

EXERCISE 7.2
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1. Find the maximum and minimum values of each of the following trigonometrical expressions:

(i) $12 \sin x - 5 \cos x$

(ii) $12 \cos x + 5 \sin x + 4$

(iii) $5 \cos x + 3 \sin (\pi/6 - x) + 4$

(iv) $\sin x - \cos x + 1$

Solution:

We know that the maximum value of $A \cos \alpha + B \sin \alpha + C$ is $C + \sqrt{(A^2 + B^2)}$,
And the minimum value is $C - \sqrt{(A^2 + B^2)}$.

(i) $12 \sin x - 5 \cos x$

Given: $f(x) = 12 \sin x - 5 \cos x$

Here, $A = -5$, $B = 12$ and $C = 0$

$$-\sqrt{((-5)^2 + 12^2)} \leq 12 \sin x - 5 \cos x \leq \sqrt{((-5)^2 + 12^2)}$$

$$-\sqrt{(25+144)} \leq 12 \sin x - 5 \cos x \leq \sqrt{(25+144)}$$

$$-\sqrt{169} \leq 12 \sin x - 5 \cos x \leq \sqrt{169}$$

$$-13 \leq 12 \sin x - 5 \cos x \leq 13$$

Hence, the maximum and minimum values of $f(x)$ are 13 and -13 respectively.

(ii) $12 \cos x + 5 \sin x + 4$

Given: $f(x) = 12 \cos x + 5 \sin x + 4$

Here, $A = 12$, $B = 5$ and $C = 4$

$$4 - \sqrt{(12^2 + 5^2)} \leq 12 \cos x + 5 \sin x + 4 \leq 4 + \sqrt{(12^2 + 5^2)}$$

$$4 - \sqrt{(144+25)} \leq 12 \cos x + 5 \sin x + 4 \leq 4 + \sqrt{(144+25)}$$

$$4 - \sqrt{169} \leq 12 \cos x + 5 \sin x + 4 \leq 4 + \sqrt{169}$$

$$-9 \leq 12 \cos x + 5 \sin x + 4 \leq 17$$

Hence, the maximum and minimum values of $f(x)$ are 17 and -9 respectively.

(iii) $5 \cos x + 3 \sin (\pi/6 - x) + 4$

Given: $f(x) = 5 \cos x + 3 \sin (\pi/6 - x) + 4$

We know that, $\sin (A - B) = \sin A \cos B - \cos A \sin B$

$$f(x) = 5 \cos x + 3 \sin (\pi/6 - x) + 4$$

$$= 5 \cos x + 3 (\sin \pi/6 \cos x - \cos \pi/6 \sin x) + 4$$

$$= 5 \cos x + 3/2 \cos x - 3\sqrt{3}/2 \sin x + 4$$

$$= 13/2 \cos x - 3\sqrt{3}/2 \sin x + 4$$

So, here $A = 13/2$, $B = -3\sqrt{3}/2$, $C = 4$

$$4 - \sqrt{[(13/2)^2 + (-3\sqrt{3}/2)^2]} \leq 13/2 \cos x - 3\sqrt{3}/2 \sin x + 4 \leq 4 + \sqrt{[(13/2)^2 + (-3\sqrt{3}/2)^2]}$$

$$4 - \sqrt{[(169/4) + (27/4)]} \leq 13/2 \cos x - 3\sqrt{3}/2 \sin x + 4 \leq 4 + \sqrt{[(169/4) + (27/4)]}$$

$$4 - 7 \leq 13/2 \cos x - 3\sqrt{3}/2 \sin x + 4 \leq 4 + 7$$

$$-3 \leq 13/2 \cos x - 3\sqrt{3}/2 \sin x + 4 \leq 11$$

Hence, the maximum and minimum values of $f(x)$ are 11 and -3 respectively.

(iv) $\sin x - \cos x + 1$

Given: $f(x) = \sin x - \cos x + 1$

So, here $A = -1$, $B = 1$ And $c = 1$

$$1 - \sqrt{[(-1)^2 + 1^2]} \leq \sin x - \cos x + 1 \leq 1 + \sqrt{[(-1)^2 + 1^2]}$$

$$1 - \sqrt{(1+1)} \leq \sin x - \cos x + 1 \leq 1 + \sqrt{(1+1)}$$

$$1 - \sqrt{2} \leq \sin x - \cos x + 1 \leq 1 + \sqrt{2}$$

Hence, the maximum and minimum values of $f(x)$ are $1 + \sqrt{2}$ and $1 - \sqrt{2}$ respectively.

2. Reduce each of the following expressions to the Sine and Cosine of a single expression:

(i) $\sqrt{3} \sin x - \cos x$

(ii) $\cos x - \sin x$

(iii) $24 \cos x + 7 \sin x$

Solution:

(i) $\sqrt{3} \sin x - \cos x$

Let $f(x) = \sqrt{3} \sin x - \cos x$

Dividing and multiplying by $\sqrt{((\sqrt{3})^2 + 1^2)}$ i.e. by 2

$$f(x) = 2(\sqrt{3}/2 \sin x - 1/2 \cos x)$$

Sine expression:

$$f(x) = 2(\cos \pi/6 \sin x - \sin \pi/6 \cos x) \text{ (since, } \sqrt{3}/2 = \cos \pi/6 \text{ and } 1/2 = \sin \pi/6)$$

We know that, $\sin A \cos B - \cos A \sin B = \sin (A - B)$

$$f(x) = 2 \sin (x - \pi/6)$$

Again,

$$f(x) = 2(\sqrt{3}/2 \sin x - 1/2 \cos x)$$

Cosine expression:

$$f(x) = 2(\sin \pi/3 \sin x - \cos \pi/3 \cos x)$$

We know that, $\cos A \cos B - \sin A \sin B = \cos (A + B)$

$$f(x) = -2 \cos(\pi/3 + x)$$

(ii) $\cos x - \sin x$

Let $f(x) = \cos x - \sin x$

Dividing and multiplying by $\sqrt{(1^2 + 1^2)}$ i.e. by $\sqrt{2}$,

$$f(x) = \sqrt{2}(1/\sqrt{2} \cos x - 1/\sqrt{2} \sin x)$$

Sine expression:

$$f(x) = \sqrt{2}(\sin \pi/4 \cos x - \cos \pi/4 \sin x) \text{ (since, } 1/\sqrt{2} = \sin \pi/4 \text{ and } 1/\sqrt{2} = \cos \pi/4)$$

We know that $\sin A \cos B - \cos A \sin B = \sin (A - B)$

$$f(x) = \sqrt{2} \sin (\pi/4 - x)$$

Again,

$$f(x) = \sqrt{2}(1/\sqrt{2} \cos x - 1/\sqrt{2} \sin x)$$

Cosine expression:

$$f(x) = 2(\cos \pi/4 \cos x - \sin \pi/4 \sin x)$$

We know that $\cos A \cos B - \sin A \sin B = \cos (A + B)$

$$f(x) = \sqrt{2} \cos (\pi/4 + x)$$

(iii) $24 \cos x + 7 \sin x$

$$\text{Let } f(x) = 24 \cos x + 7 \sin x$$

Dividing and multiplying by $\sqrt{((\sqrt{24})^2 + 7^2)} = \sqrt{625}$ i.e. by 25,

$$f(x) = 25(24/25 \cos x + 7/25 \sin x)$$

Sine expression:

$$f(x) = 25(\sin \alpha \cos x + \cos \alpha \sin x) \text{ where, } \sin \alpha = 24/25 \text{ and } \cos \alpha = 7/25$$

We know that $\sin A \cos B + \cos A \sin B = \sin (A + B)$

$$f(x) = 25 \sin (\alpha + x)$$

Cosine expression:

$$f(x) = 25(\cos \alpha \cos x + \sin \alpha \sin x) \text{ where, } \cos \alpha = 24/25 \text{ and } \sin \alpha = 7/25$$

We know that $\cos A \cos B + \sin A \sin B = \cos (A - B)$

$$f(x) = 25 \cos (\alpha - x)$$

3. Show that $\sin 100^\circ - \sin 10^\circ$ is positive.

Solution:

$$\text{Let } f(x) = \sin 100^\circ - \sin 10^\circ$$

Dividing And multiplying by $\sqrt{(1^2 + 1^2)}$ i.e. by $\sqrt{2}$,

$$f(x) = \sqrt{2}(1/\sqrt{2} \sin 100^\circ - 1/\sqrt{2} \sin 10^\circ)$$

$$f(x) = \sqrt{2}(\cos \pi/4 \sin (90+10)^\circ - \sin \pi/4 \sin 10^\circ) \text{ (since, } 1/\sqrt{2} = \cos \pi/4 \text{ and } 1/\sqrt{2} = \sin \pi/4)$$

$$f(x) = \sqrt{2}(\cos \pi/4 \cos 10^\circ - \sin \pi/4 \sin 10^\circ)$$

We know that $\cos A \cos B - \sin A \sin B = \cos (A + B)$

$$f(x) = \sqrt{2} \cos (\pi/4 + 10^\circ)$$

$$\therefore f(x) = \sqrt{2} \cos 55^\circ$$

4. Prove that $(2\sqrt{3} + 3) \sin x + 2\sqrt{3} \cos x$ lies between $-(2\sqrt{3} + \sqrt{15})$ and $(2\sqrt{3} + \sqrt{15})$.

Solution:

Let $f(x) = (2\sqrt{3} + 3) \sin x + 2\sqrt{3} \cos x$

Here, $A = 2\sqrt{3}$, $B = 2\sqrt{3} + 3$ and $C = 0$

$$-\sqrt{(2\sqrt{3})^2 + (2\sqrt{3} + 3)^2} \leq (2\sqrt{3} + 3) \sin x + 2\sqrt{3} \cos x \leq \sqrt{(2\sqrt{3})^2 + (2\sqrt{3} + 3)^2}$$

$$-\sqrt{12+12+9+12\sqrt{3}} \leq (2\sqrt{3} + 3) \sin x + 2\sqrt{3} \cos x \leq \sqrt{12+12+9+12\sqrt{3}}$$

$$-\sqrt{33+12\sqrt{3}} \leq (2\sqrt{3} + 3) \sin x + 2\sqrt{3} \cos x \leq \sqrt{33+12\sqrt{3}}$$

$$-\sqrt{15+12+6+12\sqrt{3}} \leq (2\sqrt{3} + 3) \sin x + 2\sqrt{3} \cos x \leq \sqrt{15+12+6+12\sqrt{3}}$$

We know that $(12\sqrt{3} + 6 < 12\sqrt{5})$ because the value of $\sqrt{5} - \sqrt{3}$ is more than 0.5

So if we replace, $(12\sqrt{3} + 6$ with $12\sqrt{5})$ the above inequality still holds.

So by rearranging the above expression $\sqrt{(15+12+12\sqrt{5})}$ we get, $2\sqrt{3} + \sqrt{15}$

$$-2\sqrt{3} + \sqrt{15} \leq (2\sqrt{3} + 3) \sin x + 2\sqrt{3} \cos x \leq 2\sqrt{3} + \sqrt{15}$$

Hence proved.