^2$

$$\frac{dM}{dx} = 0$$

$$\Rightarrow \frac{WL}{2} - Wx = 0$$

$$\Rightarrow \frac{WL}{2} = Wx$$

$$\Rightarrow x = \frac{L}{2}$$

$$\frac{d^2M}{dx^2} = -W < 0$$

Hence, for $x = \frac{L}{2}$, M will be maximum.

(ii) $M = \frac{Wx}{3} - \frac{W}{3}\frac{x^3}{L^2}$

$$\frac{dM}{dx} = \frac{W}{3} - W\frac{x^2}{L^2} = 0$$

$$\Rightarrow \frac{W}{3} - W\frac{x^2}{L^2} = 0$$

$$\Rightarrow \frac{W}{3} = W\frac{x^2}{L^2}$$

$$\Rightarrow x^2 = \frac{L^2}{3}$$

$$\Rightarrow x = \pm\frac{L}{\sqrt{3}}$$

$$\frac{d^2M}{dx^2} = -2W\frac{x}{L^2}$$

So,

For $x = \frac{L}{\sqrt{3}} \Rightarrow \frac{d^2M}{dx^2} = -\frac{2W}{\sqrt{3}L}$

$\Rightarrow \frac{d^2M}{dx^2} < 0$ (Condition for maximum value)

For $x = -\frac{L}{\sqrt{3}} \Rightarrow \frac{d^2M}{dx^2} = \frac{2W}{\sqrt{3}L}$

$\Rightarrow \frac{d^2M}{dx^2} > 0$ (Condition for minimum value)

Therefore, for $x = \frac{L}{\sqrt{3}}$, M will have maximum value.

7. A wire of length 28 m is to be cut into two pieces. One of the pieces is to be made into a square and the other into a circle. What should be the lengths of the two pieces so that the combined area of the circle and the square is minimum?

Solution:

Suppose the given wire, which is to be made into a square and a circle, is cut into two pieces of length x and y m respectively. Then,

$x + y = 28 \Rightarrow y = (28 - x)$

We know that perimeter of square, 4 (side) = x

Side = x/4

Area of square = $(x/4)^2 = x^2/16$

Circumference of circle, $2\pi r = y$

$r = y/2\pi$

$$\text{Area of circle} = \pi r^2 = \pi\left(\frac{y}{2\pi}\right)^2 = \frac{y^2}{4\pi}$$

Now z = Area of square + area of circle

$$\Rightarrow z=\frac{x^2}{16}+\frac{y^2}{4\pi}$$

$$\Rightarrow z=\frac{x^2}{16}+\frac{(28-x)^2}{4\pi}$$

$$\Rightarrow \frac{dz}{dx}=\frac{2x}{16}-\frac{2(28-x)}{4\pi}$$

For maximum or minimum values of z, we must have

$$\frac{dz}{dx}=0$$

From equation 1 we have

$$\Rightarrow \frac{2x}{16}-\frac{2(28-x)}{4\pi}=0.$$

$$\Rightarrow \frac{x}{4}=\frac{(28-x)}{\pi}$$

$$\Rightarrow \frac{x\pi}{4}+x=28$$

$$\Rightarrow x\left(\frac{\pi}{4}+1\right)=28$$

$$\Rightarrow x=\frac{28}{\left(\frac{\pi}{4}+1\right)}$$

$$\Rightarrow x=\frac{112}{\pi+4}$$

Again from equation 1 we have

$$\Rightarrow y=28-\frac{112}{\pi+4}.$$

$$\Rightarrow y = \frac{28\pi}{\pi + 4}$$

$$\frac{d^2z}{dx^2} = \frac{1}{8} + \frac{1}{2\pi} > 0$$

Thus, z is minimum when $x = \frac{112}{\pi + 4}$ and $y = \frac{28\pi}{\pi + 4}$.

Hence, the length of the two pieces of wire are $\frac{112}{\pi + 4}m$ and $\frac{28\pi}{\pi + 4}$ m respectively.

8. A wire of length 20 m is to be cut into two pieces. One of the pieces will be bent into shape of a square and the other into shape of an equilateral triangle. Where the wire should be cut so that the sum of the areas of the square and triangle is minimum?

Solution:

Suppose the wire, which is to be made into a square and a triangle, is cut into two pieces of length x and y respectively. Then,

x + y = 20 ⇒ y = (20 - x) …… (1)

We know that perimeter of square, 4 (side) = x

Side = x/4

Area of square = $(x/4)^2 = x^2/16$

Again we know that perimeter of triangle, 3 (side) = y.

Side = y/3

$$\text{Area of triangle } = \frac{\sqrt{3}}{4} \times (\text{ Side })^2 = \frac{\sqrt{3}}{4} \times \left(\frac{y}{3}\right)^2 = \frac{\sqrt{3}y^2}{36}$$

Now,

z = Area of square + area of triangle

$$\Rightarrow z = \frac{x^2}{16} + \frac{\sqrt{3}y^2}{36}$$

From equation 1 we have

$$\Rightarrow z=\frac{x^2}{16}+\frac{\sqrt{3}(20-x)^2}{36}.$$

$$\Rightarrow \frac{dz}{dx}=\frac{2x}{16}-\frac{2\sqrt{3}\,(20-x)}{36}$$

For maximum and minimum values of z, we have

$$\frac{dz}{dx}=0$$

$$\Rightarrow \frac{2x}{16}-\frac{\sqrt{3}\,(20-x)}{18}=0$$

$$\Rightarrow \frac{9x}{4}=\sqrt{3}\,(20-x)$$

$$\Rightarrow \frac{9x}{4}+x\sqrt{3}=20\sqrt{3}$$

$$\Rightarrow x\left(\frac{9}{4}+\sqrt{3}\right)=20\sqrt{3}$$

$$\Rightarrow x=\frac{20\sqrt{3}}{\left(\frac{9}{4}+\sqrt{3}\right)}$$

$$\Rightarrow x=\frac{80\sqrt{3}}{\left(9+4\sqrt{3}\right)}$$

From equation 1 we have

$$\Rightarrow y=20-\frac{80\sqrt{3}}{9+4\sqrt{3}}.$$

$$\Rightarrow y=\frac{180}{9+4\sqrt{3}}$$

$$\frac{d^2z}{dx^2}=\frac{1}{8}+\frac{\sqrt{3}}{18}>0$$

Thus, z is minimum when $x = \frac{80\sqrt{3}}{\left(9+4\sqrt{3}\right)}$ and $y = \frac{180}{9+4\sqrt{3}}$

Hence, the wire of length 20 m should be cut into two pieces of lengths

$\frac{80\sqrt{3}}{\left(9+4\sqrt{3}\right)}$ m and $\frac{180}{9+4\sqrt{3}} m.$

9. Given the sum of the perimeters of a square and a circle, show that the sum of their areas is least when one side of the square is equal to diameter of the circle.

Solution:

Let us say the sum of perimeter of square and circumference of circle be L

Given sum of the perimeters of a square and a circle.

Assuming, side of square = a and radius of circle = r

Then, L = 4a + 2πr ⇒ a = (L - 2πr)/4… (1)

Let the sum of area of square and circle be S

So, $S = a^2 + \pi r^2$

$\Rightarrow S = \left(\frac{L-2\pi r}{4}\right)^2 + \pi r^2$

Condition for maxima and minima

$$\frac{dS}{dr} = 0$$

$$\Rightarrow (2)(-2\pi)\left(\frac{L-2\pi r}{16}\right) + 2\pi r = 0$$

$$\Rightarrow (2)(2\pi)\left(\frac{L-2\pi r}{16}\right) = 2\pi r$$

$$\Rightarrow L - 2\pi r = 8r$$

$$(8+2\pi)r = L$$

$$\Rightarrow r = \frac{L}{8+2\pi}$$

$$\frac{d^2S}{dr^2} = \frac{\pi^2}{2} + 2\pi$$

So, for $r = \frac{L}{8+2\pi} \Rightarrow \frac{d^2S}{dr^2} = \frac{\pi^2}{2} + 2\pi > 0$

This is the condition for minima

From equation 1, we have

$$a = \frac{L - 2\pi r}{4}$$

Substituting from equation 2

$$\Rightarrow a = \frac{L - 2\pi \frac{L}{8+2\pi}}{4}$$

$$\Rightarrow a = \frac{8L + 2\pi L - 2\pi L}{4(8+2\pi)}$$

$$\Rightarrow a = \frac{8L}{4(8+2\pi)}$$

a = 2r

Side of square = diameter of the circle.

Hence, proved.

10. Find the largest possible area of a right angled triangle whose hypotenuse is 5 cm long.

Solution:

Let the base of the right angled triangle be x and its height be y. Then,

$x^2 + y^2 = 5^2$

$y^2 = 25 - x^2$

$$\Rightarrow y = \sqrt{25 - x^2}$$

As, the area of the triangle, A = ½ × x × y

$$\Rightarrow A(x) = \frac{1}{2} \times x \times \sqrt{25 - x^2}$$

$$\Rightarrow A(x) = \frac{x\sqrt{25-x^2}}{2}$$

On differentiating we get

$$\Rightarrow A'(x) = \frac{\sqrt{25-x^2}}{2} + \frac{x(-2x)}{4\sqrt{25-x^2}}$$

$$\Rightarrow A'(x) = \frac{\sqrt{25-x^2}}{2} - \frac{x^2}{2\sqrt{25-x^2}}$$

$$\Rightarrow A'(x) = \frac{25-x^2-x^2}{2\sqrt{25-x^2}}$$

$$\Rightarrow A'(x) = \frac{25-2x^2}{2\sqrt{25-x^2}}$$

For maxima or minima, we must have f'(x) = 0

A'(x) = 0

$$\Rightarrow \frac{25-2x^2}{2\sqrt{25-x^2}} = 0$$

$25 - 2x^2 = 0$

$2x^2 = 25$

$$\Rightarrow x = \frac{5}{\sqrt{2}}$$

Therefore,

$$y = \sqrt{25 - \frac{25}{2}}$$

$$= \sqrt{\frac{50-25}{2}}$$

$$= \sqrt{\frac{25}{2}}$$

$= \frac{5}{\sqrt{2}}$

Again differentiating A, we get

$$\text{Also}, A''(x) = \frac{\left[-4x\sqrt{25-x^2} - \frac{(25-2x^2)(-2x)}{2\sqrt{25-x^2}}\right]}{25-x^2}$$

$$= \frac{\left[\frac{-4x(25-x^2)+(25x-2x^3)}{\sqrt{25-x^2}}\right]}{25-x^2}$$

$$= \frac{-100x+4x^3+25x-2x^3}{(25-x^2)\sqrt{25-x^2}}$$

$$= \frac{-75x+2x^3}{(25-x^2)\sqrt{25-x^2}}$$

$$\Rightarrow A''\left(\frac{5}{\sqrt{2}}\right) = \frac{-75\left(\frac{5}{\sqrt{2}}\right)+2\left(\frac{5}{\sqrt{2}}\right)^3}{\left(25-\left(\frac{5}{\sqrt{2}}\right)^2\right)^{\frac{3}{2}}} < 0$$

So $x = \left(\frac{5}{\sqrt{2}}\right)$ is a point of maxima.

Therefore the largest possible area of the triangle

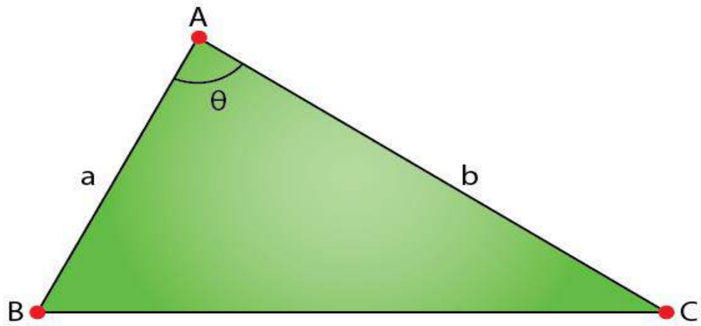
$$= \frac{1}{2} \times \left(\frac{5}{\sqrt{2}}\right) \times \left(\frac{5}{\sqrt{2}}\right) = \frac{25}{4} \text{ square units}$$

11. Two sides of a triangle have lengths 'a' and 'b' and the angle between them is θ. What value of θ will maximize the area of the triangle? Find the maximum area of the triangle also.

Solution:

It is given that two sides of a triangle have lengths a and b and the angle between them is θ.

Let the area of triangle be A



Then, $A = \frac{1}{2}ab\,\text{Sin}\,\theta$

$\Rightarrow \frac{dA}{d\theta} = \frac{1}{2}ab\,\text{Cos}\,\theta$

Condition for maxima and minima is

$$\frac{dA}{d\theta} = 0$$

$$\Rightarrow \frac{1}{2}ab\,\text{Cos}\,\theta = 0$$

$$\Rightarrow \theta = \frac{\pi}{2}$$

$$\frac{d^2A}{d\theta^2} = -\frac{1}{2}ab\,\text{Sin}\,\theta$$

So, for A to be maximum,

$$\frac{d^2A}{d\theta^2} < 0$$

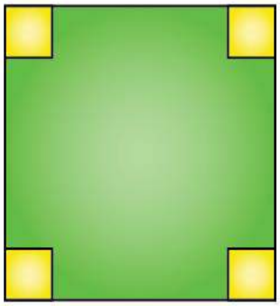
For $\theta = \frac{\pi}{2} \Rightarrow \frac{d^2A}{d\theta^2} < 0$

Hence, $\theta = \frac{\pi}{2}$ will give maximum area.

And maximum area will be $A = \frac{1}{2}ab$

12. A square piece of tin of side 18 cm is to be made into a box without top by cutting a square from each corner and folding up the flaps to form a box. What should be the side of the square to be cut off so that the volume of the box is maximum? Also, find this maximum volume

Solution:



Given side length of big square is 18 cm

Let the side length of each small square be a.

If by cutting a square from each corner and folding up the flaps we will get a cuboidal box with

Length, L = 18 – 2a

Breadth, B = 18 – 2a and

Height, H = a

Assuming, volume of box, V = LBH = $a(18 - 2a)^2$

Condition for maxima and minima is

$$\frac{dV}{da} = 0$$

$$\Rightarrow (18 - 2a)^2 + (a)(-2)(2)(18 - 2a) = 0$$

$$\Rightarrow (18 - 2a)[(18 - 2a) - 4a] = 0$$

$$\Rightarrow (18 - 2a)[18 - 6a] = 0$$

$$\Rightarrow a = 3, 9$$

$$\frac{d^2V}{da^2} = (-2)(18 - 6a) + (-6)(18 - 2a)$$

$$\frac{d^2V}{da^2} = 24a - 144$$

For a = 3, $\frac{d^2V}{da^2} = -72, \Rightarrow \frac{d^2V}{da^2} < 0$

For a = 9, $\frac{d^2V}{da^2} = 72, \Rightarrow \frac{d^2V}{da^2} > 0$

So for A to maximum

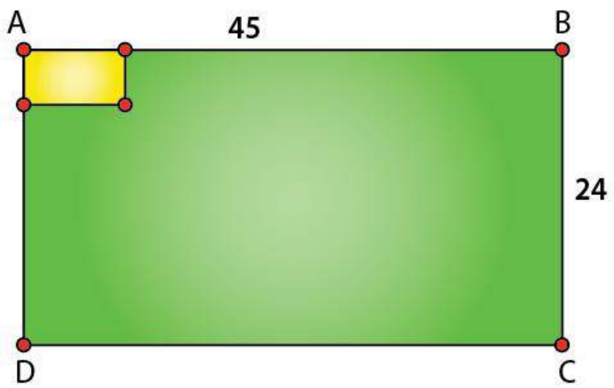
$\frac{d^2V}{da^2} < 0$

Hence, a = 3 will give maximum volume.

And maximum volume, V = a $(18 - 2a)^2$ = 432 cm^3

13. A rectangular sheet of tin 45 cm by 24 cm is to be made into a box without top, by cutting off squares from each corners and folding up the flaps. What should be the side of the square to be cut off so that the volume of the box is maximum possible?

Solution:



Given length of rectangle sheet = 45 cm
Breath of rectangle sheet = 24 cm
Let the side length of each small square be a.
If by cutting a square from each corner and folding up the flaps we will get a cuboidal box with
Length, L = 45 – 2a
Breadth, B = 24 – 2a and
Height, H = a
Assuming, volume of box, V = LBH = (45 - 2a)(24 - 2a)(a)
Condition for maxima and minima is

$\frac{dV}{da} = 0$

(45 - 2a) (24 - 2a) + (- 2) (24 - 2a) (a) + (45 - 2a) (- 2)(a) = 0

$4a^2 - 138a + 1080 + 4a^2 - 48a + 4a^2 - 90a = 0$
$12a^2 - 276a + 1080 = 0$
$a^2 - 23a + 90 = 0$
a = 5, 18

$\frac{d^2V}{da^2} = 24a - 276$

For a = 5, $\frac{d^2V}{da^2} = -156$, $\Rightarrow \frac{d^2V}{da^2} < 0$

For a = 18, $\frac{d^2V}{da^2} = +156$, $\Rightarrow \frac{d^2V}{da^2} > 0$

So for A to maximum

$\frac{d^2V}{da^2} < 0$

Hence, a = 5 will give maximum volume.

And maximum volume V = (45 - 2a) (24 - 2a) (a) = 2450 cm^3

14. A tank with rectangular base and rectangular sides, open at the top is to be constructed so that its depth is 2 m and volume is 8 m^3. If building of tank cost Rs 70 per square metre for the base and Rs 45 per square metre for sides, what is the cost of least expensive tank?

Solution:
Let the length, breath and height of tank be l, b and h respectively.
Also, assume volume of tank as V
h = 2 m (given)
V = 8 m^3
l b h = 8
2lb = 8 (given)
lb = 4
b = 4/l … (1)
Cost for building base = Rs 70/m^2
Cost for building sides = Rs 45/m^2
Cost for building the tank, C = Cost for base + cost for sides
C = lb × 70 + 2(l + b) h × 45

$C = l(4/l) \times 70 + 2(l + 4/l)(2) \times 45$ [Using (1)]

$= 280 + 180(l + 4/l)$

Condition for maxima and minima

$$\frac{dC}{dl} = 0$$

$\Rightarrow 180(1 - \frac{4}{l^2}) = 0$

$\Rightarrow \frac{4}{l^2} = 1$

$\Rightarrow l^2 = 4$

$\Rightarrow l = \pm 2$ cm

Since, l cannot be negative

So, l = 2 cm

$$\frac{d^2C}{dl^2} = 180\left(\frac{8}{l^3}\right)$$

For l = 2 $\frac{d^2C}{dl^2} = 180$

$\Rightarrow \frac{d^2C}{dl^2} > 0$

Therefore, cost will be minimum for l =2

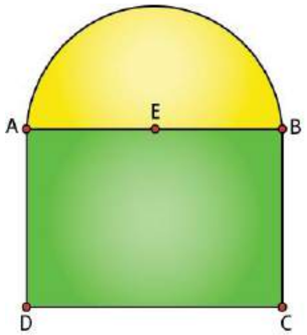
From equation 2

$C = 280 + 180(l + \frac{4}{l})$

C = Rs 1000

15. A window in the form of a rectangle is surmounted by a semi-circular opening. The total perimeter of the window is 10 m. Find the dimensions of the rectangular part of the window to admit maximum light through the whole opening.

Solution:



Let the radius of semicircle, length and breadth of rectangle be r, x and y respectively

AE = r

AB = x = 2r (semicircle is mounted over rectangle) …1

AD = y

Given Perimeter of window = 10 m

$x + 2y + \pi r = 10$

$2r + 2y + \pi r = 10$

$2y = 10 – (\pi + 2).r$

$\Rightarrow y = \frac{10-(\pi+2)r}{2}$ … 2

To admit maximum amount of light, area of window should be maximum

Assuming area of window as A

$A = xy + \frac{\pi r^2}{2}$

$\Rightarrow A = (2r)\left(\frac{10-(\pi+2)r}{2}\right) + \frac{\pi r^2}{2}$

$\Rightarrow A = 10r - \pi r^2 – 2r^2 + \frac{\pi r^2}{2}$

$\Rightarrow A = 10r – 2r^2 - \frac{\pi r^2}{2}$

Condition for maxima and minima is

$$\frac{dA}{dr} = 0$$

$\Rightarrow 10 – 4r - \pi r = 0$

$$\Rightarrow r = \frac{10}{4+\pi}$$

$$\frac{d^2A}{dr^2} = -4 - \pi < 0$$

At $r = \frac{10}{4+\pi}$ A will be maximum.

Length of rectangular part = $\frac{20}{4+\pi}$ m (from equation 1)

Breadth of rectangular part = $\frac{10-(\pi+2)r}{2}$ m (from equation 2)

$$\Rightarrow y = \frac{10 - \frac{(\pi+2)10}{4+\pi}}{2}$$

$$\Rightarrow y = \frac{10}{4+\pi}$$

16. A large window has the shape of a rectangle surmounted by an equilateral triangle. If the perimeter of the window is 12 metres find the dimensions of the rectangle that will produce the largest area of the window.

Solution:

Let the dimensions of the rectangle be x and y.

Therefore, the perimeter of window = x + y + x + x + y = 12

3x + 2y = 12

y = (12 – 3x)/2 …. (1)

Now,

$$\text{Area of the window } = xy + \frac{\sqrt{3}}{4}x^2$$

$$\Rightarrow A = x\left(\frac{12-3x}{2}\right) + \frac{\sqrt{3}}{4}x^2$$

$$\Rightarrow A = 6x - \frac{3x^2}{2} + \frac{\sqrt{3}}{4}x^2$$

$$\Rightarrow \frac{dA}{dx} = 6 - \frac{6x}{2} + \frac{2\sqrt{3}}{4}x$$

$$\Rightarrow \frac{dA}{dx} = 6 - 3x + \frac{\sqrt{3}}{2}x$$

$$\Rightarrow \frac{dA}{dx} = 6 - x\left(3 - \frac{\sqrt{3}}{2}\right)$$

For maximum and minimum values of A, we must have

$$\frac{dA}{dx} = 0$$

$$\Rightarrow 6 = x\left(3 - \frac{\sqrt{3}}{2}\right)$$

$$\Rightarrow x = \frac{12}{6-\sqrt{3}}$$

Substituting the values of x in equation 1 we get

$$y = \frac{12 - 3\left(\frac{12}{6-\sqrt{3}}\right)}{2}$$

$$\Rightarrow y = \frac{18-6\sqrt{3}}{6-\sqrt{3}}$$

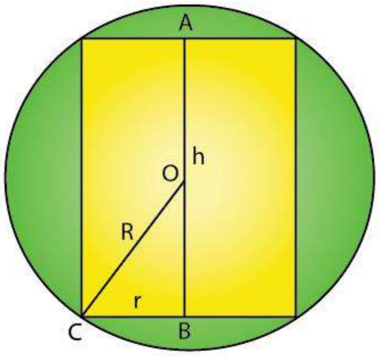
Now,

$$\frac{d^2A}{dx^2} = -3 + \frac{\sqrt{3}}{2} < 0$$

Thus, the area is maximum when $x = \frac{12}{6-\sqrt{3}}$ and $y = \frac{18-6\sqrt{3}}{6-\sqrt{3}}$.

17. Show that the height of the cylinder of maximum volume that can be inscribed in a sphere of radius R is 2R/√3.

Solution:



Let the radius, height and volume of cylinder be r, h and V respectively

Radius of sphere = R (Given)

Volume of cylinder, $V = \pi r^2 h$ ...1

$$OB = \frac{h}{2}$$

OC = R

BC = r

In triangle OBC,

$$\left(\frac{h}{2}\right)^2 + r^2 = R^2$$

$$\Rightarrow r^2 = R^2 - \left(\frac{h}{2}\right)^2 \quad ...2$$

Replacing equation 2 in equation 1, we get

$$V = \pi \left(R^2 - \left(\frac{h}{2}\right)^2\right)(h) = \pi R^2 h - \pi \frac{h^3}{4}$$

Condition for maxima and minima is

$$\frac{dV}{dh} = 0$$

$$\Rightarrow \pi R^2 - \pi \frac{3h^2}{4} = 0$$

$$\Rightarrow \pi R^2 = \pi \frac{3h^2}{4}$$

$h^2 = \frac{4}{3}R^2$

$$\Rightarrow h = \pm \frac{2}{\sqrt{3}}R$$

Since, h cannot be negative

Hence, $h = \frac{2}{\sqrt{3}}R$

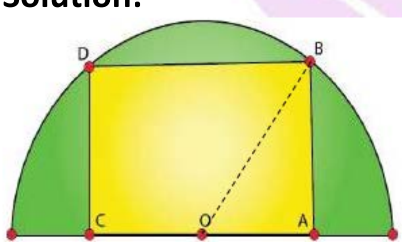
$$\frac{d^2V}{dh^2} = -\pi \frac{6h}{4}$$

For $h = \frac{2}{\sqrt{3}}R$ $\frac{d^2V}{dh^2} < 0$

V will be maximum for $h = \frac{2}{\sqrt{3}}R$

18. A rectangle is inscribed in a semi-circle of radius r with one of its sides on diameter of semi-circle. Find the dimensions of the rectangle so that its area is maximum. Find also the area.

Solution:



Let the length and breadth of rectangle ABCD be 2x and y respectively
Radius of semicircle = r (given)
In triangle OBA, where is the centre of the circle and mid-point of the side AC
$r^2 = x^2 + y^2$ (Pythagoras theorem)
$y^2 = r^2 - x^2$
$\Rightarrow y = \sqrt{r^2 - x^2}$ ... (1)
Let us say, area of rectangle = A = xy

$\Rightarrow$ A = x ($\sqrt{r^2 - x^2}$) (from equation 1)

Condition for maxima and minima is

$$\frac{dA}{dx} = 0$$

$$\Rightarrow \sqrt{r^2 - x^2} + x\left(\frac{1}{\sqrt{r^2 - x^2}}\right)\left(\frac{1}{2}\cdot(-2x)\right) = 0$$

$$\Rightarrow \sqrt{r^2 - x^2} - \left(\frac{x^2}{\sqrt{r^2 - x^2}}\right) = 0$$

$$\Rightarrow \sqrt{r^2 - x^2} = \frac{x^2}{\sqrt{r^2 - x^2}}$$

$\Rightarrow r^2 - x^2 = x^2$

$\Rightarrow 2x^2 = r^2$

$\Rightarrow x = \pm\frac{r}{\sqrt{2}}$

Since, x cannot be negative

Hence, $x = \frac{r}{\sqrt{2}}$

$$\frac{d^2A}{dx^2} = \frac{-2x}{\sqrt{r^2 - x^2}} - \left(\frac{2x\sqrt{r^2 - x^2} - x^2\frac{-2x}{\sqrt{r^2 - x^2}}}{\left(\sqrt{r^2 - x^2}\right)^2}\right)$$

For $x = \frac{r}{\sqrt{2}}$, $\frac{d^2A}{dx^2} < 0$

$\Rightarrow$ A will be maximum for $x = \frac{r}{\sqrt{2}}$

From equation 1

$y = \sqrt{r^2 - x^2} = \frac{r}{\sqrt{2}}$

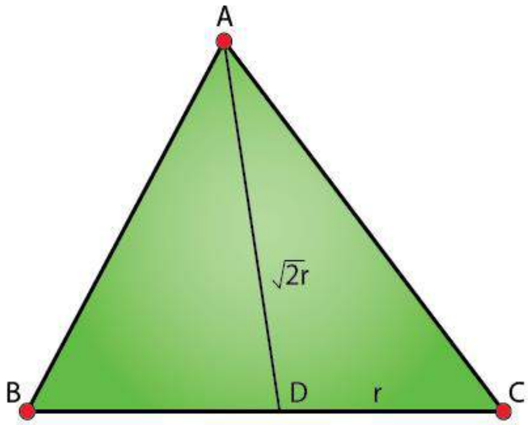
Length of rectangle = $\sqrt{2}r$ [Since, l = 2x]

Breadth of rectangle = $\frac{r}{\sqrt{2}}$

Area of rectangle = r^2

19. Prove that a conical tent of given capacity will require the least amount of canvas when the height is √2 times the radius of the base.

Solution:



Let the radius and height of cone be r and h respectively

It is given that volume of cone is fixed.

Volume of cone, $V = \frac{1}{3}\pi r^2 h$

$\Rightarrow h = \frac{3V}{\pi r^2}$ ...1

Curved surface area of cone, S = πr l (l is slant height)

Since, $l = \sqrt{r^2 + h^2}$

$$\Rightarrow l = \sqrt{r^2 + \left(\frac{3V}{\pi r^2}\right)^2}$$

$$\Rightarrow l = \sqrt{\frac{\pi^2 r^6 + 9V^2}{\pi^2 r^4}}$$

$$\Rightarrow\ l = \frac{\sqrt{\pi^2 r^6 + 9V^2}}{\pi r^2}$$

So, $S = \pi r\ \frac{\sqrt{\pi^2 r^6 + 9V^2}}{\pi r^2}$

$$\Rightarrow\ S = \frac{\sqrt{\pi^2 r^6 + 9V^2}}{r}$$

Condition for maxima and minima is

$$\frac{dS}{dr} = 0$$

$$\Rightarrow \frac{\frac{3\pi^2 r^5}{\sqrt{\pi^2 r^6 + 9V^2}} \cdot r - \sqrt{\pi^2 r^6 + 9V^2}}{r^2} = 0$$

$$\frac{3\pi^2 r^6 - \pi^2 r^6 - 9V^2}{r^2\sqrt{\pi^2 r^6 + 9V^2}} = 0$$

$$\Rightarrow\ 2\pi^2 r^6 - 9V^2 = 0$$

$$\Rightarrow\ 2\pi^2 r^6 = 9V^2 \ \dots 2$$

$$\Rightarrow\ r = \left(\frac{9V^2}{2\pi^2}\right)^{\frac{1}{6}}$$

For $r = \left(\frac{9V^2}{2\pi^2}\right)^{\frac{1}{6}}$, $\frac{d^2S}{dr^2} > 0$

$\Rightarrow$ S will be minimum for $r = \left(\frac{9V^2}{2\pi^2}\right)^{\frac{1}{6}}$

From equation 1

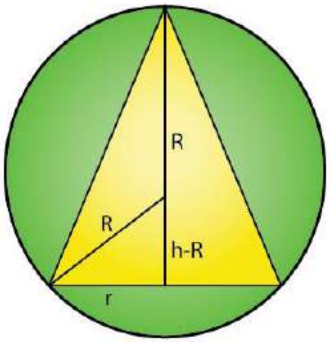
$h = \frac{3}{\pi r^2} \cdot \frac{\sqrt{2}\pi r^3}{3}$ (From equation 3)

h = √2 r

20. Show that the cone of the greatest volume which can be inscribed in a given

sphere has an altitude equal to 2/3 of the diameter of the sphere.

Solution:



Let the radius and height of cone be r and h respectively

Radius of sphere = R

$R^2 = r^2 + (h - R)^2$

$R^2 = r^2 + h^2 + R^2 - 2hR$

$r^2 = 2hR - h^2$ … (1)

Assuming volume of cone be V

Volume of cone, $V = \frac{1}{3}\pi(2hR - h^2)h$ (from equation 1)

$\Rightarrow V = \frac{1}{3}\pi(2h^2R - h^3)$

Condition for maxima and minima is

$\frac{dV}{dh} = 0$

$\Rightarrow \frac{1}{3}\pi(4hR - 3h^2) = 0$

$\Rightarrow 4hR - 3h^2 = 0$

$\Rightarrow h = \frac{4R}{3}$

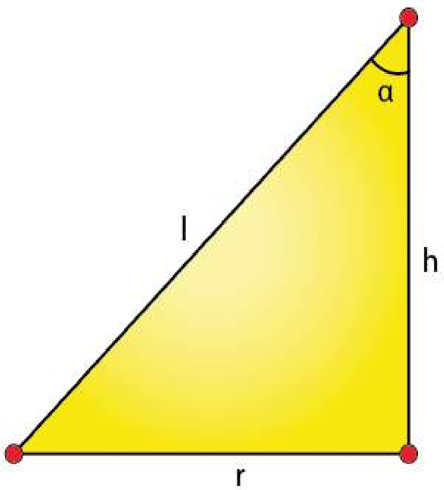
For $h = \frac{4R}{3}$, $\frac{d^2V}{dh^2} < 0$

$\Rightarrow$ V will be maximum for $h = \frac{4R}{3}$

$$h = \frac{2}{3}(2R)$$

21. Prove that the semi - vertical angle of the right circular cone of given volume and least curved surface is $\cot^{-1}\sqrt{2}$

Solution:



Let r be the radius of the base circle of the cone and l be the slant length and h be the height of the cone

Let us assume α be the semi - vertical angle of the cone.

We know that Volume of a right circular cone is given by:

$\Rightarrow V = \frac{\pi r^2 h}{3}$

Let us assume $r^2h = k$ (constant) ...... (1)

$\Rightarrow V = \frac{\pi k}{3}$

$\Rightarrow h = \frac{k}{r^2}$ ...... (2)

We know that surface area of a cone is

$\Rightarrow S = \pi rl$ ...... (3)

From the cross - section of cone we see that,

$\Rightarrow l^2 = r^2 + h^2$

$\Rightarrow l = \sqrt{r^2 + h^2}$ ...... (4)

Substituting (4) in (3), we get

$\Rightarrow S = \pi r\left(\sqrt{r^2 + h^2}\right)$

From (2)

$\Rightarrow S = \pi r\left(\sqrt{r^2 + \left(\frac{k}{r^2}\right)^2}\right)$

$\Rightarrow S = \pi r\left(\sqrt{r^2 + \left(\frac{k^2}{r^4}\right)}\right)$

$\Rightarrow S = \pi r\left(\sqrt{\frac{r^6 + k^2}{r^4}}\right)$

$\Rightarrow S = \pi r\left(\frac{\sqrt{r^6 + k^2}}{r^2}\right)$

$\Rightarrow S = \left(\frac{\pi \times \sqrt{r^6 + k^2}}{r}\right)$

Let us consider S as a function of R and we find the value of 'r' for its extremum,

Differentiating S with respect to r we get

$\Rightarrow \frac{dS}{dr} = \frac{d}{dr}\left(\frac{\pi\sqrt{r^6 + k^2}}{r}\right)$

Differentiating using U/V rule

$\Rightarrow \frac{dS}{dr} = \frac{\pi\left(r \times \frac{d\left(\sqrt{r^6 + k^2}\right)}{dr} - \left(\sqrt{r^6 + k^2}\right)\frac{dr}{dr}\right)}{r^2}$

$$\Rightarrow \frac{dS}{dr} = \frac{\pi\left(r\times\frac{1}{2\sqrt{r^6+k^2}}\times\frac{d(r^6+k^2)}{dr} - \left(\sqrt{r^6+k^2}\times 1\right)\right)}{r^2}$$

$$\Rightarrow \frac{dS}{dr} = \frac{\pi\left(\frac{r\times 6r^5}{2\sqrt{r^6+k^2}} - \sqrt{r^6+k^2}\right)}{r^2}$$

$$\Rightarrow \frac{dS}{dr} = \frac{\pi\left(\frac{3r^6}{\sqrt{r^6+k^2}} - \sqrt{r^6+k^2}\right)}{r^2}$$

$$\Rightarrow \frac{dS}{dr} = \frac{\pi\left(\frac{3r^6-(r^6+k^2)}{\sqrt{r^6+k^2}}\right)}{r^2}$$

$$\Rightarrow \frac{dS}{dr} = \frac{\pi(2r^6-k^2)}{r^2\sqrt{r^6+k^2}}$$

Equating differentiate to zero to get the relation between h and r.

$$\Rightarrow \frac{dS}{dr} = 0$$

$$\Rightarrow \frac{\pi(2r^6-k^2)}{r^2\times\sqrt{r^6+k^2}} = 0$$

Since the remainder is greater than zero only the remainder gets equal to zero

$\Rightarrow 2r^6 = k^2$

From (1)

$\Rightarrow 2r^6 = (r^2h)^2$

$\Rightarrow 2r^6 = (r^2h)^2$

$\Rightarrow 2r^6 = r^4h^2$

$\Rightarrow 2r^2 = h^2$

Since height and radius cannot be negative,

$\Rightarrow h = \sqrt{2}r$ ...... (5)

From the figure

$\Rightarrow \cot\alpha = \frac{h}{r}$

From (5)

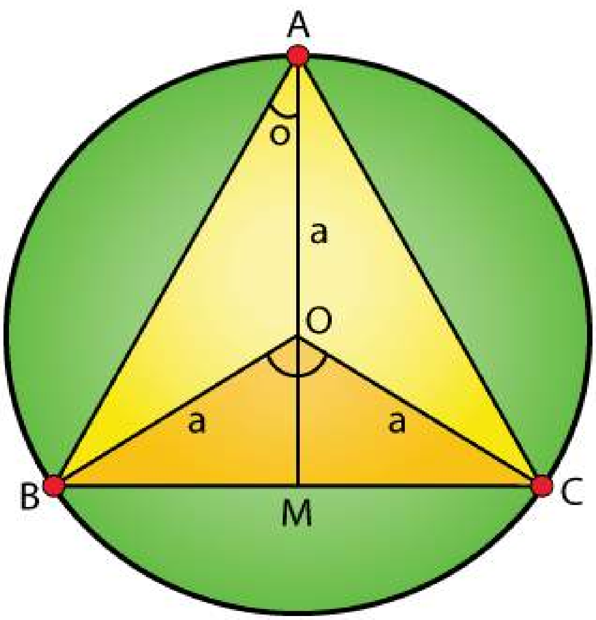
$\Rightarrow \cot\alpha = \sqrt{2}$

$\Rightarrow \alpha = \cot^{-1}\sqrt{2}$

∴ Thus proved.

22. An isosceles triangle of vertical angle 2θ is inscribed in a circle of radius a. Show that the area of the triangle is maximum when θ = π/6.

Solution:



Δ ABC is an isosceles triangle such that AB = AC.
The vertical angle BAC = 2θ
Triangle is inscribed in the circle with center O and radius a.
Draw AM perpendicular to BC.
Since, Δ ABC is an isosceles triangle, the circumcenter of the circle will lie on the perpendicular from A to BC.
Let O be the circumcenter.
BOC = 2 × 2θ = 4θ (Using central angle theorem)
COM = 2θ (Since, Δ OMB and Δ OMC are congruent triangles)

OA = OB = OC = a (radius of the circle)
In Δ OMC,
CM = asin2θ
OM = acos2θ
BC = 2CM (Perpendicular from the center bisects the chord)
BC = 2asin2θ
Height of Δ ABC = AM = AO + OM
AM = a + acos2θ

$$\text{Area of } \Delta \text{ ABC} = \frac{1}{2} \times \text{AM} \times \text{BC}$$

Differentiation this equation with respect to θ

$$\frac{dA}{d\theta} = \frac{d\left[\frac{1}{2} \times (a + a\cos 2\theta) \times (2a\sin 2\theta)\right]}{d\theta}$$

$$\frac{dA}{d\theta} = (2a\sin 2\theta)(-2a\sin 2\theta) + (a + a\cos 2\theta)(2a\cos 2\theta)$$

$$\frac{dA}{d\theta} = (-2a^2\sin^2 2\theta) + (2a^2\cos 2\theta + 2a^2\cos^2 2\theta)$$

$$\Rightarrow \frac{dA}{d\theta} = 2a^2(\cos^2 2\theta - \sin^2 2\theta) + 2a^2\cos 2\theta$$

$$\Rightarrow \frac{dA}{d\theta} = 2a^2(\cos 4\theta) + 2a^2\cos 2\theta \ (\cos^2 x - \sin^2 x = \cos 2x)$$

$$\Rightarrow \frac{d^2A}{d\theta^2} = -2 \times 4 \times a^2(\sin 4\theta) + (-4a^2\sin 2\theta)$$

Maxima or minima exists when:

$$\frac{dA}{d\theta} = 0$$

Therefore,

$$2a^2(\cos 4\theta) + 2a^2\cos 2\theta = 0$$

$$\Rightarrow \cos 4\theta + \cos 2\theta = 0$$

$\Rightarrow 2\cos^2 2\theta - 1 + \cos 2\theta = 0$

$\Rightarrow (2\cos 2\theta - 1)(\cos 2\theta + 1) = 0$

Therefore, $\Rightarrow \cos 2\theta = \frac{1}{2}$

$\Rightarrow 2\theta = \frac{\pi}{3}$

and $\cos 2\theta = -1$

$\Rightarrow 2\theta = \pi$

$\theta = \frac{\pi}{6}, \frac{\pi}{2}$

To check whether which point has a maxima, we have to check the double differentiate.

Therefore, at $\theta = \frac{\pi}{6}$:

$$\frac{d^2A}{d\theta^2} = -2 \times 4 \times a^2 \left(\sin 4 \times \frac{\pi}{6}\right) + \left(-4a^2 \sin 2 \times \frac{\pi}{6}\right)$$

$$\frac{d^2A}{d\theta^2} = -2 \times 4 \times a^2 \left(\sin \frac{2\pi}{3}\right) + \left(-4a^2 \sin \frac{\pi}{3}\right)$$

Both the sin values are positive. So the entire expression is negative. Hence there is a maxima at this point.

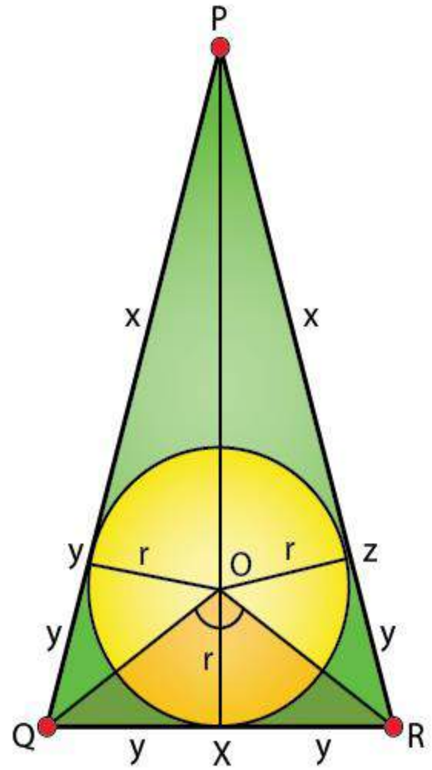
$\theta = \frac{\pi}{2}$ will not form a triangle. Hence it is discarded.

Therefore the maxima exits at:

$\theta = \frac{\pi}{6}$

23. Prove that the least perimeter of an isosceles triangle in which a circle of radius r can be inscribed is 6√3r.

Solution:



QR at X and PR at Z.

OZ, OX, OY are perpendicular to the sides PR, QR, PQ.

Here PQR is an isosceles triangle with sides PQ = PR and also from the figure,

⇒ PY = PZ = x

⇒ YQ = QX = XR = RZ = y

From the figure we can see that,

⇒ Area (ΔPQR) = Area (ΔPOR) + Area (ΔPOQ) + Area (ΔQOR)

We know that area of a triangle = $\frac{1}{2}\times$ base × height

$$\Rightarrow \frac{1}{2}\times QR\times PX = \left(\frac{1}{2}\times OZ\times PR\right) + \left(\frac{1}{2}\times OY\times PQ\right) + \left(\frac{1}{2}\times QR\times OX\right)$$

$$\frac{1}{2}\times 2y\left(r + \sqrt{x^2 + r^2}\right) = \left(\frac{1}{2}\times r\times (x + y)\right) + \left(\frac{1}{2}\times r\times (x + y)\right) + \left(\frac{1}{2}\times 2y\times r\right)$$

$$\Rightarrow y\left(r + \sqrt{x^2 + r^2}\right) = r(x + y) + yr$$

$$\Rightarrow y\left(\sqrt{x^2 + r^2}\right) = r(x + y)$$

$\Rightarrow \sqrt{x^2 + r^2} = \frac{r(x+y)}{y}$

$\Rightarrow x^2 + r^2 = \frac{r^2(x+y)^2}{y^2}$

$\Rightarrow x^2 + r^2 = r^2 + \frac{r^2x^2}{y^2} + \frac{r^2(2xy)}{y^2}$

$\Rightarrow x^2\left(1 - \frac{r^2}{y^2}\right) - \frac{r^2(2xy)}{y^2} = 0$

$\Rightarrow x\left(x\left(1 - \frac{r^2}{y^2}\right) - \frac{r^2(2y)}{y^2}\right) = 0$

$\Rightarrow x = \frac{2r^2y}{y^2-r^2}$ ...... (1)

We know that perimeter of the triangle is Per = PQ + QR + RP

⇒ PER = (x + y) + (x + y) + 2y

⇒ PER = 2x + 4y ...... (2)

From (1)

$\Rightarrow \text{PER} = \frac{4r^2y}{y^2-r^2} + 4y$

$\Rightarrow \text{PER} = \frac{4y(r^2+y^2-r^2)}{y^2-r^2}$

$\Rightarrow \text{PER} = \frac{4y^3}{y^2-r^2}$

We need perimeter to be minimum and let us PER as the function of y,

We know that for maxima and minima $\frac{d(\text{PER})}{dy} = 0$,

$\Rightarrow \frac{d(\text{PER})}{dy} = \frac{d\left(\frac{4y^3}{y^2-r^2}\right)}{dr}$

$\Rightarrow \frac{d(\text{PER})}{dy} = \frac{(y^2-r^2)\frac{d(4y^3)}{dy} - (4y^3)\frac{d(y^2-r^2)}{dy}}{(y^2-r^2)^2}$

$$\Rightarrow \frac{d(PER)}{dy} = \frac{((y^2-r^2)(12y^2))-((4y^3)(2y))}{(y^2-r^2)^2}$$

$$\Rightarrow \frac{d(PER)}{dy} = \frac{4y^4-12y^2r^2}{(y^2-r^2)^2}$$

$\Rightarrow 4y^4 - 12y^2r^2 = 0$

$\Rightarrow 4y^2 (y^2 - 3r^2) = 0$

$\Rightarrow y = \sqrt{3}r$

Differentiating PER again,

$$\Rightarrow \frac{d^2(PER)}{dy^2} = \frac{d}{dy}\left(\frac{6y^4-12y^2r^2}{(y^2-r^2)^2}\right)$$

$$\Rightarrow \frac{d^2(PER)}{dy^2} = \frac{(y^2-r^2)^2\frac{d(6y^4-12y^2r^2)}{dy}-(6y^4-12y^2r^2)\frac{d}{dy}\left((y^2-r^2)^2\right)}{(y^2-r^2)^4}$$

$$\Rightarrow \frac{d^2(PER)}{dy^2} = \frac{((y^2-r^2)^2(24y^3-24yr^2))-\left((6y^4-12y^2r^2)\left(2(y^2-r^2)(2y)\right)\right)}{(y^2-r^2)^4}$$

$$\Rightarrow \frac{d^2(PER)}{dy^2}\Big|_{y=\sqrt{3}r} = \frac{\left((3r^2-r^2)^2(72\sqrt{3}r^3-24\sqrt{3}r^3)\right)-\left((54r^4-36r^4)\left(2(3r^2-r^2)(2\sqrt{3}r)\right)\right)}{(3r^2-r^2)^4}$$

$$\Rightarrow \frac{d^2(PER)}{dy^2}\Big|_{y=\sqrt{3}r} = \frac{\left((4r^4)(48\sqrt{3}r^3)\right)-\left((18r^4)(8\sqrt{3}r^3)\right)}{16r^8}$$

$$\Rightarrow \frac{d^2(PER)}{dy^2}\Big|_{y=\sqrt{3}r} = \frac{48\sqrt{3}r^7}{16r^8}$$

$$\Rightarrow \frac{d^2(PER)}{dy^2}\Big|_{y=\sqrt{3}r} = \frac{3\sqrt{3}}{r} > 0 \text{(minima)}$$

We got minima at $y = \sqrt{3}r$.

Let's find the value of x,

$$\Rightarrow x = \frac{2r^2(\sqrt{3}r)}{(\sqrt{3}r)^2-r^2}$$

$\Rightarrow x = \sqrt{3}\, r$

⇒ PER = $2(\sqrt{3}r) + 4(\sqrt{3}r)$
⇒ PER = $6\sqrt{3}r$
∴ Thus proved