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Prove the following identities:

1. $\sqrt{[(1 - \cos 2x) / (1 + \cos 2x)]} = \tan x$

Solution:

Let us consider LHS:

$\sqrt{[(1 - \cos 2x) / (1 + \cos 2x)]}$

We know that $\cos 2x = 1 - 2 \sin^2 x$

$= 2 \cos^2 x - 1$

So,

$\sqrt{[(1 - \cos 2x) / (1 + \cos 2x)]} = \sqrt{[(1 - (1 - 2\sin^2 x)) / (1 + (2\cos^2 x - 1))]}$

$= \sqrt{[(1 - 1 + 2\sin^2 x) / (1 + 2\cos^2 x - 1)]}$

$= \sqrt{[2 \sin^2 x / 2 \cos^2 x]}$

$= \sin x/\cos x$

$= \tan x$

$=$ RHS

Hence proved.

2. $\sin 2x / (1 - \cos 2x) = \cot x$

Solution:

Let us consider LHS:

$\sin 2x / (1 - \cos 2x)$

We know that $\cos 2x = 1 - 2 \sin^2 x$

$\text{Sin } 2x = 2 \sin x \cos x$

So,

$\sin 2x / (1 - \cos 2x) = (2 \sin x \cos x) / (1 - (1 - 2\sin^2 x))$

$= (2 \sin x \cos x) / (1 - 1 + 2\sin^2 x)]$

$= [2 \sin x \cos x / 2 \sin^2 x]$

$= \cos x/\sin x$

$= \cot x$

$=$ RHS

Hence proved.

3. $\sin 2x / (1 + \cos 2x) = \tan x$

Solution:

Let us consider LHS:

$\sin 2x / (1 + \cos 2x)$

We know that $\cos 2x = 1 - 2 \sin^2 x$

$= 2 \cos^2 x - 1$

Sin 2x = 2 sin x cos x

So,

sin 2x / (1 + cos 2x) = [2 sin x cos x / (1 + (2cos^{2}x - 1))]

= [2 sin x cos x / (1 + 2cos^2 x - 1)]

= [2 sin x cos x / 2 cos^2 x]

= sin x/cos x

= tan x

= RHS

Hence proved.

4. $\sqrt{2+\sqrt{2+2\cos 4x}} = 2\cos x, 0 < x < \frac{\pi}{4}$

Solution:

Let us consider LHS:

$\sqrt{2+\sqrt{2+2\cos 4x}} = \sqrt{2+\sqrt{2+2(2\cos^2 2x-1)}}$

{since, cos 2x = 2 cos^2 x – 1 ⇒ cos 4x = 2 cos^2 2x -1}

$= \sqrt{2+\sqrt{2+4\cos^2 2x-2}}$

$= \sqrt{2+\sqrt{4\cos^2 2x}}$

$= \sqrt{2+2\cos 2x}$

$= \sqrt{2+2(2\cos^2 x-1)}$ {since, cos 2x = 2 cos^2 x – 1}

$= \sqrt{2+4\cos^2 x-2}$

$= \sqrt{4\cos^2 x}$

= 2 cos x

= RHS

Hence proved.

5. [1 – cos 2x + sin 2x] / [1 + cos 2x + sin 2x] = tan x

Solution:

Let us consider LHS:

[1 – cos 2x + sin 2x] / [1 + cos 2x + sin 2x]

We know that, cos 2x = 1 – 2 sin^2 x

= 2 cos^2 x – 1

Sin 2x = 2 sin x cos x
So,

$$= \frac{1 - (1 - 2\sin^2 x) + 2\sin x \cos x}{1 + (2\cos^2 x - 1) + 2\sin x \cos x}$$

$$= \frac{1 - 1 + 2\sin^2 x + 2\sin x \cos x}{1 + 2\cos^2 x - 1 + 2\sin x \cos x}$$

$$= \frac{2\sin^2 x + 2\sin x \cos x}{2\cos^2 x + 2\sin x \cos x}$$

$$= \frac{2\sin x(\sin x + \cos x)}{2\cos x(\cos x + \sin x)}$$

$$= \frac{\sin x}{\cos x}$$

= tan x
= RHS
Hence proved.

6. [sin x + sin 2x] / [1 + cos x + cos 2x] = tan x
Solution:
Let us consider LHS:
[sin x + sin 2x] / [1 + cos x + cos 2x]
We know that, cos 2x = $\cos^2 x - \sin^2 x$
Sin 2x = 2 sin x cos x
So,

$$\frac{\sin x + \sin 2x}{1 + \cos x + \cos 2x} = \frac{\sin x + 2\sin x \cos x}{1 + \cos x + (2\cos^2 x - 1)}$$

$$= \frac{\sin x + 2\sin x \cos x}{1 + \cos x + 2\cos^2 x - 1}$$

$$= \frac{\sin x + 2\sin x \cos x}{\cos x + 2\cos^2 x}$$

$$= \frac{\sin x(1 + 2\cos x)}{\cos x(1 + 2\cos x)}$$

$$= \frac{\sin x}{\cos x}$$

= tan x

= RHS

Hence proved.

7. cos 2x / (1 + sin 2x) = tan (π/4 - x)

Solution:

Let us consider LHS:

cos 2x / (1 + sin 2x)

We know that, $\cos 2x = \cos^2 x - \sin^2 x$

$\sin 2x = 2 \sin x \cos x$

So,

$$\frac{\cos 2x}{1+\sin 2x} = \frac{\cos^2 x - \sin^2 x}{1 + 2\sin x \cos x}$$

$$= \frac{(\cos x - \sin x)(\cos x + \sin x)}{\sin^2 x + \cos^2 x + 2\sin x \cos x}$$

(since, $a^2 - b^2 = (a - b)(a + b)$ & $\sin^2 x + \cos^2 x = 1$)

$$= \frac{(\cos x - \sin x)(\cos x + \sin x)}{(\sin x + \cos x)^2}$$

(since, $a^2 + b^2 + 2ab = (a + b)^2$)

$$= \frac{(\cos x - \sin x)(\cos x + \sin x)}{(\sin x + \cos x)(\sin x + \cos x)}$$

$$= \frac{(\cos x - \sin x)}{(\sin x + \cos x)}$$

Multiplying numerator and denominator by 1/√2

We get,

$$= \frac{\frac{1}{\sqrt{2}}(\cos x - \sin x)}{\frac{1}{\sqrt{2}}(\sin x + \cos x)}$$

$$= \frac{\left(\frac{1}{\sqrt{2}}\cos x - \frac{1}{\sqrt{2}}\sin x\right)}{\left(\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x\right)}$$

$$= \frac{\left(\sin\frac{\pi}{4}\cos x - \cos\frac{\pi}{4}\sin x\right)}{\left(\sin\frac{\pi}{4}\sin x + \cos\frac{\pi}{4}\cos x\right)}$$ (since, $1/\sqrt{2} = \sin \pi/4$)

$$= \frac{\sin\left(\frac{\pi}{4} - x\right)}{\cos\left(\frac{\pi}{4} - x\right)}$$

By using the formulas,

sin (A – B) = sin A cos B – sin B cos A

cos (A – B) = cos A cos B + sin A sin B

= tan (π/4 - x)

= RHS

Hence proved.

8. cos x / (1 - sin x) = tan (π/4 + x/2)

Solution:

Let us consider LHS:

cos x / (1 - sin x)

We know that, cos 2x = $\cos^2$ x – $\sin^2$ x

Cos x = $\cos^2$ x/2 – $\sin^2$ x/2

Sin 2x = 2 sin x cos x

Sin x = 2 sin x/2 cos x/2

So,

$$\frac{\cos x}{1 - \sin x} = \frac{\cos^2\frac{x}{2} - \sin^2\frac{x}{2}}{1 - 2\sin\frac{x}{2}\cos\frac{x}{2}}$$

$$= \frac{\left(\cos\frac{x}{2} - \sin\frac{x}{2}\right)\left(\cos\frac{x}{2} + \sin\frac{x}{2}\right)}{\sin^2\frac{x}{2} + \cos^2\frac{x}{2} + 2\sin\frac{x}{2}\cos\frac{x}{2}}$$

(By using the formula, $a^2 - b^2 = (a - b)(a + b)$ & $\sin^2 x + \cos^2 x = 1$)

$$= \frac{\left(\cos\frac{x}{2} - \sin\frac{x}{2}\right)\left(\cos\frac{x}{2} + \sin\frac{x}{2}\right)}{\left(\sin\frac{x}{2} + \cos\frac{x}{2}\right)^2}$$

(By using the formula, $a^2 + b^2 + 2ab = (a + b)^2$)

$$= \frac{\left(\cos\frac{x}{2} - \sin\frac{x}{2}\right)\left(\cos\frac{x}{2} + \sin\frac{x}{2}\right)}{\left(\sin\frac{x}{2} + \cos\frac{x}{2}\right)\left(\sin\frac{x}{2} + \cos\frac{x}{2}\right)}$$

$$= \frac{\left(\cos\frac{x}{2} - \sin\frac{x}{2}\right)}{\left(\sin\frac{x}{2} + \cos\frac{x}{2}\right)}$$

$$= \frac{\left(\cos\frac{x}{2} + \sin\frac{x}{2}\right)}{\left(\sin\frac{x}{2} - \cos\frac{x}{2}\right)}$$

Let us multiply numerator and denominator by 1/√2

We get,

$$= \frac{\frac{1}{\sqrt{2}}\left(\cos\frac{x}{2} + \sin\frac{x}{2}\right)}{\frac{1}{\sqrt{2}}\left(\sin\frac{x}{2} - \cos\frac{x}{2}\right)}$$

$$= \frac{\left(\frac{1}{\sqrt{2}}\cos\frac{x}{2} + \frac{1}{\sqrt{2}}\sin\frac{x}{2}\right)}{\left(\frac{1}{\sqrt{2}}\sin\frac{x}{2} - \frac{1}{\sqrt{2}}\cos\frac{x}{2}\right)}$$

$$= \frac{\left(\sin\frac{\pi}{4}\cos\frac{x}{2} + \cos\frac{\pi}{4}\sin\frac{x}{2}\right)}{\left(\sin\frac{\pi}{4}\sin\frac{x}{2} - \cos\frac{\pi}{4}\cos\frac{x}{2}\right)}$$ (since, 1/√2 = sin π/4)

$$= \frac{\sin\left(\frac{\pi}{4} - x\right)}{\cos\left(\frac{\pi}{4} - x\right)}$$

= tan (π/4 - x)

= RHS

Hence proved.

9. $\cos^2\frac{\pi}{8} + \cos^2\frac{3\pi}{8} + \cos^2\frac{5\pi}{8} + \cos^2\frac{7\pi}{8} = 2$

Solution:

Let us consider LHS:

$$\cos^2\frac{\pi}{8} + \cos^2\frac{3\pi}{8} + \cos^2\frac{5\pi}{8} + \cos^2\frac{7\pi}{8}$$

We know that cos 2x = $2\cos^2 x$ - 1

Cos 2x + 1 = $2\cos^2 x$

$\text{Cos}^2 x$ = (Cos 2x + 1)/2

So,

$$\cos^2\frac{\pi}{8} + \cos^2\frac{3\pi}{8} + \cos^2\frac{5\pi}{8} + \cos^2\frac{7\pi}{8}$$

$$= \frac{1+\cos\frac{2\pi}{8}}{2} + \frac{1+\cos\frac{6\pi}{8}}{2} + \frac{1+\cos\frac{10\pi}{8}}{2} + \frac{1+\cos\frac{14\pi}{8}}{2}$$

$$= \frac{1+\cos\frac{2\pi}{8}}{2} + \frac{1+\cos\left(\pi-\frac{2\pi}{8}\right)}{2} + \frac{1+\cos\left(\pi+\frac{2\pi}{8}\right)}{2} + \frac{1+\cos\left(2\pi-\frac{2\pi}{8}\right)}{2}$$

$$\left\{\because \pi - \frac{2\pi}{8} = \frac{6\pi}{8}; \pi + \frac{2\pi}{8} = \frac{10\pi}{8}; 2\pi - \frac{2\pi}{8} = \frac{14\pi}{8}\right\}$$

$$= \frac{1+\cos\frac{2\pi}{8}}{2} + \frac{1-\cos\frac{2\pi}{8}}{2} + \frac{1-\cos\frac{2\pi}{8}}{2} + \frac{1+\cos\frac{2\pi}{8}}{2}$$

{we know, cos (π – A) = - cos A, cos (π + A) = - cos A & cos (2π – A) = cos A}

$$= 2 \times \frac{1+\cos\frac{2\pi}{8}}{2} + 2 \times \frac{1-\cos\frac{2\pi}{8}}{2}$$

$$= 1 + \cos\frac{2\pi}{8} + 1 - \cos\frac{2\pi}{8}$$

= 2

= RHS

Hence proved.

10. $\sin^2\frac{\pi}{8} + \sin^2\frac{3\pi}{8} + \sin^2\frac{5\pi}{8} + \sin^2\frac{7\pi}{8} = 2$

Solution:

Let us consider LHS:

$$\sin^2\frac{\pi}{8} + \sin^2\frac{3\pi}{8} + \sin^2\frac{5\pi}{8} + \sin^2\frac{7\pi}{8}$$

We know that, cos 2x = 1 – 2sin^2 x

2sin^2 x = 1- cos 2x

Sin^2 x = (1 – cos 2x)/2

So,

$$= \frac{1-\cos\frac{2\pi}{8}}{2} + \frac{1-\cos\frac{6\pi}{8}}{2} + \frac{1-\cos\frac{10\pi}{8}}{2} + \frac{1-\cos\frac{14\pi}{8}}{2}$$

$$= \frac{1-\cos\frac{2\pi}{8}}{2} + \frac{1-\cos\left(\pi-\frac{2\pi}{8}\right)}{2} + \frac{1-\cos\left(\pi+\frac{2\pi}{8}\right)}{2} + \frac{1-\cos\left(2\pi-\frac{2\pi}{8}\right)}{2}$$

$$\left\{\because \pi - \frac{2\pi}{8} = \frac{6\pi}{8}; \pi + \frac{2\pi}{8} = \frac{10\pi}{8}; 2\pi - \frac{2\pi}{8} = \frac{14\pi}{8}\right\}$$

$$= \frac{1 - \cos\frac{2\pi}{8}}{2} + \frac{1 - \left(-\cos\frac{2\pi}{8}\right)}{2} + \frac{1 - \left(-\cos\frac{2\pi}{8}\right)}{2} + \frac{1 - \cos\frac{2\pi}{8}}{2}$$

{we know, cos (π – A) = - cos A, cos (π + A) = - cos A & cos (2π – A) = cos A}

$$= \frac{1 - \cos\frac{2\pi}{8}}{2} + \frac{1 + \cos\frac{2\pi}{8}}{2} + \frac{1 + \cos\frac{2\pi}{8}}{2} + \frac{1 - \cos\frac{2\pi}{8}}{2}$$

$$= 2 \times \frac{1 - \cos\frac{2\pi}{8}}{2} + 2 \times \frac{1 + \cos\frac{2\pi}{8}}{2}$$

$$= 1 - \cos\frac{2\pi}{8} + 1 + \cos\frac{2\pi}{8}$$

= 2

= RHS

Hence proved.

11. $(\cos \alpha + \cos \beta)^2 + (\sin \alpha + \sin \beta)^2 = 4 \cos^2 (\alpha - \beta)/2$

Solution:

Let us consider LHS:

$(\cos \alpha + \cos \beta)^2 + (\sin \alpha + \sin \beta)^2$

Upon expansion, we get,

$(\cos \alpha + \cos \beta)^2 + (\sin \alpha + \sin \beta)^2 =$

$= \cos^2 \alpha + \cos^2 \beta + 2 \cos \alpha \cos \beta + \sin^2 \alpha + \sin^2 \beta + 2 \sin \alpha \sin \beta$

$= 2 + 2 \cos \alpha \cos \beta + 2 \sin \alpha \sin \beta$

$= 2 (1 + \cos \alpha \cos \beta + \sin \alpha \sin \beta)$

$= 2 (1 + \cos (\alpha - \beta))$ [since, $\cos (A – B) = \cos A \cos B + \sin A \sin B$]

$= 2 (1 + 2 \cos^2 (\alpha - \beta)/2 - 1)$ [since, $\cos 2x = 2\cos^2 x – 1$]

$= 2 (2 \cos^2 (\alpha - \beta)/2)$

$= 4 \cos^2 (\alpha - \beta)/2$

= RHS

Hence Proved.

12. $\sin^2 (\pi/8 + x/2) – \sin^2 (\pi/8 – x/2) = 1/\sqrt{2} \sin x$

Solution:

Let us consider LHS:

$\sin^2 (\pi/8 + x/2) – \sin^2 (\pi/8 – x/2)$

we know, $\sin^2 A - \sin^2 B = \sin (A+B) \sin (A-B)$

so,

$\sin^2 (\pi/8 + x/2) - \sin^2 (\pi/8 - x/2) = \sin (\pi/8 + x/2 + \pi/8 - x/2) \sin (\pi/8 + x/2 - (\pi/8 - x/2))$

$= \sin (\pi/8 + \pi/8) \sin (\pi/8 + x/2 - \pi/8 + x/2)$

$= \sin \pi/4 \sin x$

$= 1/\sqrt{2} \sin x$ [since, since $\pi/4 = 1/\sqrt{2}$]

= RHS

Hence proved.

13. $1 + \cos^2 2x = 2 (\cos^4 x + \sin^4 x)$

Solution:

Let us consider LHS:

$1 + \cos^2 2x$

We know, $\cos 2x = \cos^2 x - \sin^2 x$

$\cos^2 x + \sin^2 x = 1$

so,

$1 + \cos^2 2x = (\cos^2 x + \sin^2 x)^2 + (\cos^2 x - \sin^2 x)^2$

$= (\cos^4 x + \sin^4 x + 2 \cos^2 x \sin^2 x) + (\cos^4 x + \sin^4 x - 2 \cos^2 x \sin^2 x)$

$= \cos^4 x + \sin^4 x + \cos^4 x + \sin^4 x$

$= 2 \cos^4 x + 2 \sin^4 x$

$= 2 (\cos^4 x + \sin^4 x)$

= RHS

Hence proved.

14. $\cos^3 2x + 3 \cos 2x = 4 (\cos^6 x - \sin^6 x)$

Solution:

Let us consider RHS:

$4 (\cos^6 x - \sin^6 x)$

Upon expansion we get,

$4 (\cos^6 x - \sin^6 x) = 4 [(\cos^2 x)^3 - (\sin^2 x)^3]$

$= 4 (\cos^2 x - \sin^2 x) (\cos^4 x + \sin^4 x + \cos^2 x \sin^2 x)$

By using the formula,

$a^3 - b^3 = (a-b) (a^2 + b^2 + ab)$

$= 4 \cos 2x (\cos^4 x + \sin^4 x + \cos^2 x \sin^2 x + \cos^2 x \sin^2 x - \cos^2 x \sin)$

We know, $\cos 2x = \cos^2 x - \sin^2 x$

So,

$= 4 \cos 2x (\cos^4 x + \sin^4 x + 2 \cos^2 x \sin^2 x - \cos^2 x \sin^2 x)$

$= 4 \cos 2x [(\cos^2 x)^2 + (\sin^2 x)^2 + 2 \cos^2 x \sin^2 x - \cos^2 x \sin^2 x]$ We know, $a^2 + b^2 + 2ab = (a + b)^2$

$= 4 \cos 2x [(1)^2 – 1/4 (4 \cos^2 x \sin^2 x)]$
$= 4 \cos 2x [(1)^2 – 1/4 (2 \cos x \sin x)^2]$

We know, $\sin 2x = 2\sin x \cos x$

$= 4 \cos 2x [(1^2) – 1/4 (\sin 2x)^2]$
$= 4 \cos 2x (1 – 1/4 \sin^2 2x)$

We know, $\sin^2 x = 1 – \cos^2 x$

$= 4 \cos 2x [1 – 1/4 (1 – \cos^2 2x)]$
$= 4 \cos 2x [1 – 1/4 + 1/4 \cos^2 2x]$
$= 4 \cos 2x [3/4 + 1/4 \cos^2 2x]$
$= 4 (3/4 \cos 2x + 1/4 \cos^3 2x)$
$= 3 \cos 2x + \cos^3 2x$
$= \cos^3 2x + 3 \cos 2x$
= LHS

Hence proved.

15. (sin 3x + sin x) sin x + (cos 3x – cos x) cos x = 0

Solution:

Let us consider LHS:

$(\sin 3x + \sin x) \sin x + (\cos 3x – \cos x) \cos x$

$= (\sin 3x) (\sin x) + \sin^2 x + (\cos 3x) (\cos x) – \cos^2 x$
$= [(\sin 3x) (\sin x) + (\cos 3x) (\cos x)] + (\sin^2 x – \cos^2 x)$
$= [(\sin 3x) (\sin x) + (\cos 3x) (\cos x)] – (\cos^2 x – \sin^2 x)$
$= \cos (3x – x) – \cos 2x$

We know, $\cos 2x = \cos^2 x – \sin^2 x$

$\cos A \cos B + \sin A \sin B = \cos(A – B)$

So,

$= \cos 2x – \cos 2x$
$= 0$
= RHS

Hence Proved.

16. $\cos^2 (\pi/4 - x) – \sin^2 (\pi/4 - x) = \sin 2x$

Solution:

Let us consider LHS:

$\cos^2 (\pi/4 - x) – \sin^2 (\pi/4 - x)$

We know, $\cos^2 A – \sin^2 A = \cos 2A$

So,

$\cos^2 (\pi/4 - x) – \sin^2 (\pi/4 - x) = \cos 2 (\pi/4 - x)$
$= \cos (\pi/2 - 2x)$

= $\sin 2x$ [since, $\cos(\pi/2 - A) = \sin A$]
= RHS

Hence proved.

17. $\cos 4x = 1 - 8\cos^2 x + 8\cos^4 x$

Solution:

Let us consider LHS:

$\cos 4x$

We know, $\cos 2x = 2\cos^2 x - 1$

So,

$\cos 4x = 2\cos^2 2x - 1$

$= 2(2\cos^2 x - 1)^2 - 1$

$= 2[(2\cos^2 x)^2 + 1^2 - 2\times 2\cos^2 x] - 1$

$= 2(4\cos^4 x + 1 - 4\cos^2 x) - 1$

$= 8\cos^4 x + 2 - 8\cos^2 x - 1$

$= 8\cos^4 x + 1 - 8\cos^2 x$

= RHS

Hence Proved.

18. $\sin 4x = 4\sin x\cos^3 x - 4\cos x\sin^3 x$

Solution:

Let us consider LHS:

$\sin 4x$

We know, $\sin 2x = 2\sin x\cos x$

$\cos 2x = \cos^2 x - \sin^2 x$

So,

$\sin 4x = 2\sin 2x\cos 2x$

$= 2(2\sin x\cos x)(\cos^2 x - \sin^2 x)$

$= 4\sin x\cos x(\cos^2 x - \sin^2 x)$

$= 4\sin x\cos^3 x - 4\sin^3 x\cos x$

= RHS

Hence proved.

19. $3(\sin x - \cos x)^4 + 6(\sin x + \cos x)^2 + 4(\sin^6 x + \cos^6 x) = 13$

Solution:

Let us consider LHS:

$3(\sin x - \cos x)^4 + 6(\sin x + \cos x)^2 + 4(\sin^6 x + \cos^6 x)$

We know, $(a + b)^2 = a^2 + b^2 + 2ab$

$(a - b)^2 = a^2 + b^2 - 2ab$

$a^3 + b^3 = (a + b)(a^2 + b^2 - ab)$

So,

$3(\sin x - \cos x)^4 + 6(\sin x + \cos x)^2 + 4(\sin^6 x + \cos^6 x) = 3\{(\sin x - \cos x)^2\}^2 + 6\{(\sin x)^2 + (\cos x)^2 + 2\sin x\cos x)\} + 4\{(\sin^2 x)^3 + (\cos^2 x)^3\}$

$= 3\{(\sin x)^2 + (\cos x)^2 - 2\sin x\cos x)\}^2 + 6(\sin^2 x + \cos^2 x + 2\sin x\cos x) + 4\{(\sin^2 x + \cos^2 x)(\sin^4 x + \cos^4 x - \sin^2 x\cos^2 x)\}$

$= 3(1 - 2\sin x\cos x)^2 + 6(1 + 2\sin x\cos x) + 4\{(1)(\sin^4 x + \cos^4 x - \sin^2 x\cos^2 x)\}$

We know, $\sin^2 x + \cos^2 x = 1$

So,

$= 3\{1^2 + (2\sin x\cos x)^2 - 4\sin x\cos x\} + 6(1 + 2\sin x\cos x) + 4\{(\sin^2 x)^2 + (\cos^2 x)^2 + 2\sin^2 x\cos^2 x - 3\sin^2 x\cos^2 x)\}$

$= 3\{1 + 4\sin^2 x\cos^2 x - 4\sin x\cos x\} + 6(1 + 2\sin x\cos x) + 4\{(\sin^2 x + \cos^2 x)^2 - 3\sin^2 x\cos^2 x)\}$

$= 3 + 12\sin^2 x\cos^2 x - 12\sin x\cos x + 6 + 12\sin x\cos x + 4\{(1)^2 - 3\sin^2 x\cos^2 x)\}$

$= 9 + 12\sin^2 x\cos^2 x + 4(1 - 3\sin^2 x\cos^2 x)$

$= 9 + 12\sin^2 x\cos^2 x + 4 - 12\sin^2 x\cos^2 x$

$= 13$

$=$ RHS

Hence proved.

20. $2(\sin^6 x + \cos^6 x) - 3(\sin^4 x + \cos^4 x) + 1 = 0$

Solution:

Let us consider LHS:

$2(\sin^6 x + \cos^6 x) - 3(\sin^4 x + \cos^4 x) + 1$

We know, $(a + b)^2 = a^2 + b^2 + 2ab$

$a^3 + b^3 = (a + b)(a^2 + b^2 - ab)$

So,

$2(\sin^6 x + \cos^6 x) - 3(\sin^4 x + \cos^4 x) + 1 = 2\{(\sin^2 x)^3 + (\cos^2 x)^3\} - 3\{(\sin^2 x)^2 + (\cos^2 x)^2\} + 1$

$= 2\{(\sin^2 x + \cos^2 x)(\sin^4 x + \cos^4 x - \sin^2 x\cos^2 x\} - 3\{(\sin^2 x)^2 + (\cos^2 x)^2 + 2\sin^2 x\cos^2 x - 2\sin^2 x\cos^2 x\} + 1$

$= 2\{(1)(\sin^4 x + \cos^4 x + 2\sin^2 x\cos^2 x - 3\sin^2 x\cos^2 x\} - 3\{(\sin^2 x + \cos^2 x)^2 - 2\sin^2 x\cos^2 x\} + 1$

We know, $\sin^2 x + \cos^2 x = 1$

$= 2\{(\sin^2 x + \cos^2 x)^2 - 3 \sin^2 x \cos^2 x\} - 3\{(1)^2 - 2\sin^2 x \cos^2 x\} + 1$

$= 2\{(1)^2 - 3 \sin^2 x \cos^2 x\} - 3(1 - 2\sin^2 x \cos^2 x) + 1$

$= 2(1 - 3 \sin^2 x \cos^2 x) - 3 + 6 \sin^2 x \cos^2 x + 1$

$= 2 - 6 \sin^2 x \cos^2 x - 2 + 6 \sin^2 x \cos^2 x$

$= 0$

= RHS

Hence proved.

21. $\cos^6 x - \sin^6 x = \cos 2x (1 - 1/4 \sin^2 2x)$

Solution:

Let us consider LHS:

$\cos^6 x - \sin^6 x$

We know, $(a + b)^2 = a^2 + b^2 + 2ab$

$a^3 - b^3 = (a - b) (a^2 + b^2 + ab)$

So,

$\cos^6 x - \sin^6 x = (\cos^2 x)^3 - (\sin^2 x)^3$

$= (\cos^2 x - \sin^2 x) (\cos^4 x + \sin^4 x + \cos^2 x \sin^2 x)$

We know, $\cos 2x = \cos^2 x - \sin^2 x$

So,

$= \cos 2x [(\cos^2 x)^2 + (\sin^2 x)^2 + 2 \cos^2 x \sin^2 x - \cos^2 x \sin^2 x]$

$= \cos 2x [(\cos^2 x)^2 + (\sin^2 x)^2 - 1/4 \times 4 \cos^2 x \sin^2 x]$

We know, $\sin^2 x + \cos^2 x = 1$

So,

$= \cos 2x [(1)^2 - 1/4 \times (2 \cos x \sin x)^2]$

We know, $\sin 2x = 2 \sin x \cos x$

So,

$= \cos 2x [1 - 1/4 \times (\sin 2x)^2]$

$= \cos 2x [1 - 1/4 \times \sin^2 2x]$

= RHS

Hence proved.

22. $\tan (\pi/4 + x) + \tan (\pi/4 - x) = 2 \sec 2x$

Solution:

Let us consider LHS:

$\tan (\pi/4 + x) + \tan (\pi/4 - x)$

We know,

tan (A+B) = (tan A + tan B)/(1- tan A tan B)
tan (A-B) = (tan A - tan B)/(1+ tan A tan B)
So,

$$tan(\frac{\pi}{4}+x)+tan(\frac{\pi}{4}-x)=\frac{tan\frac{\pi}{4}+tanx}{1-tan\frac{\pi}{4}tanx}+\frac{tan\frac{\pi}{4}-tanx}{1+tan\frac{\pi}{4}tanx}$$

We know, tan π/4 = 1
So,

$$=\frac{1+\tan x}{1-\tan x}+\frac{1-\tan x}{1+\tan x}$$

$$=\frac{(1+\tan x)^2+(1-\tan x)^2}{(1-\tan x)(1+\tan x)}$$

We know, $(a-b)(a+b)=a^2-b^2$;
$(a+b)^2=a^2+b^2+2ab$ &
$(a-b)^2=a^2+b^2-2ab$
So,

$$=\frac{1^2+\tan^2 x+2\tan x+1^2+\tan^2 x-2\tan x}{1^2-\tan^2 x}$$

$$=\frac{1+\tan^2 x+1+\tan^2 x}{1-\tan^2 x}$$

$$=\frac{2(1+\tan^2 x)}{1-\tan^2 x}$$

We know, tan x = sin x/cos x
So,

$$=\frac{2\left(1+\left(\frac{\sin x}{\cos x}\right)^2\right)}{1-\left(\frac{\sin x}{\cos x}\right)^2}$$

$$=\frac{2\left(1+\frac{\sin^2 x}{\cos^2 x}\right)}{1-\frac{\sin^2 x}{\cos^2 x}}$$

$$=\frac{2\left(\frac{\cos^2 x+\sin^2 x}{\cos^2 x}\right)}{\frac{\cos^2 x-\sin^2 x}{\cos^2 x}}$$

We know, $\cos^2 x + \sin^2 x = 1$ & $\cos 2x = \cos^2 x - \sin^2 x$

So,

$$= \frac{2\left(\frac{1}{\cos^2 x}\right)}{\frac{\cos 2x}{\cos^2 x}}$$

$$= \frac{2}{\cos 2x}$$

$= 2 \sec 2x$ (since, $1/\cos 2x = \sec 2x$)

= RHS

Hence proved.