EXERCISE 12.1 PAGE NO: 12.17

1. Find the second order derivatives of the each of the following functions:
(i) x^3 + tan x

Solution:
Given, $y = x^3 + \tan x$

We have to find $\frac{d^2y}{dx^2}$

As $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$

So let's first find $\frac{dy}{dx}$ and differentiate it again.

$\therefore \frac{dy}{dx} = \frac{d}{dx}(x^3 + \tan x) = \frac{d}{dx}(x^3) + \frac{d}{dx}(\tan x)$

$= 3x^2 + \sec^2 x$

$\therefore \frac{dy}{dx} = 3x^2 + \sec^2 x$

Differentiating again with respect to x

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\,(3x^2 + \sec^2 x) = \frac{d}{dx}(3x^2) + \frac{d}{dx}(\sec^2 x)$$

$$\frac{d^2y}{dx^2} = 6x + 2\sec x \sec x \tan x$$

$$\frac{d^2y}{dx^2} = 6x + 2\sec^2 x \tan x$$

(ii) Sin (log x)

Solution:
Let, $y = \sin(\log x)$

We have to find $\frac{d^2y}{dx^2}$

We know that $\frac{d^2y}{dx^2} = \frac{d}{dx}(\frac{dy}{dx})$

So let's first find dy/dx and differentiate it again.

$\therefore \frac{dy}{dx} = \frac{d}{dx}(\sin(\log x))$

Differentiating $\sin(\log x)$ using the chain rule,

Let, t = log x and y = sin t

$\because \frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$ [using chain rule]

$$\frac{dy}{dx} = \cos t \times \frac{1}{x}$$

$$\frac{dy}{dx} = \cos(\log x) \times \frac{1}{x}$$

Differentiating again with respect to x

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\cos(\log x) \times \frac{1}{x}\right)$$

$$\frac{d^2y}{dx^2} = \cos(\log x) \times \frac{-1}{x^2} + \frac{1}{x} \times \frac{1}{x}(-\sin(\log x))$$

Now by using product rule for differentiation we get,

$$= \frac{-1}{x^2}\cos(\log x) - \frac{1}{x^2}\sin(\log x)$$

$$\frac{d^2y}{dx^2} = \frac{-1}{x^2}\cos(\log x) - \frac{1}{x^2}\sin(\log x)$$

(iii) Log (sin x)

Solution:

Let, y = log (sin x)

We have to find $\frac{d^2y}{dx^2}$

We know $\frac{d^2y}{dx^2} = \frac{d}{dx}(\frac{dy}{dx})$

So let's first find dy/dx and differentiate it again.

$\therefore \frac{dy}{dx} = \frac{d}{dx}(\log(\sin x))$

Differentiating sin (log x) using chain rule,

Let, t = sin x and y = log t

$\because \frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$ [using chain rule]

$$\frac{dy}{dx} = \cos x \times \frac{1}{t}$$

$[\because \frac{d}{dx}\log x = \frac{1}{x}$ & $\frac{d}{dx}(\sin x) = \cos x]$

$$\frac{dy}{dx} = \frac{\cos x}{\sin x} = \cot x$$

Differentiating again with respect to x:

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(\cot x)$$

$\frac{d^2y}{dx^2} = -\operatorname{cosec}^2 x$ $[\because \frac{d}{dx}\cot x = -\operatorname{cosec}^2 x]$

$$\frac{d^2y}{dx^2} = -\operatorname{cosec}^2 x$$

(iv) e^x sin 5x

Solution:

Let, y = e^x sin 5x

Now we have to find $\frac{d^2y}{dx^2}$

We know, $\frac{d^2y}{dx^2} = \frac{d}{dx}(\frac{dy}{dx})$

So let's first find dy/dx and differentiate it again.

$$\therefore \frac{dy}{dx} = \frac{d}{dx}(e^x \sin 5x)$$

Let u = e^x and v = sin 5x

As, y = uv

Now by using product rule of differentiation:

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\therefore \frac{dy}{dx} = e^x \frac{d}{dx}(\sin 5x) + \sin 5x \frac{d}{dx} e^x$$

$$\frac{dy}{dx} = 5e^x \cos 5x + e^x \sin 5x$$

$$[\because \frac{d}{dx}(\sin ax) = a\cos ax, \text{where a is any constant } \& \frac{d}{dx}e^x = e^x]$$

Again differentiating with respect to x:

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(5e^x \cos 5x + e^x \sin 5x)$$

$$= \frac{d}{dx}(5e^x \cos 5x) + \frac{d}{dx}(e^x \sin 5x)$$

Again using the product rule

$$\frac{d^2y}{dx^2} = e^x \frac{d}{dx}(\sin 5x) + \sin 5x \frac{d}{dx} e^x + 5e^x \frac{d}{dx}(\cos 5x) + \cos 5x \frac{d}{dx}(5e^x)$$

$$\frac{d^2y}{dx^2} = 5e^x \cos 5x - 25e^x \sin 5x + e^x \sin 5x + 5e^x \cos 5x$$

$$\frac{d^2y}{dx^2} = 10e^x \cos 5x - 24e^x \sin 5x$$

(v) e^{6x} cos 3x

Solution:

Let, y = e^{6x} cos 3x

We have to find $\frac{d^2y}{dx^2}$

We know, $\frac{d^2y}{dx^2} = \frac{d}{dx}(\frac{dy}{dx})$

So let's first find dy/dx and differentiate it again.

$\therefore \frac{dy}{dx} = \frac{d}{dx}(e^{6x} \cos 3x)$

Let $u = e^{6x}$ and $v = \cos 3x$

We have, $y = uv$

Now by using product rule of differentiation

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$\therefore \frac{dy}{dx} = e^{6x}\frac{d}{dx}(\cos 3x) + \cos 3x\frac{d}{dx}e^{6x}$

$\frac{dy}{dx} = -3e^{6x}\sin 3x + 6e^{6x}\cos 3x$

$[\because \frac{d}{dx}(\cos ax) = -a\sin ax, a \text{ is any constant } \& \frac{d}{dx}e^{ax} = ae^{x}]$

Again differentiating with respect to x

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(-3e^{6x}\sin 3x + 6e^{6x}\cos 3x)$$

$$= \frac{d}{dx}(-3e^{6x}\sin 3x) + \frac{d}{dx}(6e^{6x}\cos 3x)$$

Again using the product rule

$$\frac{d^2y}{dx^2} = -3e^{6x}\frac{d}{dx}(\sin 3x) - 3\sin 3x\frac{d}{dx}e^{6x} + 6e^{6x}\frac{d}{dx}(\cos 3x) + \cos 3x\frac{d}{dx}(6e^{6x})$$

$$\frac{d^2y}{dx^2} = -9e^{6x}\cos 3x - 18e^{6x}\sin 3x - 18e^{6x}\sin 3x + 36e^{6x}\cos 3x$$

$$\frac{d^2y}{dx^2} = 27e^{6x}\cos 3x - 36e^{6x}\sin 3x$$

(vi) x^3 log x

Solution:

Let, y = x^3 log x

We have to find $\frac{d^2y}{dx^2}$

We know $\frac{d^2y}{dx^2} = \frac{d}{dx}(\frac{dy}{dx})$

So let's first find dy/dx and differentiate it again.

$\therefore \frac{dy}{dx} = \frac{d}{dx}(x^3 \log x)$

Let u = x^3 and v = log x

We have, y = uv

Now by using product rule of differentiation

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$\therefore \frac{dy}{dx} = x^3\frac{d}{dx}(\log x) + \log x\frac{d}{dx}x^3$

$$\frac{dy}{dx} = 3x^2 \log x + \frac{x^3}{x}$$

$[\because \frac{d}{dx}(\log x) = \frac{1}{x} \text{ and} \frac{d}{dx}(x^n) = nx^{n-1}]$

Again differentiating with respect to x

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(3x^2 \log x + x^2)$$

$$= \frac{d}{dx}\ (3x^2 \log x\ ) + \frac{d}{dx}\ (x^2)$$

Again using the product rule

$$\frac{d^2y}{dx^2} = 3\log x\frac{d}{dx}x^2 + 3x^2\frac{d}{dx}\log x + \frac{d}{dx}x^2$$

We know $\frac{d}{dx}(\log x) = \frac{1}{x}$ and $\frac{d}{dx}(x^n) = nx^{n-1}$

$$\frac{d^2y}{dx^2} = 6x\log x + \frac{3x^2}{x} + 2x$$

$$\frac{d^2y}{dx^2} = 6x\log x + 5x$$

(vii) $\tan^{-1}x$

Solution:

Let, $y = \tan^{-1} x$

We have to find $\frac{d^2y}{dx^2}$

As $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$

So let's first find dy/dx and differentiate it again.

$$\therefore \frac{dy}{dx} = \frac{d}{dx}(\tan^{-1}x)$$

$$\therefore \frac{dy}{dx} = \frac{1}{1+x^2}$$

Differentiating again with respect to x

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{1}{1+x^2}\right)$$

Differentiating $\frac{1}{1+x^2}$ using chain rule,

Let $t = 1 + x^2$ and $z = 1/t$

$$\because \frac{dz}{dx} = \frac{dz}{dt} \times \frac{dt}{dx}$$

$\therefore \frac{dz}{dx} = \frac{-1}{t^2} \times 2x = -\frac{2x}{1+x^2}$ $[\because \frac{d}{dx}(x^n) = nx^{n-1}]$

$\therefore \frac{d^2y}{dx^2} = -\frac{2x}{(1+x^2)^2}$

(viii) x cos x

Solution:

Let, y = x cos x

We have to find $\frac{d^2y}{dx^2}$

We know $\frac{d^2y}{dx^2} = \frac{d}{dx}(\frac{dy}{dx})$

So let's first find dy/dx and differentiate it again.

$\therefore \frac{dy}{dx} = \frac{d}{dx}(x\cos x)$

Let u = x and v = cos x

As, y = u v

Now by using product rule of differentiation:

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$\therefore \frac{dy}{dx} = x\frac{d}{dx}(\cos x) + \cos x\frac{d}{dx}x$

$$\frac{dy}{dx} = -x\sin x + \cos x$$

$[\because \frac{d}{dx}(\cos x) = -\sin x \text{ and } \frac{d}{dx}(x^n) = nx^{n-1}]$

Again differentiating with respect to x:

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(-x\sin x + \cos x)$$

$$= \frac{d}{dx}(-x\sin x) + \frac{d}{dx}\cos x$$

Again using the product rule

$$\frac{d^2y}{dx^2} = -x\frac{d}{dx}\sin x + \sin x\frac{d}{dx}(-x) + \frac{d}{dx}\cos x$$

$[\because \frac{d}{dx}(\sin x) = \cos x \text{ and } \frac{d}{dx}(x^n) = nx^{n-1}]$

$$\frac{d^2y}{dx^2} = -x\cos x - \sin x - \sin x$$

$$\frac{d^2y}{dx^2} = -x\cos x - 2\sin x$$

(ix) Log (log x)

Solution:

Let, y = log (log x)

We have to find $\frac{d^2y}{dx^2}$

We know, $\frac{d^2y}{dx^2} = \frac{d}{dx}(\frac{dy}{dx})$

So let's first find dy/dx and differentiate it again.

$\therefore \frac{dy}{dx} = \frac{d}{dx}(\log\log x)$

Let y = log t and t = log x

Using chain rule of differentiation:

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$\therefore \frac{dy}{dx} = \frac{1}{t} \times \frac{1}{x} = \frac{1}{x\log x}$

Again differentiating with respect to x:

As, $\frac{dy}{dx} = u \times v$

Where u = $\frac{1}{x}$ and v = $\frac{1}{\log x}$

Now by using product rule of differentiation:

$$\frac{d^2y}{dx^2} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\therefore \frac{d^2y}{dx^2} = \frac{1}{x}\frac{d}{dx}\left(\frac{1}{\log x}\right) + \frac{1}{\log x}\frac{d}{dx}\ \left(\frac{1}{x}\right)$$

$$\frac{d^2y}{dx^2} = -\frac{1}{x^2\ (\log x)^2} - \frac{1}{x^2 \log x}$$

$$\therefore\ \frac{d^2y}{dx^2} = -\frac{1}{x^2\ (\log x)^2} - \frac{1}{x^2 \log x}$$

2. $If\ y = e^{-x}\cos x,\ show\ that\ \frac{d^2y}{dx^2} = 2e^{-x}\sin x.$

Solution:

Let y=e^{-x} cos x

We have to find $\frac{d^2y}{dx^2}$

We have, $\frac{d^2y}{dx^2} = \frac{d}{dx}(\frac{dy}{dx})$

So let's first find dy/dx and differentiate it again.

$$\therefore \frac{dy}{dx} = \frac{d}{dx}(e^{-x}\cos x)$$

Let u = e^{-x} and v = cos x

We have, y = u v

Differentiate the above by using product rule,

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\therefore \frac{dy}{dx} = e^{-x}\frac{d}{dx}(\cos x) + \cos x\frac{dy}{dx}e^{-x}$$

$$\frac{dy}{dx} = -e^{-x}\sin x - e^{-x}\cos x$$

$$[\because \frac{d}{dx}(\cos x) = -\sin x \;\&\; \frac{d}{dx}e^{-x} = -e^{-x}]$$

Again differentiating with respect to x

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(-e^{-x}\sin x - e^{-x}\cos x)$$

$$= \frac{d}{dx}(-e^{-x}\sin x) - \frac{d}{dx}(e^{-x}\cos x)$$

Again by using product rule we get

$$\frac{d^2y}{dx^2} = -e^{-x}\frac{d}{dx}(\sin x) - \sin x\frac{d}{dx}e^{-x} - e^{-x}\frac{d}{dx}(\cos x) - \cos x\frac{d}{dx}(e^{-x})$$

$$\frac{d^2y}{dx^2} = -e^{-x}\cos x + e^{-x}\sin x + e^{-x}\sin x + e^{-x}\cos x$$

$$[\because \frac{d}{dx}(\cos x) = -\sin x, \frac{d}{dx}e^{-x} = -e^{-x}]$$

$$\frac{d^2y}{dx^2} = 2e^{-x}\sin x$$

Hence proved.

3. $\boldsymbol{If\ y = x + \tan x,\ show\ that\ \cos^2 x\frac{d^2y}{dx^2} - 2y + 2x = 0.}$

Solution:

Given y = x + tan x

Let's find $\frac{d^2y}{dx^2}$

As $\frac{d^2y}{dx^2} = \frac{d}{dx}(\frac{dy}{dx})$

So let's first find dy/dx and differentiate it again.

$\therefore \frac{dy}{dx} = \frac{d}{dx}(x + \tan x) = \frac{d}{dx}(x) + \frac{d}{dx}(\tan x) = 1 + \sec^2 x$

$\therefore \frac{dy}{dx} = 1 + \sec^2 x$

Differentiating again with respect to x

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\ (1 + \sec^2 x) = \frac{d}{dx}(1) + \frac{d}{dx}(\sec^2 x)$$

By using chain rule, we get

$$\frac{d^2y}{dx^2} = 0 + 2\sec x \sec x \tan x$$

$$\frac{d^2y}{dx^2} = 2\sec^2 x \tan x$$

As we got an expression for the second order, as we need $\cos^2 x$ term with $\frac{d^2y}{dx^2}$

Multiply both sides of equation 1 with $\cos^2 x$

We have,

$\cos^2 x \frac{d^2y}{dx^2} = 2\cos^2 x \sec^2 x \tan x$ [$\because$ cos x × sec x = 1]

$$\cos^2 x \frac{d^2y}{dx^2} = 2\tan x$$

From the given equation tan x = y – x

$$\therefore\ \cos^2 x \frac{d^2y}{dx^2} = 2(y - x)$$

$$\therefore\ \cos^2 x \frac{d^2y}{dx^2} - 2y + 2x = 0$$

4. *If* $y = x^3 \log x$, *prove that* $\frac{d^4y}{dx^4} = \frac{6}{x}$.

Solution:

Given, $y = x^3 \log x$

Let's find $\frac{d^4y}{dx^4}$

As $\frac{d^4y}{dx^4} = \frac{d}{dx}\left(\frac{d^3y}{dx^3}\right) = \frac{d}{dx}\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d}{dx}\left(\frac{d}{dx}\left(\frac{d}{dx}\left(\frac{dy}{dx}\right)\right)\right)$

So let's first find dy/dx and differentiate it again.

$\therefore \frac{dy}{dx} = \frac{d}{dx}(x^3 \log x)$

Again differentiating by using product rule, we get

$$\frac{dy}{dx} = x^3\frac{d}{dx}\log x + \log x\frac{d}{dx}x^3$$

$$\frac{dy}{dx} = \frac{x^3}{x} + 3x^2\log x$$

$$\frac{dy}{dx} = x^2(1 + 3\log x)$$

Again differentiating using product rule:

$$\frac{d^2y}{dx^2} = x^2\frac{d}{dx}(1 + 3\log x) + (1 + 3\log x)\frac{d}{dx}x^2$$

$$\frac{d^2y}{dx^2} = x^2 \times \frac{3}{x} + (1 + 3\log x) \times 2x$$

$$\frac{d^2y}{dx^2} = x(5 + 6\log\ x)$$

Again differentiating using product rule

$$\frac{d^3y}{dx^3} = x\frac{d}{dx}(5 + 6\log x) + (5 + 6\log x)\frac{d}{dx}x$$

$$\frac{d^3y}{dx^3} = x \times \frac{6}{x} + (5 + 6\log x)$$

$$\frac{d^3y}{dx^3} = 11 + 6\log x$$

Again differentiating with respect to x

$$\frac{d^4y}{dx^4} = \frac{6}{x}$$

Hence proved.

5. $If\ y = \log\ (\sin x),\ prove\ that\ \frac{d^3y}{dx^3} = 2\cos x\ cosec^3 x.$

Solution:

Given, y = log (sin x)

Let's find – $\frac{d^3y}{dx^3}$

As $\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d}{dx}(\frac{d}{dx}\left(\frac{dy}{dx}\right))$

So let's first find dy/dx and differentiate it again.

$\therefore \frac{dy}{dx} = \frac{d}{dx}(\log(\sin x))$

Differentiating log (sin x) using the chain rule,

Let, t = sin x and y = log t

$\because \frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$ [using chain rule]

$$\frac{dy}{dx} = \cos x \times \frac{1}{t}$$

$[\because \frac{d}{dx}\log x = \frac{1}{x}$ & $\frac{d}{dx}(\sin x) = \cos x]$

$$\frac{dy}{dx} = \frac{\cos x}{\sin x} = \cot x$$

Differentiating again with respect to x

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\ (\cot x)$$

$$\frac{d^2y}{dx^2} = \ -\mathrm{cosec}^2 x$$

$[\because \frac{d}{dx}\cot x = \ -\mathrm{cosec}^2 x]$

$$\frac{d^2y}{dx^2} = -\mathrm{cosec}^2 x$$

Differentiating again with respect to x:

$$\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d}{dx}\ (\ -\mathrm{cosec}^2 x)$$

Using the chain rule and $\frac{d}{dx}\mathrm{cosec}\, x = \ -\mathrm{cosec}\, x \cot x$

$$\frac{d^3y}{dx^3} = \ -2\mathrm{cosec}\, x(-\mathrm{cosec}\, x \cot x)$$

$$= 2\mathrm{cosec}^2 x \cot x \ = 2\ \mathrm{cosec}^2 x\ \frac{\cos x}{\sin x}$$

$$\therefore \frac{d^3y}{dx^3} = 2\mathrm{cosec}^3 x \cos x$$

Hence proved.

6. $If\ y = 2\sin x + 3\cos x,\ show\ that\ \frac{d^2y}{dx^2} + y = 0.$

Solution:

Given y = 2 sin x + 3 cos x

Let's find $\frac{d^2y}{dx^2}$

We know $\frac{d^2y}{dx^2} = \frac{d}{dx}(\frac{dy}{dx})$

So let's first find dy/dx and differentiate it again.

$\therefore \frac{dy}{dx} = \frac{d}{dx}(2\sin x + 3\cos x) = 2\frac{d}{dx}(\sin x) + 3\frac{d}{dx}(\cos x)$

$= 2\cos x - 3\sin x$

$\therefore \frac{dy}{dx} = 2\cos x - 3\sin x$

Differentiating again with respect to x:

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\ (2\cos x - 3\sin x) = \frac{2d}{dx}\cos x - 3\frac{d}{dx}\sin x$$

$$\frac{d^2y}{dx^2} = -2\sin x - 3\cos x$$

We have, y = 2 sin x + 3 cos x

$\therefore \frac{d^2y}{dx^2} = -(2\sin x + 3\cos x) = \ -y$

$\therefore \frac{d^2y}{dx^2} + y = 0$

Hence proved.

$$7.\ If\ y = \frac{\log x}{x},\ show\ that\ \frac{d^2y}{dx^2} = \frac{2\log x - 3}{x^3}.$$

Solution:

Given y = log x/x

Let's find $\frac{d^2y}{dx^2}$

We know $\frac{d^2y}{dx^2} = \frac{d}{dx}(\frac{dy}{dx})$

So, let's first find dy/dx and differentiate it again.

As y is the product of two functions u and v

Let u = log x and v = 1/x

Now by using product rule

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\,\frac{du}{dx}$$

$$\frac{d}{dx}\left(\frac{\log x}{x}\right) = \log x\frac{d}{dx}\frac{1}{x} + \frac{1}{x}\frac{d}{dx}\log x$$

$$\frac{dy}{dx} = -\frac{1}{x^2}\log x + \frac{1}{x^2}$$

$$\frac{dy}{dx} = \frac{1}{x^2}\,(1 - \log x)$$

Again using the product rule to find $\frac{d^2y}{dx^2}$

$$\frac{d^2y}{dx^2} = (1 - \log x)\frac{d}{dx}\frac{1}{x^2} + \frac{1}{x^2}\frac{d}{dx}(1 - \log x)$$

$$= -2\left(\frac{1 - \log x}{x^3}\right) - \frac{1}{x^3}$$

$$\therefore \frac{d^2y}{dx^2} = \frac{2\log x - 3}{x^3}$$

Hence proved.

8. $\boldsymbol{If\ x = a\ sec\ \theta,\ y = b\tan\theta,\ prove\ that\ \frac{d^2y}{dx^2} = -\frac{b^4}{a^2y^3}.}$

Solution:

If y = f (θ) and x = g (θ) that is y is a function of θ and x is also some other function of θ.

Then dy/dθ = f' (θ) and dx/dθ = g' (θ)

We can write: $\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$

Given,

x = a sec θ ......equation 1

y = b tan θ ......equation 2

We have to prove $\frac{d^2y}{dx^2} = -\frac{b^4}{a^2y^3}$.

Let's find $\frac{d^2y}{dx^2}$

As, $\frac{d^2y}{dx^2} = \frac{d}{dx}(\frac{dy}{dx})$

So, let's first find dy/dx using parametric form and differentiate it again.

$\frac{dx}{d\theta} = \frac{d}{d\theta} a \sec\theta = a \sec\theta \tan\theta$ .....equation 3

Similarly, $\frac{dy}{d\theta} = b \sec^2\theta$...... equation 4

$[\because \frac{d}{dx}\sec x = \sec x \tan x, \frac{d}{dx}\tan x = \sec^2 x]$

$$\therefore \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{b\sec^2\theta}{a\sec\theta\tan\theta} = \frac{b}{a}\operatorname{cosec}\theta$$

Differentiating again with respect to x

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(\frac{b}{a}\operatorname{cosec}\theta)$$

$\frac{d^2y}{dx^2} = -\frac{b}{a}\operatorname{cosec}\theta\cot\theta\frac{d\theta}{dx}$ .....equation 5 [using chain rule]

From equation 3:

$$\frac{dx}{d\theta} = a\sec\theta\tan\theta$$

$\therefore \frac{d\theta}{dx} = \frac{1}{a\sec\theta\tan\theta}$

Putting the value in equation 5

$$\frac{d^2y}{dx^2} = -\frac{b}{a}\operatorname{cosec}\theta\cot\theta\frac{1}{a\sec\theta\tan\theta}$$

$$\frac{d^2y}{dx^2} = \frac{-b}{a^2\tan^3\theta}$$

From equation 1:

y = b tan θ

$\therefore \frac{d^2y}{dx^2} = \frac{-b}{\frac{a^2y^3}{b^3}} = -\frac{b^4}{a^2y^3}$.

9. $If\ x = a(\cos\theta + \theta\sin\theta),\ y = a(\sin\theta - \theta\cos\theta),\ prove\ that$ $\frac{d^2x}{d\theta^2} = a(\cos\theta - \theta\sin\theta), \frac{d^2y}{d\theta^2} = a(\sin\theta + \theta\cos\theta)\ and\ \frac{d^2y}{dx^2} = \frac{sec^3\theta}{a\theta}.$

Solution:

If y = f (θ) and x = g (θ), that is y is a function of θ and x is also some other function of θ.

Then dy/dθ = f' (θ) and dx/dθ = g' (θ)

We can write: $\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$

Given,

x =a (cos θ + θ sin θ) ……equation 1

y =a (sin θ – θ cos θ) ……equation 2

Let's find $\frac{d^2y}{dx^2}$

We know $\frac{d^2y}{dx^2} = \frac{d}{dx}(\frac{dy}{dx})$

$$\frac{dx}{d\theta} = \frac{d}{d\theta} a(\cos\theta + \theta\sin\theta)$$

$$= a(-\sin\theta + \theta\cos\theta + \sin\theta)$$

= a θ cos θ... Equation 4

Again differentiating with respect to θ using product rule

$$\frac{d^2x}{d\theta^2} = a(-\theta\sin\theta + \cos\theta)$$

$$\therefore \frac{d^2x}{d\theta^2} = a(\cos\theta - \theta\sin\theta)$$

Similarly,

$$\frac{dy}{d\theta} = \frac{d}{d\theta} a(\sin\theta - \theta\cos\theta) = a\frac{d}{d\theta}\sin\theta - a\frac{d}{d\theta}(\theta\cos\theta)$$

$$= a\cos\theta + a\theta\sin\theta - a\cos\theta$$

$\therefore \frac{dy}{d\theta} = a\theta\sin\theta$ .... equation 5

Again differentiating with respect to θ using product rule

$$\frac{d^2x}{d\theta^2} = a(\theta\cos\theta + \sin\theta)$$

$$\therefore \frac{d^2x}{d\theta^2} = a(\sin\theta + \theta\cos\theta)$$

$$\because \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$$

Using equation 4 and 5, we have

$$\frac{dy}{dx} = \frac{a\theta\sin\theta}{a\theta\cos\theta} = \tan\theta$$

We have $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$

Again differentiating with respect to x

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\tan\theta$$

$$= \sec^2\theta\frac{d\theta}{dx}$$

$$\because \frac{dx}{d\theta} = a\theta\cos\theta \Rightarrow \frac{d\theta}{dx} = \frac{1}{a\theta\cos\theta}$$

Putting a value in the above equation we get

$$\frac{d^2y}{dx^2} = \sec^2\theta \times \frac{1}{a\theta\cos\theta}$$

$$\frac{d^2y}{dx^2} = \frac{\sec^3\theta}{a\theta}$$

10. $If\ y = e^x \cos x,\ prove\ that\ \frac{d^2y}{dx^2} = 2e^x \cos\left(x + \frac{\Pi}{2}\right).$

Solution:

Given, $y = e^x \cos x$

We have to find $\frac{d^2y}{dx^2}$

As $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$

So let's first find dy/dx and differentiate it again.

$$\therefore \frac{dy}{dx} = \frac{d}{dx}(e^x\cos x)$$

Let $u = e^x$ and $v = \cos x$

As, $y = u\,v$

Now by using product rule we get

$\therefore \frac{dy}{dx} = e^x \frac{d}{dx}(\cos x) + \cos x \frac{dy}{dx} e^x$

$\frac{dy}{dx} = -e^x \sin x + e^x \cos x$ $[\because \frac{d}{dx}(\cos x) = -\sin x \ \& \ \frac{d}{dx} e^x = e^x]$

Again differentiating with respect to x

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(-e^x \sin x + e^x \cos x)$$

$$= \frac{d}{dx}(-e^x \sin x) + \frac{d}{dx}(e^x \cos x)$$

Again using the product rule

$$\frac{d^2y}{dx^2} = -e^x \frac{d}{dx}(\sin x) - \sin x \frac{d}{dx} e^x + e^x \frac{d}{dx}(\cos x) + \cos x \frac{d}{dx}(e^x)$$

$$\frac{d^2y}{dx^2} = -e^x \cos x - e^x \sin x - e^x \sin x + e^x \cos x$$

$[\because \frac{d}{dx}(\cos x) = -\sin x, \frac{d}{dx} e^{-x} = -e^{-x}]$

$\frac{d^2y}{dx^2} = -2e^x \sin x$ $[\because -\sin x = \cos(x + \pi/2)]$

$$\frac{d^2y}{dx^2} = -2e^x \cos(x + \frac{\pi}{2})$$

Hence proved.

11. $If\ x = a\cos\theta\ y = b\sin\theta,\ show\ that\ \frac{d^2y}{dx^2} = -\frac{b^4}{a^2y^3}.$

Solution:

Given,

x = a cos θ ……equation 1

y = b sin θ ……equation 2

If y = f (θ) and x = g (θ) that is y is a function of θ and x is also some other function of θ.

Then dy/dθ = f' (θ) and dx/dθ = g' (θ)

We can write $\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$

Now we have to prove $\frac{d^2y}{dx^2} = -\frac{b^4}{a^2y^3}$.

Let's find $\frac{d^2y}{dx^2}$

As $\frac{d^2y}{dx^2} = \frac{d}{dx}(\frac{dy}{dx})$

So, let's first find dy/dx using parametric form and differentiate it again.

$\frac{dx}{d\theta} = \frac{d}{d\theta} a\cos\theta = -a\sin\theta$ .....equation 3

Similarly, $\frac{dy}{d\theta} = b\cos\theta$ ...... equation 4

[∵ $\frac{d}{dx}\cos x = -\sin x\tan x, \frac{d}{dx}\sin x = \cos x$

$$\therefore \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = -\frac{b\cos\theta}{a\sin\theta} = -\frac{b}{a}\cot\theta$$

Differentiating again with respect to x

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(-\frac{b}{a}\cot\theta)$$

By using chain rule, we get

$\frac{d^2y}{dx^2} = \frac{b}{a}\mathrm{cosec}^2\theta\,\frac{d\theta}{dx}$ .....equation 5

From equation 3

$$\frac{dx}{d\theta} = -a\sin\theta$$

$\therefore \frac{d\theta}{dx} = \frac{-1}{a\sin\theta}$

Putting the value in equation 5

$$\frac{d^2y}{dx^2} = -\frac{b}{a}\operatorname{cosec}^2\theta \frac{1}{a\sin\theta}$$

$$\frac{d^2y}{dx^2} = \frac{-b}{a^2\sin^3\theta}$$

From equation 1:

y = b sin θ

$$\therefore \frac{d^2y}{dx^2} = \frac{-b}{\frac{a^2y^3}{b^3}} = -\frac{b^4}{a^2y^3}$$

Hence proved.

12. *If* $x = a(1-\cos^3\theta), y = s\sin^3\theta,$ *prove that* $\frac{d^2y}{dx^2} = \frac{32}{27a}$ *at* $\theta = \frac{\pi}{6}$.

Solution:

Given,

x = a (1 – cos ^{3}θ) …… equation 1

y = a sin ^{3} θ …… equation 2

If y = f (θ) and x = g (θ) that is y is a function of θ and x is also some other function of θ.

Then dy/dθ = f' (θ) and dx/dθ = g' (θ)

We can write $\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$

Let's find $\frac{d^2y}{dx^2}$

We know $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$

Let's find $\frac{d^2y}{dx^2}$

We know $\frac{d^2y}{dx^2} = \frac{d}{dx}(\frac{dy}{dx})$

So, let's first find dy/dx using parametric form and differentiate it again.

Now by using chain rule,

$\frac{dx}{d\theta} = \frac{d}{d\theta} a\,(1 - \cos^3\theta) = 3\,a\cos^2\theta \sin\,\theta$ .....equation 3

Similarly,

$\frac{dy}{d\theta} = \frac{d}{d\theta} a \sin^3\theta = 3\,a\sin^2\theta\cos\theta$ ......equation 4

$[\because \frac{d}{dx}\cos x = -\sin x \,\&\, \frac{d}{dx}\cos x = \sin x]$

$$\therefore \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{3\,a\sin^2\theta\cos\theta}{3\,a\cos^2\theta\sin\,\theta} = \tan\theta$$

Differentiating again with respect to x

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(\tan\theta)$$

$\frac{d^2y}{dx^2} = \sec^2\theta \frac{d\theta}{dx}$ ..... Equation 5

From equation 3

$$\frac{dx}{d\theta} = 3\,a\cos^2\theta\sin\,\theta$$

$$\therefore \frac{d\theta}{dx} = \frac{1}{3\,a\cos^2\theta\sin\,\theta}$$

Putting the value in equation 5

$$\frac{d^2y}{dx^2} = \sec^2\theta\frac{1}{3\,a\cos^2\theta\sin\,\theta}$$

$$\frac{d^2y}{dx^2} = \frac{1}{3\,a\cos^4\theta\sin\theta}$$

Put θ = π/6

$$\left(\frac{d^2y}{dx^2}\right)\text{at}\left(x=\frac{\pi}{6}\right) = \frac{1}{3\,a\cos^4\frac{\pi}{6}\sin\frac{\pi}{6}} = \frac{1}{3a\left(\frac{\sqrt{3}}{2}\right)^4\frac{1}{2}}$$

$$\therefore \left(\frac{d^2y}{dx^2}\right)\text{at}\left(x=\frac{\pi}{6}\right) = \frac{32}{27\,a}$$

Hence proved

13. $If\ x = a(\theta + \sin\theta), y = a(1 + \cos\theta),\ prove\ that\ \frac{d^2y}{dx^2} = -\frac{a}{y^2}.$

Solution:

Given,

x = a (θ + sin θ) ……equation 1

y = a (1+ cos θ) ……equation 2

If y = f (θ) and x = g (θ) that is y is a function of θ and x is also some other function of θ.

Then dy/dθ = f’ (θ) and dx/dθ = g’ (θ)

We can write $\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$

Let’s find $\frac{d^2y}{dx^2}$

As, $\frac{d^2y}{dx^2} = \frac{d}{dx}(\frac{dy}{dx})$

So, let’s first find dy/dx using parametric form and differentiate it again.

$$\frac{dx}{d\theta} = \frac{d}{d\theta}a\,(\theta\ +\ \sin\theta) = a(1+\cos\theta) = y$$

Similarly,

$\frac{dy}{d\theta} = \frac{d}{d\theta} a\,(1 + \cos\theta) = -a\sin\theta$ ...... equation 4

$[\because \frac{d}{dx}\cos x = -\sin x, \frac{d}{dx}\sin x = \cos x$

$\therefore \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{-a\sin\theta}{a(1+\cos\theta)} = \frac{-\sin\theta}{(1+\cos\theta)} = \frac{-a\sin\theta}{y}$

Differentiating again with respect to x

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = -a\frac{d}{dx}\left(\frac{\sin\theta}{y}\right)$$

Using product rule and chain rule together, we get

$$\frac{d^2y}{dx^2} = -a\left(\frac{\sin\theta}{-y^2}\frac{dy}{dx} + \frac{1}{y}\cos\theta\frac{d\theta}{dx}\right)$$

By using equation 3 and 5

$$\frac{d^2y}{dx^2} = -a\left(\frac{\sin\theta}{-y^2}\frac{(-a\sin\theta)}{y} + \frac{1}{y}\cos\theta\frac{1}{y}\right)$$

$$\frac{d^2y}{dx^2} = -a\left(\frac{a\sin^2\theta}{y^3} + \frac{1}{y^2}\cos\theta\right)$$

$$\frac{d^2y}{dx^2} = -\frac{a}{y^2}\left(\frac{a\sin^2\theta}{a\,(1+\cos\theta)} + \cos\theta\right)$$

$$\frac{d^2y}{dx^2} = -\frac{a}{y^2}\left(\frac{1-\cos^2\theta}{(1+\cos\theta)} + \cos\theta\right)$$

$$\frac{d^2y}{dx^2} = -\frac{a}{y^2}\left(\frac{(1-\cos\theta)(1+\cos\theta)}{(1+\cos\theta)} + \cos\theta\right)$$

$$\frac{d^2y}{dx^2} = -\frac{a}{y^2}(1-\cos\theta + \cos\theta)$$

$$\therefore \frac{d^2y}{dx^2} = -\frac{a}{y^2}$$

Hence proved.

14. $If\ x = a(\theta - \sin\theta), y = a(1 + \cos\theta),\ find\ \frac{d^2y}{dx^2}.$

Solution:

Given,

x = a (θ – sin θ) ……equation 1

y = a (1+ cos θ) ……equation 2

If y = f (θ) and x = g (θ) that is y is a function of θ and x is also some other function of θ.

Then dy/dθ = f’ (θ) and dx/dθ = g’ (θ)

We can write $\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$

Now we have to find $\frac{d^2y}{dx^2}$

As, $\frac{d^2y}{dx^2} = \frac{d}{dx}(\frac{dy}{dx})$

So, let’s first find dy/dx using parametric form and differentiate it again.

$\frac{dx}{d\theta} = \frac{d}{d\theta} a\ (\theta - \sin\theta) = a(1 - \cos\theta)$ …..equation 3

Similarly,

$\frac{dy}{d\theta} = \frac{d}{d\theta} a\ (1 + \cos\theta) = -a\sin\theta$ ……equation 4

$\therefore \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{-a\sin\theta}{a(1-\cos\theta)} = \frac{-\sin\theta}{(1-\cos\theta)}$ ….. Equation 5

Differentiating again with respect to x, we get

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = -\frac{d}{dx}\left(\frac{\sin\theta}{1 - \cos\theta}\right)$$

Using product rule and chain rule together, we get

$$\frac{d^2y}{dx^2} = \{-\frac{1}{1-\cos\theta}\frac{d}{d\theta}\sin\theta - \sin\theta\frac{d}{d\theta}\frac{1}{(1-\cos\theta)}\}\frac{d\theta}{dx}$$

Apply chain rule to determine $\frac{d}{d\theta}\frac{1}{(1-\cos\theta)}$

$$\frac{d^2y}{dx^2} = \left\{\frac{-\cos\theta}{1-\cos\theta} + \frac{\sin^2\theta}{(1-\cos\theta)^2}\right\}\frac{1}{a(1-\cos\theta)}$$

$$\frac{d^2y}{dx^2} = \left\{\frac{-\cos\theta(1-\cos\theta)+\sin^2\theta}{(1-\cos\theta)^2}\right\}\frac{1}{a(1-\cos\theta)}$$

$$\frac{d^2y}{dx^2} = \left\{\frac{-\cos\theta+\cos^2\theta+\sin^2\theta}{(1-\cos\theta)^2}\right\}\frac{1}{a(1-\cos\theta)}$$

$$\frac{d^2y}{dx^2} = \left\{\frac{1-\cos\theta}{(1-\cos\theta)^2}\right\}\frac{1}{a(1-\cos\theta)} [\because \cos^2\theta + \sin^2\theta = 1]$$

$$\frac{d^2y}{dx^2} = \frac{1}{a(1-\cos\theta)^2}$$

We know $1 - \cos\theta = 2\sin^2\theta/2$

$$\frac{d^2y}{dx^2} = \frac{1}{a\left(2\sin^2\frac{\theta}{2}\right)^2}$$

$$\therefore \frac{d^2y}{dx^2} = \frac{1}{4a}\operatorname{cosec}^4\frac{\theta}{2}$$

15. $If\ x = a(1-\cos\theta), y = a(\theta+\sin\theta),\ prove\ that\ \frac{d^2y}{dx^2} = -\frac{1}{a}\ at\ \theta = \frac{\pi}{2}.$

Solution:

Given,

y = a (θ + sin θ) ……equation 1

x = a (1– cos θ) ……equation 2

If y = f (θ) and x = g (θ) that is y is a function of θ and x is also some other function of θ.

Then dy/dθ = f′ (θ) and dx/dθ = g′ (θ)

We can write $\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$

Now we have to prove $\frac{d^2y}{dx^2} = -\frac{1}{a}$ at $\theta = \frac{\pi}{2}$.

Let's find $\frac{d^2y}{dx^2}$

As $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$

So, let's first find dy/dx using parametric form and differentiate it again.

$\frac{dy}{d\theta} = \frac{d}{d\theta}a\,(\theta + \sin\theta) = a(1 + \cos\theta)$ .....equation 3

Similarly,

$\frac{dx}{d\theta} = \frac{d}{d\theta}a\,(1 - \cos\theta) = a\sin\theta$ ......equation 4

$[\because \frac{d}{dx}\cos x = -\sin x, \frac{d}{dx}\sin x = \cos x]$

$\therefore \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{a(1+\cos\theta)}{a\sin\theta} = \frac{(1+\cos\theta)}{\sin\theta}$ .....equation 5

Differentiating again with respect to x

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{(1+\cos\theta)}{\sin\theta}\right) = \frac{d}{dx}(1+\cos\theta)\mathrm{cosec}\,\theta$$

Using product rule and chain rule together we get

$$\frac{d^2y}{dx^2} = \left\{\mathrm{cosec}\,\theta\frac{d}{d\theta}(1+\cos\theta) + (1+\cos\theta)\frac{d}{d\theta}\mathrm{cosec}\,\theta\right\}\frac{d\theta}{dx}$$

$$\frac{d^2y}{dx^2} = \{\mathrm{cosec}\,\theta(-\sin\theta) + (1+\cos\theta)(-\mathrm{cosec}\,\theta\cot\theta)\}\frac{1}{a\sin\theta}$$

$$\frac{d^2y}{dx^2} = \{-1 - \mathrm{cosec}\theta\cot\theta - \cot^2\theta\}\frac{1}{a\sin\theta}$$

As we have to find $\frac{d^2y}{dx^2} = -\frac{1}{a}$ at $\theta = \frac{\pi}{2}$

∴ put θ = π/2 in above equation:

$$\frac{d^2y}{dx^2} = \{-1 - \operatorname{cosec}\frac{\pi}{2}\cot\frac{\pi}{2} - \cot^2\frac{\pi}{2}\}\frac{1}{a\sin\frac{\pi}{2}}$$

$$= \frac{\{-1 - 0 - 0\}1}{a}$$

$$\frac{d^2y}{dx^2} = -\frac{1}{a}$$

16. $If\ x = a(1 - \cos\theta), y = a(\theta + \sin\theta),\ prove\ that\ \frac{d^2y}{dx^2} = -\frac{1}{a}\ at\ \theta = \frac{\pi}{2}.$

Solution:

Given,

y = a (θ + sin θ) ……equation 1

x = a (1+ cos θ) ……equation 2

If y = f (θ) and x = g (θ) that is y is a function of θ and x is also some other function of θ.

Then dy/dθ = f' (θ) and dx/dθ = g' (θ)

We can write: $\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$

Given,

y = a (θ + sin θ) ……equation 1

x = a (1+ cos θ) ……equation 2

Now we have to prove $\frac{d^2y}{dx^2} = -\frac{1}{a}$ at $\theta = \frac{\pi}{2}$.

Let's find $\frac{d^2y}{dx^2}$

We know, $\frac{d^2y}{dx^2} = \frac{d}{dx}(\frac{dy}{dx})$

So, let's first find dy/dx using parametric form and differentiate it again.

$\frac{dy}{d\theta} = \frac{d}{d\theta}a\,(\theta + \sin\theta) = a(1 + \cos\theta)$ .....equation 3

Similarly,

$\frac{dx}{d\theta} = \frac{d}{d\theta}a\,(1 + \cos\theta) = -a\sin\theta$ ......equation 4

$[\because \frac{d}{dx}\cos x = -\sin x, \frac{d}{dx}\sin x = \cos x]$

$\therefore \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{a(1+\cos\theta)}{-a\sin\theta} = -\frac{(1+\cos\theta)}{\sin\theta}$ .....equation 5

Differentiating again with respect to x

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(-\frac{(1+\cos\theta)}{\sin\theta}\right) = -\frac{d}{dx}(1+\cos\theta)\mathrm{cosec}\,\theta$$

Using product rule and chain rule together

$$\frac{d^2y}{dx^2} = -\{\mathrm{cosec}\,\theta\frac{d}{d\theta}(1+\cos\theta) + (1+\cos\theta)\frac{d}{d\theta}\mathrm{cosec}\,\theta\}\frac{d\theta}{dx}$$

$$\frac{d^2y}{dx^2} = -\{\mathrm{cosec}\,\theta(-\sin\theta) + (1+\cos\theta)(-\mathrm{cosec}\,\theta\cot\theta)\}\frac{1}{(-a\sin\theta)}$$

$$\frac{d^2y}{dx^2} = \{-1 - \mathrm{cosec}\theta\cot\theta - \cot^2\theta\}\frac{1}{a\sin\theta}$$

As we have to find $\frac{d^2y}{dx^2} = -\frac{1}{a}$ at $\theta = \frac{\pi}{2}$

$\therefore$ put θ = π/2 in above equation:

$$\frac{d^2y}{dx^2} = \{-1 - \mathrm{cosec}\frac{\pi}{2}\cot\frac{\pi}{2} - \cot^2\frac{\pi}{2}\}\frac{1}{a\sin\frac{\pi}{2}} = \frac{\{-1-0-0\}1}{a}$$

$$\frac{d^2y}{dx^2} = -\frac{1}{a}$$

17. *If* $x = \cos\theta, y = \sin^3\theta)$, *prove that* $y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 3\sin^2\theta(5\cos^2\theta - 1)$.

Solution:

Given,

$y = \sin^3\theta$ ......equation 1

$x = \cos\theta$ ......equation 2

If y = f (θ) and x = g (θ), that is y is a function of θ and x is also some other function of θ.

Then dy/dθ = f' (θ) and dx/dθ = g' (θ)

We can write: $\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$

To prove: $y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 3\sin^2\theta\,(5\cos^2\theta - 1)$

Now we have to find $\frac{d^2y}{dx^2}$

We know, $\frac{d^2y}{dx^2} = \frac{d}{dx}(\frac{dy}{dx})$

So, let's first find dy/dx using parametric form and differentiate it again.

$\frac{dx}{d\theta} = -\sin\theta$ ....equation 3

Applying chain rule to differentiate $\sin^3\theta$, then

$\frac{dy}{d\theta} = 3\sin^2\theta\cos\theta$ ....equation 4

$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{3\sin^2\theta\cos\theta}{-\sin\theta} = -3\sin\theta\cos\theta$ ....equation 5

Again differentiating with respect to x

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(\frac{dy}{dx})$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(-3\sin\theta\cos\theta)$$

Applying product rule and chain rule together, we get

$$\frac{d^2y}{dx^2} = -3\{\sin\theta\frac{d}{d\theta}\cos\theta + \cos\theta\frac{d}{d\theta}\sin\theta\}\frac{d\theta}{dx}$$

Put the value of dθ/dx

$$\frac{d^2y}{dx^2} = 3\{-\sin^2\theta + \cos^2\theta\}\frac{1}{\sin\theta}$$

Multiplying y both sides to approach towards the expression we want to prove

$$y\frac{d^2y}{dx^2} = 3\{-\sin^2\theta + \cos^2\theta\}\frac{y}{\sin\theta}$$

Substitute the value of y

$$y\frac{d^2y}{dx^2} = 3\{-\sin^2\theta + \cos^2\theta\}\sin^2\theta$$

Adding equation 5 and squaring we get

$$y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 3\{-\sin^2\theta + \cos^2\theta\}\sin^2\theta + 9\sin^2\theta\cos^2\theta$$

$$y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 3\sin^2\theta\{-\sin^2\theta + \cos^2\theta + 3\cos^2\theta\}$$

$$y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 3\sin^2\theta\{5\cos^2\theta - 1\}$$

18. $If\ y = \sin(\sin x),\ prove\ that\ \frac{d^2y}{dx^2} + \tan x.\frac{dy}{dx} + y\cos^2 x = 0.$

Solution:

Given, y = sin (sin x) ……equation 1

To prove: $\frac{d^2y}{dx^2} + \tan x . \frac{dy}{dx} + y\cos^2 x = 0$

Now we have to find $\frac{d^2y}{dx^2}$

We know $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$

So, first we have to find dy/dx

$$\frac{dy}{dx} = \frac{d}{dx}\sin(\sin x)$$

Using chain rule, we will differentiate the above expression

Let t = sin x $\Longrightarrow \frac{dt}{dx} = \cos x$

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx}$$

$\frac{dy}{dx} = \cos t \cos x = \cos(\sin x)\cos x$ ....equation 2

Again differentiating with respect to x applying product rule, we get

$$\frac{d^2y}{dx^2} = \cos x \frac{d}{dx}\cos(\sin x) + \cos(\sin x)\frac{d}{dx}\cos x$$

Using chain rule we get

$$\frac{d^2y}{dx^2} = -\cos x \cos x \sin(\sin x) - \sin x \cos(\sin x)$$

$$\frac{d^2y}{dx^2} = -y\cos^2 x - \tan x \cos x \cos(\sin x)$$

And using equation 2, we have:

$$\frac{d^2y}{dx^2} = -y\cos^2 x - \tan x \frac{dy}{dx}$$

$$\frac{d^2y}{dx^2} + y\cos^2 x + \tan x \frac{dy}{dx} = 0$$