

EXERCISE 3.1

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1. Determine whether the following operation define a binary operation on the given set or not:

(i) '*' on \mathbb{N} defined by $a * b = a^b$ for all $a, b \in \mathbb{N}$.

(ii) 'O' on \mathbb{Z} defined by $a O b = a^b$ for all $a, b \in \mathbb{Z}$.

(iii) '*' on \mathbb{N} defined by $a * b = a + b - 2$ for all $a, b \in \mathbb{N}$

(iv) ' \times_6 ' on $S = \{1, 2, 3, 4, 5\}$ defined by $a \times_6 b =$ Remainder when a b is divided by 6.

(v) '+ $_6$ ' on $S = \{0, 1, 2, 3, 4, 5\}$ defined by $a +_6 b$

$$= \begin{cases} a + b, & \text{if } a + b < 6 \\ a + b - 6, & \text{if } a + b \geq 6 \end{cases}$$

(vi) ' \odot ' on \mathbb{N} defined by $a \odot b = a^b + b^a$ for all $a, b \in \mathbb{N}$

(vii) '*' on \mathbb{Q} defined by $a * b = (a - 1) / (b + 1)$ for all $a, b \in \mathbb{Q}$

Solution:

(i) Given '*' on \mathbb{N} defined by $a * b = a^b$ for all $a, b \in \mathbb{N}$.

Let $a, b \in \mathbb{N}$. Then,

$a^b \in \mathbb{N}$ [$\because a^b \neq 0$ and a, b is positive integer]

$\Rightarrow a * b \in \mathbb{N}$

Therefore,

$a * b \in \mathbb{N}, \forall a, b \in \mathbb{N}$

Thus, * is a binary operation on \mathbb{N} .

(ii) Given 'O' on \mathbb{Z} defined by $a O b = a^b$ for all $a, b \in \mathbb{Z}$.

Both $a = 3$ and $b = -1$ belong to \mathbb{Z} .

$\Rightarrow a * b = 3^{-1}$

$= 1/3 \notin \mathbb{Z}$

Thus, * is not a binary operation on \mathbb{Z} .

(iii) Given '*' on \mathbb{N} defined by $a * b = a + b - 2$ for all $a, b \in \mathbb{N}$

If $a = 1$ and $b = 1$,

$a * b = a + b - 2$

$= 1 + 1 - 2$

$= 0 \notin \mathbb{N}$

Thus, there exist $a = 1$ and $b = 1$ such that $a * b \notin \mathbb{N}$

So, * is not a binary operation on \mathbb{N} .

(iv) Given ' \times_6 ' on $S = \{1, 2, 3, 4, 5\}$ defined by $a \times_6 b =$ Remainder when $a b$ is divided by 6. Consider the composition table,

\times_6	1	2	3	4	5
1	1	2	3	4	5
2	2	4	0	2	4
3	3	0	3	0	3
4	4	2	0	4	2
5	5	4	3	2	1

Here all the elements of the table are not in S .

\Rightarrow For $a = 2$ and $b = 3$,

$a \times_6 b = 2 \times_6 3 =$ remainder when 6 divided by 6 = $0 \notin S$

Thus, \times_6 is not a binary operation on S .

(v) Given ' $+_6$ ' on $S = \{0, 1, 2, 3, 4, 5\}$ defined by $a +_6 b$

$$= \begin{cases} a + b, & \text{if } a + b < 6 \\ a + b - 6, & \text{if } a + b \geq 6 \end{cases}$$

Consider the composition table,

$+_6$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

Here all the elements of the table are not in S .

\Rightarrow For $a = 2$ and $b = 3$,

$a \times_6 b = 2 \times_6 3 = \text{remainder when 6 divided by 6} = 0 \neq$
 Thus, \times_6 is not a binary operation on S.

(vi) Given ' \odot ' on N defined by $a \odot b = a^b + b^a$ for all $a, b \in \mathbb{N}$

Let $a, b \in \mathbb{N}$. Then,

$$a^b, b^a \in \mathbb{N}$$

$$\Rightarrow a^b + b^a \in \mathbb{N} \quad [\because \text{Addition is binary operation on } \mathbb{N}]$$

$$\Rightarrow a \odot b \in \mathbb{N}$$

Thus, \odot is a binary operation on N.

(vii) Given '*' on Q defined by $a * b = (a - 1) / (b + 1)$ for all $a, b \in \mathbb{Q}$

If $a = 2$ and $b = -1$ in Q,

$$a * b = (a - 1) / (b + 1)$$

$$= (2 - 1) / (-1 + 1)$$

$$= 1/0 \text{ [which is not defined]}$$

For $a = 2$ and $b = -1$

$a * b$ does not belong to Q

So, * is not a binary operation in Q.

2. Determine whether or not the definition of * given below gives a binary operation.

In the event that * is not a binary operation give justification of this.

(i) On \mathbb{Z}^+ , defined * by $a * b = a - b$

(ii) On \mathbb{Z}^+ , define * by $a * b = ab$

(iii) On R, define * by $a * b = ab^2$

(iv) On \mathbb{Z}^+ define * by $a * b = |a - b|$

(v) On \mathbb{Z}^+ define * by $a * b = a$

(vi) On R, define * by $a * b = a + 4b^2$

Here, \mathbb{Z}^+ denotes the set of all non-negative integers.

Solution:

(i) Given On \mathbb{Z}^+ , defined * by $a * b = a - b$

If $a = 1$ and $b = 2$ in \mathbb{Z}^+ , then

$$a * b = a - b$$

$$= 1 - 2$$

$$= -1 \notin \mathbb{Z}^+ \text{ [because } \mathbb{Z}^+ \text{ is the set of non-negative integers]}$$

For $a = 1$ and $b = 2$,

$$a * b \notin \mathbb{Z}^+$$

Thus, $*$ is not a binary operation on Z^+ .

(ii) Given Z^+ , define $*$ by $a * b = a b$

Let $a, b \in Z^+$

$\Rightarrow a, b \in Z^+$

$\Rightarrow a * b \in Z^+$

Thus, $*$ is a binary operation on R .

(iii) Given on R , define by $a * b = ab^2$

Let $a, b \in R$

$\Rightarrow a, b^2 \in R$

$\Rightarrow ab^2 \in R$

$\Rightarrow a * b \in R$

Thus, $*$ is a binary operation on R .

(iv) Given on Z^+ define $*$ by $a * b = |a - b|$

Let $a, b \in Z^+$

$\Rightarrow |a - b| \in Z^+$

$\Rightarrow a * b \in Z^+$

Therefore,

$a * b \in Z^+, \forall a, b \in Z^+$

Thus, $*$ is a binary operation on Z^+ .

(v) Given on Z^+ define $*$ by $a * b = a$

Let $a, b \in Z^+$

$\Rightarrow a \in Z^+$

$\Rightarrow a * b \in Z^+$

Therefore, $a * b \in Z^+ \forall a, b \in Z^+$

Thus, $*$ is a binary operation on Z^+ .

(vi) Given On R , define $*$ by $a * b = a + 4b^2$

Let $a, b \in R$

$\Rightarrow a, 4b^2 \in R$

$\Rightarrow a + 4b^2 \in R$

$\Rightarrow a * b \in R$

Therefore, $a * b \in R, \forall a, b \in R$

Thus, $*$ is a binary operation on R .

3. Let $*$ be a binary operation on the set I of integers, defined by $a * b = 2a + b - 3$. Find the value of $3 * 4$.

Solution:

$$\text{Given } a * b = 2a + b - 3$$

$$3 * 4 = 2(3) + 4 - 3$$

$$= 6 + 4 - 3$$

$$= 7$$

4. Is $*$ defined on the set $\{1, 2, 3, 4, 5\}$ by $a * b = \text{LCM of } a \text{ and } b$ a binary operation? Justify your answer.

Solution:

LCM	1	2	3	4	5
1	1	2	3	4	5
2	2	2	6	4	10
3	3	5	3	12	15
4	4	4	12	4	20
5	5	10	15	20	5

In the given composition table, all the elements are not in the set $\{1, 2, 3, 4, 5\}$. If we consider $a = 2$ and $b = 3$, $a * b = \text{LCM of } a \text{ and } b = 6 \notin \{1, 2, 3, 4, 5\}$. Thus, $*$ is not a binary operation on $\{1, 2, 3, 4, 5\}$.

5. Let $S = \{a, b, c\}$. Find the total number of binary operations on S .

Solution:

Number of binary operations on a set with n elements is n^{n^2}

Here, $S = \{a, b, c\}$

Number of elements in $S = 3$

Number of binary operations on a set with 3 elements is 3^{3^2}

EXERCISE 3.2

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1. Let '*' be a binary operation on \mathbb{N} defined by $a * b = \text{l.c.m.}(a, b)$ for all $a, b \in \mathbb{N}$

(i) Find $2 * 4$, $3 * 5$, $1 * 6$.

(ii) Check the commutativity and associativity of '*' on \mathbb{N} .

Solution:

(i) Given $a * b = \text{l.c.m.}(a, b)$

$$2 * 4 = \text{l.c.m.}(2, 4)$$

$$= 4$$

$$3 * 5 = \text{l.c.m.}(3, 5)$$

$$= 15$$

$$1 * 6 = \text{l.c.m.}(1, 6)$$

$$= 6$$

(ii) We have to prove commutativity of *

Let $a, b \in \mathbb{N}$

$$a * b = \text{l.c.m.}(a, b)$$

$$= \text{l.c.m.}(b, a)$$

$$= b * a$$

Therefore

$$a * b = b * a \quad \forall a, b \in \mathbb{N}$$

Thus * is commutative on \mathbb{N} .

Now we have to prove associativity of *

Let $a, b, c \in \mathbb{N}$

$$a * (b * c) = a * \text{l.c.m.}(b, c)$$

$$= \text{l.c.m.}(a, (b, c))$$

$$= \text{l.c.m.}(a, b, c)$$

$$(a * b) * c = \text{l.c.m.}(a, b) * c$$

$$= \text{l.c.m.}((a, b), c)$$

$$= \text{l.c.m.}(a, b, c)$$

Therefore

$$a * (b * c) = (a * b) * c, \quad \forall a, b, c \in \mathbb{N}$$

Thus, * is associative on \mathbb{N} .

2. Determine which of the following binary operation is associative and which is

commutative:

(i) $*$ on N defined by $a * b = 1$ for all $a, b \in N$

(ii) $*$ on Q defined by $a * b = (a + b)/2$ for all $a, b \in Q$

Solution:

(i) We have to prove commutativity of $*$

Let $a, b \in N$

$$a * b = 1$$

$$b * a = 1$$

Therefore,

$$a * b = b * a, \text{ for all } a, b \in N$$

Thus $*$ is commutative on N .

Now we have to prove associativity of $*$

Let $a, b, c \in N$

$$\text{Then } a * (b * c) = a * (1)$$

$$= 1$$

$$(a * b) * c = (1) * c$$

$$= 1$$

Therefore $a * (b * c) = (a * b) * c$ for all $a, b, c \in N$

Thus, $*$ is associative on N .

(ii) First we have to prove commutativity of $*$

Let $a, b \in N$

$$a * b = (a + b)/2$$

$$= (b + a)/2$$

$$= b * a$$

Therefore, $a * b = b * a, \forall a, b \in N$

Thus $*$ is commutative on N .

Now we have to prove associativity of $*$

Let $a, b, c \in N$

$$a * (b * c) = a * (b + c)/2$$

$$= [a + (b + c)]/2$$

$$= (2a + b + c)/2$$

$$\text{Now, } (a * b) * c = (a + b)/2 * c$$

$$= [(a + b)/2 + c] / 2$$

$$= (a + b + 2c)/4$$

Thus, $a * (b * c) \neq (a * b) * c$

If $a = 1, b = 2, c = 3$

$$1 * (2 * 3) = 1 * (2 + 3)/2$$

$$= 1 * (5/2)$$

$$= [1 + (5/2)]/2$$

$$= 7/4$$

$$(1 * 2) * 3 = (1 + 2)/2 * 3$$

$$= 3/2 * 3$$

$$= [(3/2) + 3]/2$$

$$= 4/9$$

Therefore, there exist $a = 1, b = 2, c = 3 \in \mathbb{N}$ such that $a * (b * c) \neq (a * b) * c$

Thus, $*$ is not associative on \mathbb{N} .

3. Let A be any set containing more than one element. Let $'*$ ' be a binary operation on A defined by $a * b = b$ for all $a, b \in A$. Is $'*$ ' commutative or associative on A ?

Solution:

Let $a, b \in A$

Then, $a * b = b$

$b * a = a$

Therefore $a * b \neq b * a$

Thus, $*$ is not commutative on A

Now we have to check associativity:

Let $a, b, c \in A$

$$a * (b * c) = a * c$$

$$= c$$

Therefore

$$a * (b * c) = (a * b) * c, \forall a, b, c \in A$$

Thus, $*$ is associative on A

4. Check the commutativity and associativity of each of the following binary operations:

(i) $'*$ ' on \mathbb{Z} defined by $a * b = a + b + a b$ for all $a, b \in \mathbb{Z}$

(ii) $'*$ ' on \mathbb{N} defined by $a * b = 2^{ab}$ for all $a, b \in \mathbb{N}$

(iii) $'*$ ' on \mathbb{Q} defined by $a * b = a - b$ for all $a, b \in \mathbb{Q}$

(iv) $'\odot'$ on \mathbb{Q} defined by $a \odot b = a^2 + b^2$ for all $a, b \in \mathbb{Q}$

(v) $'\circ'$ on \mathbb{Q} defined by $a \circ b = (ab/2)$ for all $a, b \in \mathbb{Q}$

(vi) $'**'$ on \mathbb{Q} defined by $a ** b = ab^2$ for all $a, b \in \mathbb{Q}$

- (vii) $*$ on Q defined by $a * b = a + a b$ for all $a, b \in Q$
 (viii) $*$ on R defined by $a * b = a + b - 7$ for all $a, b \in R$
 (ix) $*$ on Q defined by $a * b = (a - b)^2$ for all $a, b \in Q$
 (x) $*$ on Q defined by $a * b = a b + 1$ for all $a, b \in Q$
 (xi) $*$ on N defined by $a * b = a^b$ for all $a, b \in N$
 (xii) $*$ on Z defined by $a * b = a - b$ for all $a, b \in Z$
 (xiii) $*$ on Q defined by $a * b = (ab/4)$ for all $a, b \in Q$
 (xiv) $*$ on Z defined by $a * b = a + b - ab$ for all $a, b \in Z$
 (xv) $*$ on Q defined by $a * b = \gcd(a, b)$ for all $a, b \in Q$

Solution:

(i) First we have to check commutativity of $*$

Let $a, b \in Z$

$$\begin{aligned} \text{Then } a * b &= a + b + ab \\ &= b + a + ba \\ &= b * a \end{aligned}$$

Therefore,

$$a * b = b * a, \forall a, b \in Z$$

Now we have to prove associativity of $*$

Let $a, b, c \in Z$, Then,

$$\begin{aligned} a * (b * c) &= a * (b + c + b c) \\ &= a + (b + c + b c) + a (b + c + b c) \\ &= a + b + c + b c + a b + a c + a b c \\ (a * b) * c &= (a + b + a b) * c \\ &= a + b + a b + c + (a + b + a b) c \\ &= a + b + a b + c + a c + b c + a b c \end{aligned}$$

Therefore,

$$a * (b * c) = (a * b) * c, \forall a, b, c \in Z$$

Thus, $*$ is associative on Z .

(ii) First we have to check commutativity of $*$

Let $a, b \in N$

$$\begin{aligned} a * b &= 2^{ab} \\ &= 2^{ba} \\ &= b * a \end{aligned}$$

Therefore, $a * b = b * a, \forall a, b \in N$

Thus, $*$ is commutative on N

Now we have to check associativity of $*$

Let $a, b, c \in \mathbb{N}$

$$\begin{aligned} \text{Then, } a * (b * c) &= a * (2^{bc}) \\ &= 2^{a+2^{bc}} \end{aligned}$$

$$\begin{aligned} (a * b) * c &= (2^{ab}) * c \\ &= 2^{ab+2^c} \end{aligned}$$

Therefore, $a * (b * c) \neq (a * b) * c$

Thus, $*$ is not associative on \mathbb{N}

(iii) First we have to check commutativity of $*$

Let $a, b \in \mathbb{Q}$, then

$$a * b = a - b$$

$$b * a = b - a$$

Therefore, $a * b \neq b * a$

Thus, $*$ is not commutative on \mathbb{Q}

Now we have to check associativity of $*$

Let $a, b, c \in \mathbb{Q}$, then

$$\begin{aligned} a * (b * c) &= a * (b - c) \\ &= a - (b - c) \end{aligned}$$

$$= a - b + c$$

$$\begin{aligned} (a * b) * c &= (a - b) * c \\ &= a - b - c \end{aligned}$$

Therefore,

$$a * (b * c) \neq (a * b) * c$$

Thus, $*$ is not associative on \mathbb{Q}

(iv) First we have to check commutativity of \odot

Let $a, b \in \mathbb{Q}$, then

$$a \odot b = a^2 + b^2$$

$$= b^2 + a^2$$

$$= b \odot a$$

Therefore, $a \odot b = b \odot a, \forall a, b \in \mathbb{Q}$

Thus, \odot on \mathbb{Q}

Now we have to check associativity of \odot

Let $a, b, c \in \mathbb{Q}$, then

$$\begin{aligned} a \odot (b \odot c) &= a \odot (b^2 + c^2) \\ &= a^2 + (b^2 + c^2)^2 \end{aligned}$$

$$= a^2 + b^4 + c^4 + 2b^2c^2$$

$$(a \odot b) \odot c = (a^2 + b^2) \odot c$$

$$= (a^2 + b^2)^2 + c^2$$

$$= a^4 + b^4 + 2a^2b^2 + c^2$$

Therefore,

$$(a \odot b) \odot c \neq a \odot (b \odot c)$$

Thus, \odot is not associative on \mathbb{Q} .

(v) First we have to check commutativity of o

Let $a, b \in \mathbb{Q}$, then

$$a o b = (ab/2)$$

$$= (b a/2)$$

$$= b o a$$

Therefore, $a o b = b o a, \forall a, b \in \mathbb{Q}$

Thus, o is commutative on \mathbb{Q}

Now we have to check associativity of o

Let $a, b, c \in \mathbb{Q}$, then

$$a o (b o c) = a o (b c/2)$$

$$= [a (b c/2)]/2$$

$$= [a (b c/2)]/2$$

$$= (a b c)/4$$

$$(a o b) o c = (ab/2) o c$$

$$= [(ab/2) c] / 2$$

$$= (a b c)/4$$

Therefore $a o (b o c) = (a o b) o c, \forall a, b, c \in \mathbb{Q}$

Thus, o is associative on \mathbb{Q} .

(vi) First we have to check commutativity of $*$

Let $a, b \in \mathbb{Q}$, then

$$a * b = ab^2$$

$$b * a = ba^2$$

Therefore,

$$a * b \neq b * a$$

Thus, $*$ is not commutative on \mathbb{Q}

Now we have to check associativity of $*$

Let $a, b, c \in \mathbb{Q}$, then

$$a * (b * c) = a * (bc^2)$$

$$= a (bc^2)^2$$

$$= ab^2 c^4$$

$$(a * b) * c = (ab^2) * c$$

$$= ab^2c^2$$

Therefore $a * (b * c) \neq (a * b) * c$

Thus, $*$ is not associative on Q .

(vii) First we have to check commutativity of $*$

Let $a, b \in Q$, then

$$a * b = a + ab$$

$$b * a = b + ba$$

$$= b + ab$$

Therefore, $a * b \neq b * a$

Thus, $*$ is not commutative on Q .

Now we have to prove associativity on Q .

Let $a, b, c \in Q$, then

$$a * (b * c) = a * (b + b c)$$

$$= a + a (b + b c)$$

$$= a + ab + a b c$$

$$(a * b) * c = (a + a b) * c$$

$$= (a + a b) + (a + a b) c$$

$$= a + a b + a c + a b c$$

Therefore $a * (b * c) \neq (a * b) * c$

Thus, $*$ is not associative on Q .

(viii) First we have to check commutativity of $*$

Let $a, b \in R$, then

$$a * b = a + b - 7$$

$$= b + a - 7$$

$$= b * a$$

Therefore,

$$a * b = b * a, \text{ for all } a, b \in R$$

Thus, $*$ is commutative on R

Now we have to prove associativity of $*$ on R .

Let $a, b, c \in R$, then

$$a * (b * c) = a * (b + c - 7)$$

$$= a + b + c - 7 - 7$$

$$= a + b + c - 14$$

$$(a * b) * c = (a + b - 7) * c$$

$$= a + b - 7 + c - 7$$

$$= a + b + c - 14$$

Therefore,

$$a * (b * c) = (a * b) * c, \text{ for all } a, b, c \in \mathbb{R}$$

Thus, $*$ is associative on \mathbb{R} .

(ix) First we have to check commutativity of $*$

Let $a, b \in \mathbb{Q}$, then

$$a * b = (a - b)^2$$

$$= (b - a)^2$$

$$= b * a$$

Therefore,

$$a * b = b * a, \text{ for all } a, b \in \mathbb{Q}$$

Thus, $*$ is commutative on \mathbb{Q}

Now we have to prove associativity of $*$ on \mathbb{Q}

Let $a, b, c \in \mathbb{Q}$, then

$$a * (b * c) = a * (b - c)^2$$

$$= a * (b^2 + c^2 - 2bc)$$

$$= (a - b^2 - c^2 + 2bc)^2$$

$$(a * b) * c = (a - b)^2 * c$$

$$= (a^2 + b^2 - 2ab) * c$$

$$= (a^2 + b^2 - 2ab - c)^2$$

$$\text{Therefore, } a * (b * c) \neq (a * b) * c$$

Thus, $*$ is not associative on \mathbb{Q} .

(x) First we have to check commutativity of $*$

Let $a, b \in \mathbb{Q}$, then

$$a * b = ab + 1$$

$$= ba + 1$$

$$= b * a$$

Therefore

$$a * b = b * a, \text{ for all } a, b \in \mathbb{Q}$$

Thus, $*$ is commutative on \mathbb{Q}

Now we have to prove associativity of $*$ on \mathbb{Q}

Let $a, b, c \in \mathbb{Q}$, then

$$a * (b * c) = a * (bc + 1)$$

$$= a (b c + 1) + 1$$

$$= a b c + a + 1$$

$$(a * b) * c = (ab + 1) * c$$

$$= (ab + 1) c + 1$$

$$= a b c + c + 1$$

Therefore, $a * (b * c) \neq (a * b) * c$

Thus, $*$ is not associative on Q .

(xi) First we have to check commutativity of $*$

Let $a, b \in N$, then

$$a * b = a^b$$

$$b * a = b^a$$

Therefore, $a * b \neq b * a$

Thus, $*$ is not commutative on N .

Now we have to check associativity of $*$

$$a * (b * c) = a * (b^c)$$

$$= a^{b^c}$$

$$(a * b) * c = (a^b) * c$$

$$= (a^b)^c$$

$$= a^{bc}$$

Therefore, $a * (b * c) \neq (a * b) * c$

Thus, $*$ is not associative on N

(xii) First we have to check commutativity of $*$

Let $a, b \in Z$, then

$$a * b = a - b$$

$$b * a = b - a$$

Therefore,

$$a * b \neq b * a$$

Thus, $*$ is not commutative on Z .

Now we have to check associativity of $*$

Let $a, b, c \in Z$, then

$$a * (b * c) = a * (b - c)$$

$$= a - (b - c)$$

$$= a - (b + c)$$

$$(a * b) * c = (a - b) - c$$

$$= a - b - c$$

Therefore, $a * (b * c) \neq (a * b) * c$

Thus, $*$ is not associative on Z

(xiii) First we have to check commutativity of $*$

Let $a, b \in Q$, then

$$a * b = (ab/4)$$

$$= (ba/4)$$

$$= b * a$$

Therefore, $a * b = b * a$, for all $a, b \in Q$

Thus, $*$ is commutative on Q

Now we have to check associativity of $*$

Let $a, b, c \in Q$, then

$$a * (b * c) = a * (b c/4)$$

$$= [a (b c/4)]/4$$

$$= (a b c/16)$$

$$(a * b) * c = (ab/4) * c$$

$$= [(ab/4) c]/4$$

$$= a b c/16$$

Therefore,

$$a * (b * c) = (a * b) * c \text{ for all } a, b, c \in Q$$

Thus, $*$ is associative on Q .

(xiv) First we have to check commutativity of $*$

Let $a, b \in Z$, then

$$a * b = a + b - ab$$

$$= b + a - ba$$

$$= b * a$$

Therefore, $a * b = b * a$, for all $a, b \in Z$

Thus, $*$ is commutative on Z .

Now we have to check associativity of $*$

Let $a, b, c \in Z$

$$a * (b * c) = a * (b + c - b c)$$

$$= a + b + c - b c - ab - ac + a b c$$

$$(a * b) * c = (a + b - a b) c$$

$$= a + b - ab + c - (a + b - ab) c$$

$$= a + b + c - ab - ac - bc + a b c$$

Therefore,

$$a * (b * c) = (a * b) * c, \text{ for all } a, b, c \in \mathbb{Z}$$

Thus, $*$ is associative on \mathbb{Z} .

(xv) First we have to check commutativity of $*$

Let $a, b \in \mathbb{N}$, then

$$a * b = \gcd(a, b)$$

$$= \gcd(b, a)$$

$$= b * a$$

Therefore, $a * b = b * a$, for all $a, b \in \mathbb{N}$

Thus, $*$ is commutative on \mathbb{N} .

Now we have to check associativity of $*$

Let $a, b, c \in \mathbb{N}$

$$a * (b * c) = a * [\gcd(a, b)]$$

$$= \gcd(a, b, c)$$

$$(a * b) * c = [\gcd(a, b)] * c$$

$$= \gcd(a, b, c)$$

Therefore,

$$a * (b * c) = (a * b) * c, \text{ for all } a, b, c \in \mathbb{N}$$

Thus, $*$ is associative on \mathbb{N} .

5. If the binary operation \circ is defined by $a \circ b = a + b - ab$ on the set $\mathbb{Q} - \{-1\}$ of all rational numbers other than -1 , show that \circ is commutative on $\mathbb{Q} - \{-1\}$.

Solution:

Let $a, b \in \mathbb{Q} - \{-1\}$.

$$\text{Then } a \circ b = a + b - ab$$

$$= b + a - ba$$

$$= b \circ a$$

Therefore,

$$a \circ b = b \circ a \text{ for all } a, b \in \mathbb{Q} - \{-1\}$$

Thus, \circ is commutative on $\mathbb{Q} - \{-1\}$

6. Show that the binary operation $*$ on \mathbb{Z} defined by $a * b = 3a + 7b$ is not commutative?

Solution:

Let $a, b \in \mathbb{Z}$

$$a * b = 3a + 7b$$

$$b * a = 3b + 7a$$

Thus, $a * b \neq b * a$

Let $a = 1$ and $b = 2$

$$1 * 2 = 3 \times 1 + 7 \times 2$$

$$= 3 + 14$$

$$= 17$$

$$2 * 1 = 3 \times 2 + 7 \times 1$$

$$= 6 + 7$$

$$= 13$$

Therefore, there exist $a = 1, b = 2 \in \mathbb{Z}$ such that $a * b \neq b * a$

Thus, $*$ is not commutative on \mathbb{Z} .

7. On the set \mathbb{Z} of integers a binary operation $*$ is defined by $a * b = ab + 1$ for all $a, b \in \mathbb{Z}$. Prove that $*$ is not associative on \mathbb{Z} .

Solution:

Let $a, b, c \in \mathbb{Z}$

$$a * (b * c) = a * (bc + 1)$$

$$= a(bc + 1) + 1$$

$$= abc + a + 1$$

$$(a * b) * c = (ab + 1) * c$$

$$= (ab + 1)c + 1$$

$$= abc + c + 1$$

Thus, $a * (b * c) \neq (a * b) * c$

Thus, $*$ is not associative on \mathbb{Z} .

EXERCISE 3.3

PAGE NO: 3.15

1. Find the identity element in the set I^+ of all positive integers defined by $a * b = a + b$ for all $a, b \in I^+$.

Solution:

Let e be the identity element in I^+ with respect to $*$ such that

$$a * e = a = e * a, \forall a \in I^+$$

$$a * e = a \text{ and } e * a = a, \forall a \in I^+$$

$$a + e = a \text{ and } e + a = a, \forall a \in I^+$$

$$e = 0, \forall a \in I^+$$

Thus, 0 is the identity element in I^+ with respect to $*$.

2. Find the identity element in the set of all rational numbers except -1 with respect to $*$ defined by $a * b = a + b + ab$

Solution:

Let e be the identity element in I^+ with respect to $*$ such that

$$a * e = a = e * a, \forall a \in Q - \{-1\}$$

$$a * e = a \text{ and } e * a = a, \forall a \in Q - \{-1\}$$

$$a + e + ae = a \text{ and } e + a + ea = a, \forall a \in Q - \{-1\}$$

$$e + ae = 0 \text{ and } e + ea = 0, \forall a \in Q - \{-1\}$$

$$e(1 + a) = 0 \text{ and } e(1 + a) = 0, \forall a \in Q - \{-1\}$$

$$e = 0, \forall a \in Q - \{-1\} \text{ [because } a \text{ not equal to } -1]$$

Thus, 0 is the identity element in $Q - \{-1\}$ with respect to $*$.

EXERCISE 3.4

PAGE NO: 3.25

1. Let $*$ be a binary operation on Z defined by $a * b = a + b - 4$ for all $a, b \in Z$.

(i) Show that $*$ is both commutative and associative.

(ii) Find the identity element in Z

(iii) Find the invertible element in Z .

Solution:

(i) First we have to prove commutativity of $*$

Let $a, b \in Z$. then,

$$a * b = a + b - 4$$

$$= b + a - 4$$

$$= b * a$$

Therefore,

$$a * b = b * a, \forall a, b \in Z$$

Thus, $*$ is commutative on Z .

Now we have to prove associativity of Z .

Let $a, b, c \in Z$. then,

$$a * (b * c) = a * (b + c - 4)$$

$$= a + b + c - 4 - 4$$

$$= a + b + c - 8$$

$$(a * b) * c = (a + b - 4) * c$$

$$= a + b - 4 + c - 4$$

$$= a + b + c - 8$$

Therefore,

$$a * (b * c) = (a * b) * c, \text{ for all } a, b, c \in Z$$

Thus, $*$ is associative on Z .

(ii) Let e be the identity element in Z with respect to $*$ such that

$$a * e = a = e * a \quad \forall a \in Z$$

$$a * e = a \text{ and } e * a = a, \quad \forall a \in Z$$

$$a + e - 4 = a \text{ and } e + a - 4 = a, \quad \forall a \in Z$$

$$e = 4, \quad \forall a \in Z$$

Thus, 4 is the identity element in Z with respect to $*$.

(iii) Let $a \in Z$ and $b \in Z$ be the inverse of a . Then,

$$a * b = e = b * a$$

$$a * b = e \text{ and } b * a = e$$

$$a + b - 4 = 4 \text{ and } b + a - 4 = 4$$

$$b = 8 - a \in \mathbb{Z}$$

Thus, $8 - a$ is the inverse of $a \in \mathbb{Z}$

2. Let $*$ be a binary operation on \mathbb{Q}_0 (set of non-zero rational numbers) defined by $a * b = (3ab/5)$ for all $a, b \in \mathbb{Q}_0$. Show that $*$ is commutative as well as associative. Also, find its identity element, if it exists.

Solution:

First we have to prove commutativity of $*$

Let $a, b \in \mathbb{Q}_0$

$$a * b = (3ab/5)$$

$$= (3ba/5)$$

$$= b * a$$

Therefore, $a * b = b * a$, for all $a, b \in \mathbb{Q}_0$

Now we have to prove associativity of $*$

Let $a, b, c \in \mathbb{Q}_0$

$$a * (b * c) = a * (3bc/5)$$

$$= [a (3bc/5)] / 5$$

$$= 3abc/25$$

$$(a * b) * c = (3ab/5) * c$$

$$= [(3ab/5)c] / 5$$

$$= 3abc/25$$

Therefore $a * (b * c) = (a * b) * c$, for all $a, b, c \in \mathbb{Q}_0$

Thus $*$ is associative on \mathbb{Q}_0

Now we have to find the identity element

Let e be the identity element in \mathbb{Z} with respect to $*$ such that

$$a * e = a = e * a \quad \forall a \in \mathbb{Q}_0$$

$$a * e = a \text{ and } e * a = a, \quad \forall a \in \mathbb{Q}_0$$

$$3ae/5 = a \text{ and } 3ea/5 = a, \quad \forall a \in \mathbb{Q}_0$$

$$e = 5/3 \quad \forall a \in \mathbb{Q}_0 \text{ [because } a \text{ is not equal to } 0]$$

Thus, $5/3$ is the identity element in \mathbb{Q}_0 with respect to $*$.

3. Let $*$ be a binary operation on $\mathbb{Q} - \{-1\}$ defined by $a * b = a + b + ab$ for all $a, b \in \mathbb{Q} - \{-1\}$. Then,

- (i) Show that $*$ is both commutative and associative on $Q - \{-1\}$
 (ii) Find the identity element in $Q - \{-1\}$
 (iii) Show that every element of $Q - \{-1\}$ is invertible. Also, find inverse of an arbitrary element.

Solution:

(i) First we have to check commutativity of $*$

Let $a, b \in Q - \{-1\}$

Then $a * b = a + b + ab$

$= b + a + ba$

$= b * a$

Therefore,

$a * b = b * a, \forall a, b \in Q - \{-1\}$

Now we have to prove associativity of $*$

Let $a, b, c \in Q - \{-1\}$, Then,

$a * (b * c) = a * (b + c + bc)$

$= a + (b + c + bc) + a(b + c + bc)$

$= a + b + c + bc + ab + ac + abc$

$(a * b) * c = (a + b + ab) * c$

$= a + b + ab + c + (a + b + ab)c$

$= a + b + ab + c + ac + bc + abc$

Therefore,

$a * (b * c) = (a * b) * c, \forall a, b, c \in Q - \{-1\}$

Thus, $*$ is associative on $Q - \{-1\}$.

(ii) Let e be the identity element in I^+ with respect to $*$ such that

$a * e = a = e * a, \forall a \in Q - \{-1\}$

$a * e = a$ and $e * a = a, \forall a \in Q - \{-1\}$

$a + e + ae = a$ and $e + a + ea = a, \forall a \in Q - \{-1\}$

$e + ae = 0$ and $e + ea = 0, \forall a \in Q - \{-1\}$

$e(1 + a) = 0$ and $e(1 + a) = 0, \forall a \in Q - \{-1\}$

$e = 0, \forall a \in Q - \{-1\}$ [because a not equal to -1]

Thus, 0 is the identity element in $Q - \{-1\}$ with respect to $*$.

(iii) Let $a \in Q - \{-1\}$ and $b \in Q - \{-1\}$ be the inverse of a . Then,

$a * b = e = b * a$

$a * b = e$ and $b * a = e$

$$a + b + ab = 0 \text{ and } b + a + ba = 0$$

$$b(1 + a) = -a \text{ Q } \{-1\}$$

$$b = -a/1 + a \text{ Q } \{-1\} \text{ [because } a \text{ not equal to } -1]$$

Thus, $-a/1 + a$ is the inverse of $a \in \mathbb{Q} - \{-1\}$

4. Let $A = \mathbb{R}_0 \times \mathbb{R}$, where \mathbb{R}_0 denote the set of all non-zero real numbers. A binary operation 'O' is defined on A as follows: $(a, b) O (c, d) = (ac, bc + d)$ for all $(a, b), (c, d) \in \mathbb{R}_0 \times \mathbb{R}$.

(i) Show that 'O' is commutative and associative on A

(ii) Find the identity element in A

(iii) Find the invertible element in A.

Solution:

(i) Let $X = (a, b)$ and $Y = (c, d) \in A, \forall a, c \in \mathbb{R}_0$ and $b, d \in \mathbb{R}$

Then, $X O Y = (ac, bc + d)$

And $Y O X = (ca, da + b)$

Therefore,

$$X O Y = Y O X, \forall X, Y \in A$$

Thus, O is not commutative on A.

Now we have to check associativity of O

Let $X = (a, b), Y = (c, d)$ and $Z = (e, f), \forall a, c, e \in \mathbb{R}_0$ and $b, d, f \in \mathbb{R}$

$$X O (Y O Z) = (a, b) O (ce, de + f)$$

$$= (ace, bce + de + f)$$

$$(X O Y) O Z = (ac, bc + d) O (e, f)$$

$$= (ace, (bc + d)e + f)$$

$$= (ace, bce + de + f)$$

Therefore, $X O (Y O Z) = (X O Y) O Z, \forall X, Y, Z \in A$

(ii) Let $E = (x, y)$ be the identity element in A with respect to O, $\forall x \in \mathbb{R}_0$ and $y \in \mathbb{R}$ Such that,

$$X O E = X = E O X, \forall X \in A$$

$$X O E = X \text{ and } E O X = X$$

$$(ax, bx + y) = (a, b) \text{ and } (xa, ya + b) = (a, b)$$

Considering $(ax, bx + y) = (a, b)$

$$ax = a$$

$$x = 1$$

$$\text{And } bx + y = b$$

$$y = 0 \text{ [since } x = 1]$$

Considering $(xa, ya + b) = (a, b)$

$$xa = a$$

$$x = 1$$

And $ya + b = b$

$$y = 0 \text{ [since } x = 1]$$

Therefore $(1, 0)$ is the identity element in A with respect to O .

(iii) Let $F = (m, n)$ be the inverse in $A \forall m \in R_0$ and $n \in R$

$$X O F = E \text{ and } F O X = E$$

$$(am, bm + n) = (1, 0) \text{ and } (ma, na + b) = (1, 0)$$

Considering $(am, bm + n) = (1, 0)$

$$am = 1$$

$$m = 1/a$$

And $bm + n = 0$

$$n = -b/a \text{ [since } m = 1/a]$$

Considering $(ma, na + b) = (1, 0)$

$$ma = 1$$

$$m = 1/a$$

And $na + b = 0$

$$n = -b/a$$

Therefore the inverse of $(a, b) \in A$ with respect to O is $(1/a, -b/a)$

EXERCISE 3.5
PAGE NO: 3.33
1. Construct the composition table for \times_4 on set $S = \{0, 1, 2, 3\}$.
Solution:

 Given that \times_4 on set $S = \{0, 1, 2, 3\}$

Here,

$$1 \times_4 1 = \text{remainder obtained by dividing } 1 \times 1 \text{ by } 4 \\ = 1$$

$$0 \times_4 1 = \text{remainder obtained by dividing } 0 \times 1 \text{ by } 4 \\ = 0$$

$$2 \times_4 3 = \text{remainder obtained by dividing } 2 \times 3 \text{ by } 4 \\ = 2$$

$$3 \times_4 3 = \text{remainder obtained by dividing } 3 \times 3 \text{ by } 4 \\ = 1$$

So, the composition table is as follows:

\times_4	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

2. Construct the composition table for $+_5$ on set $S = \{0, 1, 2, 3, 4\}$
Solution:

$$1 +_5 1 = \text{remainder obtained by dividing } 1 + 1 \text{ by } 5 \\ = 2$$

$$3 +_5 1 = \text{remainder obtained by dividing } 3 + 1 \text{ by } 5 \\ = 2$$

$$4 +_5 1 = \text{remainder obtained by dividing } 4 + 1 \text{ by } 5 \\ = 3$$

So, the composition table is as follows:

$+_5$	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

3. Construct the composition table for \times_6 on set $S = \{0, 1, 2, 3, 4, 5\}$.

Solution:

Here,

$$1 \times_6 1 = \text{remainder obtained by dividing } 1 \times 1 \text{ by } 6 \\ = 1$$

$$3 \times_6 4 = \text{remainder obtained by dividing } 3 \times 4 \text{ by } 6 \\ = 0$$

$$4 \times_6 5 = \text{remainder obtained by dividing } 4 \times 5 \text{ by } 6 \\ = 2$$

So, the composition table is as follows:

\times_6	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	0	2	4
3	0	3	0	3	0	3
4	0	4	2	0	4	2
5	0	5	4	3	2	1

4. Construct the composition table for \times_5 on set $Z_5 = \{0, 1, 2, 3, 4\}$

Solution:

Here,

$$1 \times_5 1 = \text{remainder obtained by dividing } 1 \times 1 \text{ by } 5 \\ = 1$$

$$3 \times_5 4 = \text{remainder obtained by dividing } 3 \times 4 \text{ by } 5 \\ = 2$$

$$4 \times_5 4 = \text{remainder obtained by dividing } 4 \times 4 \text{ by } 5 \\ = 1$$

So, the composition table is as follows:

\times_5	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

5. For the binary operation \times_{10} set $S = \{1, 3, 7, 9\}$, find the inverse of 3.

Solution:

Here,

$$1 \times_{10} 1 = \text{remainder obtained by dividing } 1 \times 1 \text{ by } 10 \\ = 1$$

$$3 \times_{10} 7 = \text{remainder obtained by dividing } 3 \times 7 \text{ by } 10 \\ = 1$$

$$7 \times_{10} 9 = \text{remainder obtained by dividing } 7 \times 9 \text{ by } 10 \\ = 3$$

So, the composition table is as follows:

\times_{10}	1	3	7	9
1	1	3	7	9
3	3	9	1	7
7	7	1	9	3
9	9	7	3	1

From the table we can observe that elements of first row are same as the top-most row.

So, $1 \in S$ is the identity element with respect to \times_{10}

Now we have to find inverse of 3

$$3 \times_{10} 7 = 1$$

So the inverse of 3 is 7.