

EXERCISE 9.1

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1. Test the continuity of the following function at the origin:

$$f(x) = egin{cases} rac{x}{|x|}, x
eq 0 \\ 1, x = 0 \end{cases}$$

Solution:

Given

$$f(x) = \begin{cases} \frac{x}{|x|}, & x \neq 0 \\ 1, & x = 0 \end{cases}$$

Consider LHL at x = 0

$$\lim_{x o0^{-}}f\left(x
ight) =\lim_{h o0}f\left(0-h
ight) =\lim_{h o0}f\left(-h
ight)$$

$$\lim_{h\rightarrow 0}\frac{-h}{|-h|}=\lim_{h\rightarrow 0}\frac{-h}{h}=\lim_{h\rightarrow 0}-1=-1$$

Consider RHL at x = 0

$$\lim_{x\rightarrow0^{+}}f\left(x\right) =\lim_{h\rightarrow0}f\left(0+h\right) =\lim_{h\rightarrow0}f\left(h\right)$$

$$\lim_{h\to 0}\frac{h}{|h|}=\lim_{h\to 0}\frac{h}{h}=\lim_{h\to 0}1=1$$

$$\therefore \lim_{x \to 0^{+}} f\left(x\right) \neq \lim_{x \to 0^{-}} f\left(x\right)$$

Hence LHL ≠ RHL

Hence f(x) is discontinuous at origin.

2. A function f(x) is defined as

$$f(x) = egin{cases} rac{x^2 - x - 6}{x - 3}, & if \ x
eq 3 \\ 5, & if \ x = 3 \end{cases}$$

Show that f(x) is continuous at x = 3.

Solution:

Given



$$f(x) = \begin{cases} \frac{x^2 - x - 6}{x - 3}, & \text{if } x \neq 3\\ 5, & \text{if } x = 3 \end{cases}$$

Consider LHL at x = 3

$$\lim_{x\to 3^{-}} f(x) = \lim_{h\to 0} f(3-h)$$

$$\lim_{h \to 0} \frac{(3-h)^2 - (3-h) - 6}{(3-h) - 3} = \lim_{h \to 0} \frac{9 + h^2 - 6h - 3 + h - 6}{-h} = \lim_{h \to 0} \frac{h^2 - 5h}{-h} = \lim_{h \to 0} (5-h) = 5$$

Consider RHL at x = 3

$$\lim_{x\to 3^+} f(x) = \lim_{h\to 0} f(3+h)$$

$$\lim_{h \to 0} \frac{(3+h)^2 - (3+h) - 6}{(3+h) - 3} = \lim_{h \to 0} \frac{9+h^2 + 6h - 3 - h - 6}{h} = \lim_{h \to 0} \frac{h^2 + 5h}{h} = \lim_{h \to 0} (5+h) = 5$$

Now, f(3) = 5

$$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^-} f(x) = f(3)$$

Hence f(x) is continuous at x = 3

3. A function f(x) is defined as

$$f(x) = \begin{cases} \frac{x^2 - 9}{x - 3}, & \text{if } x \neq 3 \\ 6, & \text{if } x = 3 \end{cases}$$

Show that f(x) is continuous at x = 3.

Solution:

Given

$$f(x) = \begin{cases} \frac{x^2 - 9}{x - 3}; & \text{if } x \neq 3 \\ 6; & \text{if } x = 3 \end{cases}$$

Consider LHL at x = 3



$$\lim_{x o 3^{-}} f(x) = \lim_{h o 0} f(3-h)$$

$$\lim_{h \to 0} \frac{(3-h)^2 - 9}{(3-h) - 3} = \lim_{h \to 0} \frac{3^2 + h^2 - 6h - 9}{3 - h - 3} = \lim_{h \to 0} \frac{h^2 - 6h}{-h} = \lim_{h \to 0} \frac{h(h-6)}{-h} = \lim_{h \to 0} (6-h) = 6$$

Consider RHL at x = 3

$$= \lim_{x \to 3^+} f(x) = \lim_{h \to 0} f(3+h)$$

$$\lim_{x\to 3^+} f(x) = \lim_{h\to 0} f(3+h)$$

$$\lim_{h \to 0} \frac{(3+h)^2 - 9}{3+h-3} = \lim_{h \to 0} \frac{3^2 + h^2 + 6h - 9}{h} = \lim_{h \to 0} \frac{h^2 + 6h}{h} = \lim_{h \to 0} \frac{h(6+h)}{h} = \lim_{h \to 0} (6+h) = 6$$

We have f(3) = 6

$$\lim_{x\to 3^{-}} f(x) = \lim_{x\to 3^{+}} f(x) = f(3)$$

Hence f(x) is continuous at x = 3

$$4. \ f(x) = \begin{cases} \frac{x^2 - 1}{x - 1}; \ if \ x \neq 1 \\ 2; \ if \ x = 1 \end{cases}$$

Find whether f(x) is continuous at x = 1.

Solution:

Given

$$f(x) = \begin{cases} \frac{x^2 - 1}{x - 1}; & if \ x \neq 1 \\ 2; & if \ x = 1 \end{cases}$$

Consider LHL at x = 1

$$\lim_{x\to 1^-}f(x)=\lim_{h\to 0}f(1-h)$$

$$\lim_{h \to 0} \frac{\left(1-h\right)^2 - 1}{\left(1-h\right) - 1} = \lim_{h \to 0} \frac{1+h^2 - 2h - 1}{1-h - 1} = \lim_{h \to 0} \frac{h^2 - 2h}{-h} = \lim_{h \to 0} \frac{h\left(h - 2\right)}{-h} = \lim_{h \to 0} \left(2-h\right) = 2$$



Consider RHL at x = 1

$$\lim_{x \to 1^+} f(x) = \lim_{h \to 0} f(1+h)$$

$$\lim_{h \to 0} \frac{(1+h)^2 - 1}{(1+h) - 1} = \lim_{h \to 0} \frac{1 + h^2 + 2h - 1}{1 + h - 1} = \lim_{h \to 0} \frac{h^2 + 2h}{h} = \lim_{h \to 0} \frac{h(h+2)}{h} = \lim_{h \to 0} (2+h) = 2$$

Given f(1) = 2

$$\therefore \lim_{x \to 1^{-}} f(x) = \lim_{x \to 1^{+}} f(x) = f(1)$$

Hence f(x) is continuous at x = 1

5. If
$$f(x) = \begin{cases} \frac{\sin 3x}{x}; & when \ x \neq 0 \\ 1; & when \ x = 0 \end{cases}$$

Find whether f(x) is continuous at x = 0.

Solution:

Given

$$f(x) = \begin{cases} \frac{\sin 3x}{x}; & when \ x \neq 0 \\ 1; & when \ x = 0 \end{cases}$$

Consider LHL at x = 0

$$\lim_{x o 0^{-}} f(x) = \lim_{h o 0} f(0-h) = \lim_{h o 0} f(-h)$$

$$\lim_{h \to 0} \frac{\sin(-3h)}{-h} = \lim_{h \to 0} \frac{-\sin(3h)}{-h} = \lim_{h \to 0} \frac{3\sin(3h)}{3h} = 3\lim_{h \to 0} \frac{\sin(3h)}{3h} = 3 \cdot 1 = 3$$

Consider RHL at x = 0

$$\lim_{x o 0^+} f\left(x
ight) = \lim_{h o 0} f\left(h
ight)$$

$$\lim_{h \to 0} \frac{\sin 3h}{h} = \lim_{h \to 0} \frac{3\sin 3h}{3h} = 3\lim_{h \to 0} \frac{\sin (3h)}{3h} = 3 \cdot 1 = 3$$



Given f(0) = 1

f(x) to be continuous at x = a

But here,

$$\lim_{x o a^{-}} f(x) = \lim_{x o a^{+}} f(x) = f(a)$$

$$\lim_{x o 0^{-}} f\left(x
ight) = \lim_{x o 0^{+}} f\left(x
ight)
eq f\left(0
ight)$$

Hence f(x) is discontinuous at x = 0

$$6.\,If\,\,f(x)=\left\{ \begin{matrix} e^{\frac{1}{x}};\,\,when\,\,x\neq0\\ 1;\,\,when\,\,x=0 \end{matrix} \right.$$

Find whether f(x) is continuous at x = 0.

Solution:

Given

$$f(x) = egin{pmatrix} e^{rac{1}{x}}, ifx
eq 0 \ 1, ifx = 0 \end{pmatrix}$$

Consider LHL at x = 0

$$\lim_{x o0^{-}}f\left(x
ight) =\lim_{h o0}f\left(0-h
ight) =\lim_{h o0}f\left(-h
ight)$$

$$\lim_{h \to 0} e^{\frac{-1}{h}} = \lim_{h \to 0} \left(\frac{1}{e^{\frac{1}{h}}}\right) = \frac{1}{\lim_{h \to 0} e^{\frac{1}{h}}} = 0$$

Consider RHL at x = 0

$$\lim_{x o 0^+} f\left(x
ight) = \lim_{h o 0} f\left(h
ight)$$

$$\lim_{h\to 0}e^{\frac{1}{h}}=\infty$$

We have f (0) = 1



It is known that for a function f(x) to be continuous at x = a

$$\lim_{x \to a^{-}} f\left(x\right) = \lim_{x \to a^{+}} f\left(x\right) = f\left(a\right)$$

But

$$\lim_{x o 0^{-}} f\left(x
ight)
eq \lim_{x o 0^{+}} f\left(x
ight)$$

Hence f(x) is discontinuous at x = 0

$$7. \ Let \ f(x) = \begin{cases} \frac{1-cosx}{x^2}, \ when \ x \neq 0 \\ 1, \ when \ x = 0. \end{cases} Show \ that \ f(x) \ is \ discontinuous \ at \ x = 0$$

Solution:

Given

$$f(x) = \begin{cases} \frac{1-\cos x}{x^2}, & when \ x \neq 0\\ 1, & when \ x = 0. \end{cases}$$

Consider,

$$\lim_{x o 0} f(x) = \lim_{x o 0} \left(rac{1 - cosx}{x^2}
ight)$$

$$\Rightarrow \lim_{x o 0} f(x) = \lim_{x o 0} \left(rac{2 \sin^2 rac{x}{2}}{x^2}
ight)$$

$$\Rightarrow \lim_{x o 0} f(x) = \lim_{x o 0} \left(rac{2 \sin^2 rac{x}{2}}{4 \left(rac{x^2}{4}
ight)}
ight)$$

$$\Rightarrow \lim_{x \to 0} f(x) = \lim_{x \to 0} \left(\frac{2\left(\sin\frac{x}{2}\right)^2}{4\left(\frac{x}{2}\right)^2} \right)$$



$$\Rightarrow \lim_{x o 0} f(x) = rac{2}{4} \lim_{x o 0} \left(rac{sinrac{x}{2}}{rac{x}{2}}
ight)^2$$

$$\Rightarrow\lim_{x
ightarrow0}f\left(x
ight) =rac{1}{2}\cdot1^{2}=rac{1}{2}$$

We have f(0) = 1

$$\lim_{x \to 0} f(x) \neq f(0)$$

Thus f(x) is discontinuous at x = 0

 $8. \, Show \, \, that \, f(x) = \left\{ \begin{matrix} \frac{x-|x|}{2}, \, \, when \, \, x \neq 0 \\ 2, \, \, when \, x = 0. \end{matrix} \right. \, is \, discontinuous \, at \, x = 0$

Solution:

Given

$$f(x) = egin{cases} rac{x-|x|}{2}, \ when \ x
eq 0 \ 2, \ when x = 0. \end{cases}$$

The given function can be written as

$$f\left(x
ight) = \left\{ egin{array}{l} rac{x-x}{2}, when x > 0 \ rac{x+x}{2}, when x < 0 \ 2, when x = 0 \end{array}
ight.$$

$$f(x) = \left\{ egin{aligned} 0, when x > 0 \ x, when x < 0 \ 2, when x = 0 \end{aligned}
ight.$$

Consider LHL at x = 0

$$\lim_{x o 0^-}f(x)=\lim_{h o 0}f(0-h)=\lim_{h o 0}f(-h)$$

$$=\lim_{h\to 0}\left(-h\right)=0$$



Consider LHL at x = 0

$$\lim_{x o 0^-} f(x) = \lim_{h o 0} f(0-h) = \lim_{h o 0} f(-h)$$

$$=\lim_{h\to 0}\left(-h\right)=0$$

Consider RHL at x = 0

$$\lim_{x o 0^+}f\left(x
ight)=\lim_{h o 0}f\left(0+h
ight)=\lim_{h o 0}f\left(h
ight)$$

$$\lim_{h\to 0} 0 = 0$$

And we have f(0) = 2

$$\lim_{x \to 0^{-}} f\left(x\right) = \lim_{x \to 0^{+}} f\left(x\right)
eq f\left(0\right)$$

Hence, f(x) is discontinuous at x = 0

9. Show that
$$f(x) = \begin{cases} \frac{|x-a|}{x-a}, & when \ x \neq a \\ 1, & when \ x = a. \end{cases}$$
 is discontinuous at $x = a$

Solution:

Given

$$f(x) = egin{cases} rac{|x-a|}{x-a}, \ when \ x
eq a \ 1, \ when x = a. \end{cases}$$

The given function can be written as

$$f(x) = \left\{ egin{aligned} rac{x-a}{x-a}, when x > a \ rac{a-x}{x-a}, when x < a \ 1, when x = a \end{aligned}
ight.$$

$$\Rightarrow f(x) = egin{cases} 1, when x > a \ -1, when x < a \ 1, when x = a \end{cases}$$



$$\Rightarrow f(x) = egin{pmatrix} 1, when x \geq a \ -1, when x < a \end{pmatrix}$$

Consider LHL at x = a

$$\lim_{x o a^{-}} f\left(x
ight) = \lim_{h o 0} f\left(a - h
ight)$$

$$=\lim_{h\to 0} (-1) = -1$$

Consider RHL at x = a

$$\lim_{x \to a^+} f(x) = \lim_{h \to 0} f(a+h)$$

$$\lim_{h\to 0} (1) = 1$$

$$\lim_{x o a^{-}} f(x)
eq \lim_{x o a^{+}} f(x)$$

Thus f(x) is discontinuous at x = a

10. Discuss the continuity of the following functions at the indicated point(s):

(i)
$$f(x) = \begin{cases} |x|\cos\left(\frac{1}{x}\right), & x \neq 0 \ 0, & x = 0 \end{cases}$$
 at $x = 0$

Solution:

Given

$$f(x) = egin{cases} |x|\cos\left(rac{1}{x}
ight), & x
eq 0 \ 0, & x = 0 \end{cases}$$

Consider,

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} |x| \cos\left(\frac{1}{x}\right)$$

$$\Rightarrow \lim_{x \to 0} f\left(x\right) = \lim_{x \to 0} |x| \lim_{x \to 0} \cos\left(\frac{1}{x}\right)$$



$$\Rightarrow\lim_{x
ightarrow0}f\left(x
ight) =0 imes\lim_{x
ightarrow0}\cos\!\left(rac{1}{x}
ight) =0$$

$$\Rightarrow\lim_{x
ightarrow0}f\left(x
ight) =f\left(0
ight)$$

Hence f(x) is continuous at x = 0

(ii)
$$f(x)=egin{cases} x^2\sin\Bigl(rac{1}{x}\Bigr), & x
eq 0 \ 0, & x=0 \end{cases}$$
 at $x=0$

Solution:

Given

$$f(x) = egin{cases} x^2 \sin\Bigl(rac{1}{x}\Bigr), & x
eq 0 \ 0, & x = 0 \end{cases}$$

Consider,

$$\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right) = \lim_{x\to 0} x^2 \lim_{x\to 0} \sin\left(\frac{1}{x}\right) = 0 \times \lim_{x\to 0} \sin\left(\frac{1}{x}\right) = 0$$

$$\Rightarrow\lim_{x
ightarrow0}f\left(x
ight) =f\left(0
ight)$$

Hence f(x) is continuous at x = 0

(iii)
$$f(x) = egin{cases} (x-a)\sin\Bigl(rac{1}{x-a}\Bigr), & x
eq a \ 0, & x = a \end{cases}$$

Solution:

Given

$$f(x) = egin{cases} (x-a)\sin\Bigl(rac{1}{x-a}\Bigr), & x
eq a \ 0, & x = a \end{cases}$$



Now substitute x - a = y in above equation then we get,

$$\lim_{x \to a} (x - a) \sin\left(\frac{1}{x - a}\right) = \lim_{y \to 0} y \sin\left(\frac{1}{y}\right)$$

$$=\lim_{y\to 0}y\lim_{y\to 0}\sin\left(\frac{1}{y}\right)=0\times\lim_{y\to 0}\sin\left(\frac{1}{y}\right)=0$$

$$\Rightarrow \lim_{x \to a} f(x) = f(a) = 0$$

Hence f(x) is continuous at x = a

(iv)
$$f(x) = \left\{egin{array}{ll} rac{e^x-1}{\log(1+2x)}, if & x
eq a \ 7, if & x = 0 \end{array}
ight.$$

Solution:

Given

$$f\left(x
ight) = \left\{ egin{array}{ll} rac{e^{x}-1}{\log(1+2x)}, if & x
eq a \ 7, if & x = 0 \end{array}
ight.$$

Consider,

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{e^x - 1}{\log(1 + 2x)}$$

$$\Rightarrow \lim_{x o 0} f(x) = \lim_{x o 0} rac{e^x - 1}{rac{2x \log(1 + 2x)}{2x}}$$

$$\Rightarrow \lim_{x \to 0} f\left(x\right) = \frac{1}{2} \lim_{x \to 0} \frac{\left(\frac{e^x - 1}{x}\right)}{\left(\frac{\log(1 + 2x)}{2x}\right)}$$



$$\Rightarrow \lim_{x \to 0} f(x) = \frac{1}{2} \times \frac{\left(\lim_{x \to 0} \frac{e^x - 1}{x}\right)}{\left(\lim_{x \to 0} \frac{\log(1 + 2x)}{2x}\right)} = \frac{1}{2} \times \frac{1}{1} = \frac{1}{2}$$

And we have f(0) = 7

$$\Rightarrow \lim_{x \to 0} f(x) \neq f(0)$$

Hence f(x) is discontinuous at x = 0

$$extstyle \left\{ egin{aligned} & rac{1-x^n}{1-x}, & x
eq 1 \ & n-1, & x=1 \end{aligned}
ight. n \in N \ at x = 1$$

Solution:

Given

$$f(x) = \left\{ egin{aligned} rac{1-x^n}{1-x}, & x
eq 1 \ n-1, & x=1 \end{aligned}
ight. n \in N$$

Clearly, f(1) = n - 1

$$\underset{\text{LHL} \,=\, h \to 0}{\lim} \, f(1-h) \,=\, \underset{h \to 0}{\lim} \, \frac{{}^{1-(1-h)^n}}{{}^{1-(1-h)}} \,=\, \underset{h \to 0}{\lim} \, \frac{{}^{1-(1-h)^n}}{h}$$

Using binomial theorem we get

$$(1-h)^n = \sum_{k=0}^n \binom{n}{k} (-h)^k 1^{n-k}$$

$$(1-h)_{n}^{n} = 1 - n h + \binom{n}{2} h^{2} - \dots$$

LHL =

$$\lim_{h\to 0}\frac{^{1-1+nh-\binom{n}{2}h^2+\cdots higher deg\, terms}}{h}=\lim_{h\to 0}\{n-\binom{n}{2}h+\binom{n}{3}h^2-\cdots higher deg\, terms\}$$

Putting h=0 we get,



LHL = n

$$\underset{\mathsf{RHL} \,=\, h \to 0}{\lim} \, f(1+h) \,=\, \underset{h \to 0}{\lim} \, \frac{{}^{1-(1+h)^n}}{{}^{1-(1+h)}} =\, \underset{h \to 0}{\lim} \, \frac{{}^{1-(1+h)^n}}{{}^{-h}}$$

Using binomial expansion as used above we get the following expression

Similarly,

RHL =

$$\lim_{h\to 0} \frac{1-1-nh-\binom{n}{2}h^2-\cdots higher deg terms}{-h} = \lim_{h\to 0} \{n+\binom{n}{2}h+\binom{n}{3}h^2 - higher deg terms\}$$

··· higher deg terms

Putting h=0 we get,

RHL = n

Thus RHL = LHL \neq f (1)

Hence f (x) is discontinuous at x=1

$$\left(\mathsf{Vi}
ight) f\left(x
ight) = \left\{ egin{array}{ll} rac{\left| x^2 - 1
ight|}{x - 1}, for & x
eq 1 \ 2, for & x = 1 \end{array}
ight.$$

Solution:

Given

$$f\left(x
ight) = \left\{egin{array}{l} rac{|x^2-1|}{x-1}, for & x
eq 1 \ 2, for & x = 1 \end{array}
ight.$$

Clearly, f(1) = 2

$$\underset{\text{LHL} = h \to 0}{\lim} \, f(1-h) = \underset{h \to 0}{\lim} \frac{|\, (1-h)^2 - 1|}{1-h-1} = \underset{h \to 0}{\lim} \frac{|\, 1+h^2 - 2h - 1|}{-h} = \underset{h \to 0}{\lim} \frac{|h(h-2)|}{-h}$$

Since h is positive no which is very close to 0

∴ (h-2) is negative and hence h (h-2) is also negative.

$$|h(h-2)| = -h(h-2)$$



$$_{\text{LHL}}=\lim_{h\to 0}\frac{^{-h(h-2)}}{^{-h}}=\ \lim_{h\to 0}(h-2)=\ -2$$

$$\underset{\mathsf{RHL} \, = \, h \to 0}{\lim} \, f(1 + h) \, = \, \lim_{h \to 0} \frac{\mid (1 + h)^2 - 1 \mid}{1 + h - 1} \, = \, \lim_{h \to 0} \frac{\mid 1 + h^2 + 2h - 1 \mid}{h} \, = \, \lim_{h \to 0} \frac{\mid h(h + 2) \mid}{h}$$

Since h is a positive no which is very close to 0

(h+2) is positive and hence h (h-2) is also positive.

$$\therefore RHL = \lim_{h \to 0} \frac{h(h+2)}{h} = \lim_{h \to 0} (h+2) = 2$$

Clearly, LHL ≠ RHL

Hence f(x) is discontinuous at x=1

(Vii)
$$f(x)=egin{cases} rac{2|x|+x^2}{x}, & x
eq 0 \ 0, & x=0 \end{cases}$$

Solution:

Given

$$f\left(x
ight)=\left\{egin{array}{ll} rac{2|x|+x^2}{x}, & x
eq 0\ 0, & x=0 \end{array}
ight.$$

Clearly, f(0) = 0

$$\lim_{h \to 0} f(0 - h) = \lim_{h \to 0} f(-h) = \lim_{h \to 0} \frac{2|-h| + (-h)^2}{-h}$$

$$\lim_{h\to 0} \frac{2h+h^2}{-h} = \lim_{h\to 0} (-2-h) = -2$$

$$\mathsf{RHL} = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} f(h) = \lim_{h \to 0} \frac{2|h| + (h)^2}{h}$$

$$\lim_{h \to 0} \frac{2h + h^2}{h} = \lim_{h \to 0} (2 + h) = 2$$



Clearly, LHL ≠ RHL ≠ f (0)

Hence f(x) is discontinuous at x=0

$$(viii) \; f(x) = \left\{ egin{aligned} |x-a|sinrac{1}{x-a}, \; when \; x
eq a \ 0, \; when \; x = a. \end{aligned}
ight. \; at \; x = a$$

Solution:

Given

$$f(x) = \left\{ egin{aligned} |x-a|sinrac{1}{x-a}, & when & x
eq a \ 0, & when & x = a. \end{aligned}
ight.$$

Clearly, f(a) = 0

$$\underset{\text{LHL} = \ h \rightarrow 0}{\lim} f(a-h) = \underset{h \rightarrow 0}{\lim} |(a-h-a)| \sin \left(\frac{1}{a-h-a}\right)$$

$$\lim_{h\to 0} \lvert -h \rvert \, sin \left(\frac{1}{-h} \right) = \lim_{h\to 0} h \, sin \left(\frac{1}{h} \right) = 0$$

$$\mathsf{RHL} = \lim_{h \to 0} f(a+h) = \ \lim_{h \to 0} |a+h-a| \sin\left(\frac{1}{a+h-a}\right) = \ \lim_{h \to 0} |h| \sin\left(\frac{1}{h}\right)$$

$$\lim_{h\to 0} h sin\left(\frac{1}{h}\right) = 0$$

Since whatever is value of h, sin (1/h) is going to range from -1 to 1

As $h \rightarrow 0$, i.e. approximately 0

Clearly, LHL = RHL = f(a)

Hence f(x) is continuous at x = 0

$$11. \ Show \ that \ f(x) = \begin{cases} 1+x^2, \ if \ 0 \leq x \leq 1 \\ 2-x, \ if \ x > 1. \end{cases} \ is \ discontinuous \ at \ x = 1$$

Solution:

Given



$$f(x) = egin{cases} 1+x^2, if & 0 \leq x \leq 1 \ 2-x, if & x > 1 \end{cases}$$

Consider LHL at x = 1

$$\lim_{x o 1^-} f(x) = \lim_{h o 0} f(1-h)$$

$$=\lim_{h o 0}\left(1+\left(1-h
ight)^{2}
ight)=\lim_{h o 0}\left(2+h^{2}-2h
ight)=2$$

Now again consider RHL at x = 1

$$\lim_{x o 1^+} f\left(x
ight) = \lim_{h o 0} f\left(1+h
ight)$$

$$=\lim_{h\to 0}\left(2-(1+h)\right)=\lim_{h\to 0}\left(1-h\right)=1$$

$$\lim_{x o 1^{-}} f(x)
eq \lim_{x o 1^{+}} f(x)$$

Hence f(x) is discontinuous at x = 1

$$12. \, Show \, that \, f(x) = \begin{cases} \frac{sin3x}{tan2x}, \, \, if \, \, x < 0 \\ \\ \frac{3}{2}, \, \, if \, \, x = 0 \quad \, is \, continuous \, \, at \, \, x = 0 \\ \\ \frac{\log(1+3x)}{e^{2x}-1, \, if \, \, x > 0} \end{cases}$$

Solution:

Given

$$f(x) = egin{cases} rac{sin3x}{tan2x}, & if \ x < 0 \ & rac{3}{2}, & if \ x = 0 \ & rac{\log(1+3x)}{e^{2x}-1, & if \ x > 0 \end{cases}$$



Consider LHL at x = 0

$$\lim_{x \to 0^{-}} f(x) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} f(-h)$$

$$=\lim_{h\to 0}\left(\frac{\sin 3\left(-h\right)}{\tan 2\left(-h\right)}\right)=\lim_{h\to 0}\left(\frac{\sin 3h}{\tan 2h}\right)=\lim_{h\to 0}\left(\frac{\frac{3\sin 3h}{3h}}{\frac{2\tan 2h}{2h}}\right)$$

$$=\frac{\lim_{h\to 0}\left(\frac{3\sin 3h}{3h}\right)}{\lim_{h\to 0}\left(\frac{2\tan 2h}{2h}\right)}=\frac{3\lim_{h\to 0}\left(\frac{\sin 3h}{3h}\right)}{2\lim_{h\to 0}\left(\frac{\tan 2h}{2h}\right)}=\frac{3\times 1}{2\times 1}=\frac{3}{2}$$

Consider RHL at x = 0

$$\lim_{x\to 0^+}f(x)=\lim_{h\to 0}f(0+h)=\lim_{h\to 0}f(h)$$

$$= \lim_{h \to 0} \left(\frac{\log(1+3h)}{e^{2h}-1} \right) = \lim_{h \to 0} \left(\frac{3h \frac{\log(1+3h)}{3h}}{\frac{2h(e^{2h}-1)}{2h}} \right)$$

$$=\frac{3}{2}\lim_{h\to 0}\left(\frac{\frac{\log(1+3h)}{3h}}{\frac{(e^{2h}-1)}{2h}}\right)=\frac{3}{2}\frac{\lim_{h\to 0}\left(\frac{\log(1+3h)}{3h}\right)}{\lim_{h\to 0}\left(\frac{(e^{2h}-1)}{2h}\right)}=\frac{3\times 1}{2\times 1}=\frac{3}{2}$$

We have f(0) = 3/2

$$\lim_{x o 0^{-}}f\left(x
ight) =\lim_{x o 0^{+}}f\left(x
ight) =f\left(0
ight)$$

Thus f(x) is continuous at x = 0

13. Find the value of a for which the function f defined by

$$f(x) = egin{cases} a \sin rac{\pi}{2}(x+1), x \leq 0 \ is \ continuous \ at \ x = 0 \ rac{tan \ x - sin \ x}{x^3}, x > 0 \end{cases}$$

Solution:



Given

$$f(x) = \begin{cases} a \sin \frac{\pi}{2}(x+1), x \leq 0 \\ \frac{\tan x - \sin x}{x^3}, x > 0 \end{cases}$$

Consider LHL at x = 0

$$\lim_{x\to 0^-}f\left(x\right)=\lim_{h\to 0}f\left(0-h\right)=\lim_{h\to 0}f\left(-h\right)=\lim_{h\to 0}a\sin\frac{\pi}{2}(-h+1)=a\sin\frac{\pi}{2}=a$$

Now again consider RHL at x = 0

$$\lim_{x o 0^+}f\left(x
ight)=\lim_{h o 0}f\left(0+h
ight)=\lim_{h o 0}f\left(h
ight)=\lim_{h o 0}rac{ an h-\sin h}{h^3}$$

$$\Rightarrow \lim_{x\rightarrow 0^{+}}f\left(x\right)=\lim_{h\rightarrow 0}\frac{\frac{\sin h}{\cos h}-\sin h}{h^{3}}$$

$$\Rightarrow \lim_{x \to 0^{+}} f(x) = \lim_{h \to 0} \frac{\frac{\sin h}{\cos h} (1 - \cos h)}{h^{3}}$$

$$\Rightarrow\lim_{x
ightarrow0^{+}}f\left(x
ight) =\lim_{h
ightarrow0}rac{\left(1-\cos h
ight) an h}{h^{3}}$$

$$\Rightarrow \lim_{x \rightarrow 0^{+}} f\left(x\right) = \lim_{h \rightarrow 0} \frac{2 \sin^{2} \frac{h}{2} \tan h}{4 \frac{h^{2}}{4} \times h}$$

$$\Rightarrow \lim_{x
ightarrow 0^+} f(x) = rac{2}{4} \lim_{h
ightarrow 0} rac{\sin^2 rac{h}{2} an h}{rac{h^2}{4} imes h}$$

$$\Rightarrow \lim_{x \to 0^+} f(x) = \frac{1}{2} \lim_{h \to 0} \left(\frac{\sin \frac{h}{2}}{\frac{h}{2}} \right)^2 \lim_{h \to 0} \frac{\tan h}{h}$$



$$\Rightarrow \lim_{x\rightarrow 0^{+}}f\left(x\right) =\frac{1}{2}\times 1\times 1$$

$$\Rightarrow\lim_{x
ightarrow0^{+}}f\left(x
ight) =rac{1}{2}$$

If f(x) is continuous at x = 0, then

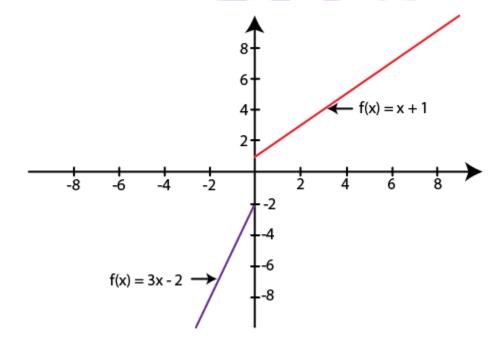
$$\lim_{x o 0^{-}} f\left(x
ight) = \lim_{x o 0^{+}} f\left(x
ight)$$

$$\Rightarrow a = \frac{1}{2}$$

14. Examine the continuity of the function
$$f(x)=egin{cases} 3x-2,\,x\leq 0 \ x+1,x>0 \end{cases}$$
 at $x=0$

Also sketch the graph of this function.

Solution:





Given

$$f(x) = \begin{cases} 3x - 2, & x \le 0 \\ x + 1, & x > 0 \end{cases} at x = 0$$

The given function can be written as

$$f(x) = egin{cases} 3x - 2, x < 0 \ 3(0) - 2, x = 0 \ x + 1, x > 0 \end{cases}$$

$$\Rightarrow f(x) = egin{cases} 3x-2, x < 0 \ -2, x = 0 \ x+1, x > 0 \end{cases}$$

Consider LHL at x = 0

$$=\lim_{x\to 0^{-}}f\left(x\right)=\lim_{h\to 0}f\left(0-h\right)=\lim_{h\to 0}f\left(-h\right)$$

$$\lim_{h\to 0} 3\left(-h\right) - 2 = -2$$

Now again consider RHL at x = 0

$$=\lim_{x\rightarrow0^{+}}f\left(x\right) =\lim_{h\rightarrow0}f\left(0+h\right) =\lim_{h\rightarrow0}f\left(h\right)$$

$$\lim_{h o 0} \left(h + 1
ight) = 1$$

$$\lim_{x \to 0^{-}} f(x)
eq \lim_{x \to 0^{+}} f(x)$$

Hence f(x) is discontinuous at x = 0

15. Discuss the continuity of the function

$$f(x) = \left\{ egin{array}{l} x, x > 0 \\ 1, x = 0 \\ -x, x < 0 \end{array}
ight. at the point x = 0$$

Solution:





Given

$$f(x) = \begin{cases} x, x > 0 \\ 1, x = 0 \\ -x, x < 0 \end{cases} at the point x = 0$$

Consider LHL at x = 0

$$=\lim_{x\rightarrow0^{-}}f\left(x\right) =\lim_{h\rightarrow0}f\left(0-h\right) =\lim_{h\rightarrow0}f\left(-h\right)$$

$$\lim_{h\to 0}-(-h)=0$$

Consider RHL at x = 0

$$\lim_{x \to 0^+} f(x) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} f(h)$$

$$\lim_{h\to 0}\left(h\right)=0$$

And we have f(0) = 1

$$\lim_{x \to 0^{-}} f(x) = \lim_{x \to 0^{+}} f(x)
eq f(0)$$

Hence f(x) is discontinuous at x = 0

16. Discuss the continuity of the function

$$f(x) = \begin{cases} x, 0 \le x < \frac{1}{2} \\ 12, x = \frac{1}{2} \\ 1 - x, \frac{1}{2} < x \le 1 \end{cases} \text{ at the point } x = \frac{1}{2}$$

Solution:

Given

$$f(x) = \begin{cases} x, 0 \le x < \frac{1}{2} \\ 12, x = \frac{1}{2} \\ 1 - x, \frac{1}{2} < x \le 1 \end{cases}$$



Consider LHL at x = 1/2

$$\lim_{x o rac{1}{2}^-}f\left(x
ight)=\lim_{h o 0}f\left(rac{1}{2}-h
ight)$$

$$\lim_{h o 0}\left(rac{1}{2}-h
ight)=rac{1}{2}$$

Again consider RHL at x = 1/2

$$\lim_{x o rac{1}{2}^+} f(x) = \lim_{h o 0} f\left(rac{1}{2} + h
ight)$$

$$\lim_{h o 0} \left(1 - \left(rac{1}{2} + h
ight)
ight) = rac{1}{2}$$

We have $f(1/2) = \frac{1}{2}$

$$\lim_{x \to \frac{1}{2}^{-}} f\left(x\right) = \lim_{x \to \frac{1}{2}^{+}} f\left(x\right) = f\left(\frac{1}{2}\right)$$

Hence f(x) is continuous at $x = \frac{1}{2}$

17. Discuss the continuity of

$$f(x) = egin{cases} 2x - 1, x < 0 \ 2x + 1, x \ge 0 \end{cases} at x = 0$$

Solution:

Given

$$f(x) = \begin{cases} 2x - 1, & x < 0 \\ 2x + 1, & x \ge 0 \end{cases} at x = 0$$

Consider LHL at x = 0

$$\lim_{x o0^{-}}f\left(x
ight) =2\left(0
ight) -1=-1$$



Again consider RHL at x = 0

$$\lim_{x
ightarrow0^{+}}f\left(x
ight) =2\left(0
ight) +1=1$$

$$\Rightarrow \lim_{x\rightarrow 0^{-}}f\left(x\right) \neq \lim_{x\rightarrow 0+}f\left(x\right)$$

Hence f(x) is discontinuous at x = 0

18. For what value of k is the function

$$f(x) = egin{cases} rac{x^2-1}{x-1}, x
eq 1 \ k, x = 1 \end{cases} \ continuous \ at \ x = 1?$$

Solution:

Given

$$f(x) = \begin{cases} \frac{x^2 - 1}{x - 1}, & x \neq 1 \\ k, & x = 1 \end{cases}$$

If f(x) is continuous at x = 1, then

$$\lim_{x\to 1}f(x)=f(1)$$

$$\lim_{x\to 1}\frac{x^2-1}{x-1}=k$$

$$\lim_{x \to 1} \frac{\left(x - 1\right)\left(x + 1\right)}{x - 1} = k$$

$$\lim_{x\to 1}\left(x+1\right)=k$$

$$k = 2$$

19. Determine the value of the constant k so that the function



$$f(x) = egin{cases} rac{x^2-3x+2}{x-1}, x
eq 1 \\ k, x = 1 \end{cases} \ continuous \ at \ x = 1$$

Solution:

Given

$$f(x) = \begin{cases} \frac{x^2 - 3x + 2}{x - 1}, & x \neq 1 \\ k, & x = 1 \end{cases}$$

If f(x) is continuous at x = 1, then

$$\lim_{x\to 1}f(x)=f(1)$$

$$\lim_{x\to 1}\frac{x^2-3x+2}{x-1}=k$$

$$\lim_{x\to 1}\frac{\left(x-2\right)\left(x-1\right)}{x-1}=k$$

$$\lim_{x\to 1}\left(x-2\right)=k$$

$$k = -1$$

20. For what value of k is the function

$$f(x) = egin{cases} rac{\sin 5x}{3x}, & if \ x
eq 0 \ k, & if \ x = 0 \end{cases} continuous \ at \ x = 0?$$

Solution:

Given

$$f(x) = \begin{cases} \frac{\sin 5x}{3x}, & \text{if } x \neq 0 \\ k, & \text{if } x = 0 \end{cases}$$

If f(x) is continuous at x = 0, then we have



$$\lim_{x\to 0}f(x)=f(0)$$

$$\lim_{x \to 0} \frac{\sin 5x}{3x} = k$$

$$\lim_{x\to 0}\frac{5\sin 5x}{3\times 5x}=k$$

$$\frac{5}{3}\lim_{x\to 0}\frac{\sin 5x}{5x}=k$$

$$\frac{5}{3} \times 1 = k$$

$$k=rac{5}{3}$$

21. Determine the value of the constant k so that the function

$$f(x) = egin{cases} kx^2, & if & x \leq 2 \\ 3, & if & x > 2 \end{cases}$$
 is continuous at $x = 2$

Solution:

Given

$$f(x) = \begin{cases} kx^2, \ if \ x \leq 2 \\ 3, \ if \ x > 2 \end{cases}$$

If f(x) is continuous at x = 2, then we have

$$\lim_{x o 2^{-}}f\left(x
ight) =\lim_{x o 2^{+}}f\left(x
ight) =f\left(2
ight)$$

Now,

$$\lim_{x o 2^-}f(x)=\lim_{h o 0}f(2-h)=\lim_{h o 0}k(2-h)^2=4k$$



$$f(2) = 3$$

From the above equation we can write as

$$4k = 3$$

$$\Rightarrow k = rac{3}{4}$$

22. Determine the value of the constant k so that the function

$$f(x) = \left\{ egin{array}{ll} rac{\sin 2x}{5x}, & if \ x
eq 0 \ k, & if \ x = 0 \end{array}
ight. is continuous \ at \ x = 0$$

Solution:

Given

$$f(x) = \begin{cases} \frac{\sin 2x}{5x}, & \text{if } x \neq 0 \\ k, & \text{if } x = 0 \end{cases}$$

If f(x) is continuous at x = 0

$$\lim_{x \to 0} f(x) = f(0)$$

$$\lim_{x\to 0}\frac{\sin 2x}{5x}=k$$

$$\lim_{x\to 0}\frac{2\sin 2x}{5\times 2x}=k$$

$$\frac{2}{5}\lim_{x\to 0}\frac{\sin 2x}{2x}=k$$

$$rac{2}{5} imes 1=k$$

$$k=rac{2}{5}$$



23. Find the values of a so that the function

$$f(x) = \left\{ \begin{matrix} ax+5, \ if \ x \leq 2 \\ x-1, \ if \ x > 2 \end{matrix} \right. \ is \ continuous \ at \ x = 2$$

Solution:

Given

$$f(x) = \begin{cases} ax + 5, & \text{if } x \leq 2 \\ x - 1, & \text{if } x > 2 \end{cases}$$

Consider LHL at x = 2

$$\lim_{h\to 0}a\left(2-h\right)+5=2a+5$$

Now again consider

$$\lim_{x o 2^+} f\left(x
ight) = \lim_{h o 0} f\left(2+h
ight)$$

$$\lim_{h\to 0}\left(2+h-1\right)$$

$$f(2) = a(2) + 5 = 2a + 5$$

Since f(x) is continuous at x = 2 we have

$$\lim_{x o2^{-}}f\left(x
ight) =\lim_{x o2^{+}}f\left(x
ight) =f\left(2
ight)$$

$$2a + 5 = 1$$

$$2a = -4$$

$$a = -2$$