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Question 1: Find the cube of each of the following binomial expressions:
(i) (1/x + y/3)

(ii) (3/x – 2/x^2)

(iii) (2x + 3/x)

(iv) (4 – 1/3x)

Solution:

[Using identities: $(a + b)^3 = a^3 + b^3 + 3ab(a + b)$ and $(a - b)^3 = a^3 - b^3 - 3ab(a - b)$]

(i)

$$\left(\frac{1}{x} + \frac{y}{3}\right)^3 = \left(\frac{1}{x}\right)^3 + \left(\frac{y}{3}\right)^3 + 3\left(\frac{1}{x}\right)\left(\frac{y}{3}\right)\left(\frac{1}{x} + \frac{y}{3}\right)$$

$$= \frac{1}{x^3} + \frac{y^3}{27} + 3 \times \frac{1}{x} \times \frac{y}{3}\left(\frac{1}{x} + \frac{y}{3}\right)$$

$$= \frac{1}{x^3} + \frac{y^3}{27} + \frac{y}{x}\left(\frac{1}{x} + \frac{y}{3}\right)$$

$$= \frac{1}{x^3} + \frac{y^3}{27} + \left(\frac{y}{x} \times \frac{1}{x}\right) + \left(\frac{y}{x} \times \frac{y}{3}\right)$$

$$= \frac{1}{x^3} + \frac{y^3}{27} + \frac{y}{x^2} + \frac{y^2}{3x}$$

(ii)

$$\left(\frac{3}{x} - \frac{2}{x^2}\right)^3 = \left(\frac{3}{x}\right)^3 - \left(\frac{2}{x^2}\right)^3 - 3\left(\frac{3}{x}\right)\left(\frac{2}{x^2}\right)\left(\frac{3}{x} - \frac{2}{x^2}\right)$$

$$= \frac{27}{x^3} - \frac{8}{x^6} - 3 \times \frac{3}{x} \times \frac{2}{x^2}\left(\frac{3}{x} - \frac{2}{x^2}\right)$$

$$= \frac{27}{x^3} - \frac{8}{x^6} - \frac{18}{x^3}\left(\frac{3}{x} - \frac{2}{x^2}\right)$$

$$= \frac{27}{x^3} - \frac{8}{x^6} - \frac{54}{x^4} + \frac{36}{x^5}$$

(iii)

$(2x + \frac{3}{x})^3$

$= 8x^3 + \frac{27}{x^3} + \frac{18x}{x}(2x + \frac{3}{x})$

$= 8x^3 + \frac{27}{x^3} + \frac{18x}{x}(2x + \frac{3}{x})$

$= 8x^3 + \frac{27}{x^3} + (18 \times 2x) + (18 \times \frac{3}{x})$

$= 8x^3 + \frac{27}{x^3} + 36x + \frac{54}{x}$

(iv)

$(4 - \frac{1}{3x})^3 = 4^3 - (\frac{1}{3x})^3 - 3(4)(\frac{1}{3x})(4 - \frac{1}{3x})$

$= 64 - \frac{1}{27x^3} - \frac{4}{x}(4 - \frac{1}{3x})$

$= 64 - \frac{1}{27x^3} - \frac{16}{x} + \frac{4}{3x^2}$

Question 2: Simplify each of the following:
(i) $(x + 3)^3 + (x - 3)^3$
(ii) $(x/2 + y/3)^3 - (x/2 - y/3)^3$
(iii) $(x + 2/x)^3 + (x - 2/x)^3$
(iv) $(2x - 5y)^3 - (2x + 5y)^3$

Solution:
[Using identities:
$a^3 + b^3 = (a + b)(a^2 + b^2 - ab)$
$a^3 - b^3 = (a - b)(a^2 + b^2 + ab)$
$(a + b)(a-b) = a^2 - b^2$
$(a + b)^2 = a^2 + b^2 + 2ab$ and
$(a - b)^2 = a^2 + b^2 - 2ab$]

(i) $(x + 3)^3 + (x - 3)^3$
Here $a = (x + 3)$, $b = (x - 3)$

$= (x + 3 + x - 3)[(x + 3)^2 + (x - 3)^2 - (x + 3)(x - 3)]$

$= 2x[(x^2 + 9 + 6x) + (x^2 + 9 - 6x) - x^2 + 9]$

$= 2x[(x^2 + 9 + 6x + x^2 + 9 - 6x - x^2 + 9)]$

$= 2x(x^2 + 27)$

$= 2x^3 + 54x$

(ii) $(x/2 + y/3)^3 - (x/2 - y/3)^3$

Here a = (x/2 + y/3) and b = (x/2 – y/3)

$$= [(\frac{x}{2} + \frac{y}{3}) - (\frac{x}{2} - \frac{y}{3})][(\frac{x}{2} + \frac{y}{3})^2 + (\frac{x}{2} - \frac{y}{3})^2 + (\frac{x}{2} + \frac{y}{3})(\frac{x}{2} - \frac{y}{3})]$$

$$= \frac{2y}{3}[(\frac{x^2}{4} + \frac{y^2}{9} + \frac{2xy}{6}) + (\frac{x^2}{4} + \frac{y^2}{9} - \frac{2xy}{6}) + \frac{x^2}{4} - \frac{y^2}{9}]$$

$$= \frac{2y}{3}[\frac{x^2}{4} + \frac{y^2}{9} + \frac{x^2}{4} + \frac{x^2}{4}]$$

$$= \frac{2y}{3}[\frac{3x^2}{4} + \frac{y^2}{9}]$$

$$= \frac{x^2y}{2} + \frac{2y^3}{27}$$

(iii) $(x + 2/x)^3 + (x - 2/x)^3$

Here a = (x + 2/x) and b = (x – 2/x)

$= (x + \frac{2}{x} + x - \frac{2}{x})[(x + \frac{2}{x})^2 + (x - \frac{2}{x})^2 - ((x + \frac{2}{x})(x - \frac{2}{x}))]$

$= (2x)[(x^2 + \frac{4}{x^2} + \frac{4x}{x}) + (x^2 + \frac{4}{x^2} - \frac{4x}{x}) - (x^2 - \frac{4}{x^2})$

$= (2x)[(x^2 + \frac{4}{x^2} + \frac{4}{x^2} + \frac{4}{x^2})$

$= (2x)[(x^2 + \frac{12}{x^2})$

$= 2x^3 + \frac{24}{x}$

(iv) $(2x – 5y)^3 – (2x + 5y)^3$

Here a = (2x – 5y) and b = 2x + 5y

$= (2x - 5y - 2x - 5y)[(2x - 5y)^2 + (2x + 5y)^2 + ((2x - 5y)(2x + 5y))]$

$= (-10y)[(4x^2 + 25y^2 - 20xy) + (4x^2 + 25y^2 + 20xy) + 4x^2 - 25y^2]$

$= (-10y)[4x^2 + 4x^2 + 4x^2 + 25y^2]$

$= (-10y)[12x^2 + 25y^2\}$

$= -120x^2y - 250y^3$

Question 3: If a + b = 10 and ab = 21, find the value of $a^3 + b^3$.
Solution:
a + b = 10, ab = 21 (given)
Choose a + b = 10
Cubing both sides,
$(a + b)^3 = (10)^3$
$a^3 + b^3 + 3ab(a + b) = 1000$
$a^3 + b^3 + 3 \times 21 \times 10 = 1000$ (using given values)
$a^3 + b^3 + 630 = 1000$
$a^3 + b^3 = 1000 – 630 = 370$
or $a^3 + b^3 = 370$

Question 4: If a – b = 4 and ab = 21, find the value of $a^3 - b^3$.
Solution:
a – b = 4, ab= 21 (given)
Choose a – b = 4
Cubing both sides,
$(a – b)^3 = (4)^3$
$a^3 – b^3 – 3ab (a – b) = 64$
$a^3 – b^3 - 3 \times 21 \times 4 = 64$ (using given values)
$a^3 – b^3 – 252 = 64$
$a^3 – b^3 = 64 + 252$
= 316
Or $a^3 – b^3 = 316$

Question 5: If x + 1/x = 5, find the value of $x^3 + 1/x^3$.
Solution:
Given: x + 1/x = 5
Apply Cube on x + 1/x

$$(x+\frac{1}{x})^3 = x^3 + \frac{1}{x^3} + 3(x \times \frac{1}{x})(x+\frac{1}{x})$$

$$5^3 = x^3 + \frac{1}{x^3} + 3(x + \frac{1}{x})$$

$$125 = x^3 + \frac{1}{x^3} + 3(5)$$

$$125 = x^3 + \frac{1}{x^3} + 15$$

$$125 - 15 = x^3 + \frac{1}{x^3}$$

$$x^3 + \frac{1}{x^3} = 110$$

Question 6: If x - 1/x = 7, find the value of $x^3 - 1/x^3$.
Solution:
Given: x - 1/x = 7
Apply Cube on x - 1/x

$(x - \frac{1}{x})^3 = x^3 - \frac{1}{x^3} - 3(x \times \frac{1}{x})(x - \frac{1}{x})$

$7^3 = x^3 - \frac{1}{x^3} - 3(x - \frac{1}{x})$

$343 = x^3 - \frac{1}{x^3} - (3 \times 7)$

$343 + 21 = x^3 - \frac{1}{x^3}$

$x^3 - \frac{1}{x^3} = 364$

Question 7: If x - 1/x = 5, find the value of x^3 - 1/x^3 .

Solution:
Given: x - 1/x = 5
Apply Cube on x - 1/x

$(x - \frac{1}{x})^3 = x^3 - \frac{1}{x^3} - 3(x \times \frac{1}{x})(x - \frac{1}{x})$

$5^3 = x^3 - \frac{1}{x^3} - 3(x - \frac{1}{x})$

$125 = x^3 - \frac{1}{x^3} - (3 \times 5)$

$125 = x^3 - \frac{1}{x^3} - 15$

$125 + 15 = x^3 - \frac{1}{x^3}$

$x^3 - \frac{1}{x^3} = 140$

Question 8: If (x^2 + 1/x^2) = 51, find the value of x^3 - 1/x^3.

Solution:
We know that: (x – y)^2 = x^2 + y^2 – 2xy

Replace y with 1/x, we get

(x – 1/x)^2 = x^2 + 1/x^2 – 2

Since $(x^2 + 1/x^2) = 51$ (given)

$(x - 1/x)^2 = 51 - 2 = 49$

or $(x - 1/x) = \pm 7$

Now, Find $x^3 - 1/x^3$

We know that, $x^3 - y^3 = (x - y)(x^2 + y^2 + xy)$

Replace y with 1/x, we get

$x^3 - 1/x^3 = (x - 1/x)(x^2 + 1/x^2 + 1)$

Use $(x - 1/x) = 7$ and $(x^2 + 1/x^2) = 51$

$x^3 - 1/x^3 = 7 \times 52 = 364$

$x^3 - 1/x^3 = 364$

Question 9: If $(x^2 + 1/x^2) = 98$, find the value of $x^3 + 1/x^3$.

Solution:
We know that: $(x + y)^2 = x^2 + y^2 + 2xy$

Replace y with 1/x, we get

$(x + 1/x)^2 = x^2 + 1/x^2 + 2$

Since $(x^2 + 1/x^2) = 98$ (given)

$(x + 1/x)^2 = 98 + 2 = 100$

or $(x + 1/x) = \pm 10$

Now, Find $x^3 + 1/x^3$

We know that, $x^3 + y^3 = (x + y)(x^2 + y^2 - xy)$

Replace y with 1/x, we get

$x^3 + 1/x^3 = (x + 1/x)(x^2 + 1/x^2 - 1)$

Use $(x + 1/x) = 10$ and $(x^2 + 1/x^2) = 98$

$x^3 + 1/x^3 = 10 \times 97 = 970$

$x^3 + 1/x^3 = 970$

Question 10: If $2x + 3y = 13$ and $xy = 6$, find the value of $8x^3 + 27y^3$.
Solution:
Given: $2x + 3y = 13$, $xy = 6$
Cubing $2x + 3y = 13$ both sides, we get
$(2x + 3y)^3 = (13)^3$
$(2x)^3 + (3y)^3 + 3(2x)(3y)(2x + 3y) = 2197$
$8x^3 + 27y^3 + 18xy(2x + 3y) = 2197$
$8x^3 + 27y^3 + 18 \times 6 \times 13 = 2197$
$8x^3 + 27y^3 + 1404 = 2197$
$8x^3 + 27y^3 = 2197 - 1404 = 793$
$8x^3 + 27y^3 = 793$

Question 11: If $3x – 2y = 11$ and $xy = 12$, find the value of $27x^3 – 8y^3$.
Solution:
Given: $3x – 2y = 11$ and $xy = 12$
Cubing $3x – 2y = 11$ both sides, we get
$(3x – 2y)^3 = (11)^3$
$(3x)^3 – (2y)^3 – 3(3x)(2y)(3x – 2y) = 1331$
$27x^3 – 8y^3 – 18xy(3x - 2y) = 1331$
$27x^3 – 8y^3 – 18 \times 12 \times 11 = 1331$
$27x^3 – 8y^3 – 2376 = 1331$
$27x^3 – 8y^3 = 1331 + 2376 = 3707$
$27x^3 – 8y^3 = 3707$