EXERCISE 9.1

1. Test the continuity of the following function at the origin:

$$f(x)=\begin{cases}\frac{x}{|x|}, x\neq 0\\ 1, x=0\end{cases}$$

Solution:

Given

$$f(x)=\begin{cases}\frac{x}{|x|}, x\neq 0\\ 1, x=0\end{cases}$$

Consider LHL at x = 0

$$\lim_{x\to 0^-} f(x)=\lim_{h\to 0} f(0-h)=\lim_{h\to 0} f(-h)$$

$$\lim_{h\to 0}\frac{-h}{|-h|}=\lim_{h\to 0}\frac{-h}{h}=\lim_{h\to 0}-1=-1$$

Consider RHL at x = 0

$$\lim_{x\to 0^+} f(x)=\lim_{h\to 0} f(0+h)=\lim_{h\to 0} f(h)$$

$$\lim_{h\to 0}\frac{h}{|h|}=\lim_{h\to 0}\frac{h}{h}=\lim_{h\to 0}1=1$$

$$\therefore \lim_{x\to 0^+} f(x)\neq \lim_{x\to 0^-} f(x)$$

Hence LHL ≠ RHL

Hence f(x) is discontinuous at origin.

2. A function f(x) is defined as

$$f(x)=\begin{cases}\frac{x^2-x-6}{x-3},\ if\ x\neq 3\\ 5,\ if\ x=3\end{cases}$$

Show that f(x) is continuous at x = 3.

Solution:

Given

$$f(x)=\begin{cases}\frac{x^2-x-6}{x-3}, & if\ x\neq 3\\ 5, & if\ x=3\end{cases}$$

Consider LHL at x = 3

$$\lim_{x\to 3^-} f(x)=\lim_{h\to 0} f(3-h)$$

$$\lim_{h\to 0}\frac{(3-h)^2-(3-h)-6}{(3-h)-3}=\lim_{h\to 0}\frac{9+h^2-6h-3+h-6}{-h}=\lim_{h\to 0}\frac{h^2-5h}{-h}=\lim_{h\to 0}(5-h)=5$$

Consider RHL at x = 3

$$\lim_{x\to 3^+} f(x)=\lim_{h\to 0} f(3+h)$$

$$\lim_{h\to 0}\frac{(3+h)^2-(3+h)-6}{(3+h)-3}=\lim_{h\to 0}\frac{9+h^2+6h-3-h-6}{h}=\lim_{h\to 0}\frac{h^2+5h}{h}=\lim_{h\to 0}(5+h)=5$$

Now, f (3) = 5

$$\therefore \lim_{x\to 3^+} f(x)=\lim_{x\to 3^-} f(x)=f(3)$$

Hence f(x) is continuous at x = 3

3. A function f(x) is defined as

$$f(x)=\begin{cases}\frac{x^2-9}{x-3}, & if\ x\neq 3\\ 6, & if\ x=3\end{cases}$$

Show that f(x) is continuous at x = 3.

Solution:

Given

$$f(x)=\begin{cases}\frac{x^2-9}{x-3}; & if\ x\neq 3\\ 6; & if\ x=3\end{cases}$$

Consider LHL at x = 3

$$\lim_{x \to 3^-} f(x) = \lim_{h \to 0} f(3-h)$$

$$\lim_{h \to 0} \frac{(3-h)^2 - 9}{(3-h) - 3} = \lim_{h \to 0} \frac{3^2 + h^2 - 6h - 9}{3 - h - 3} = \lim_{h \to 0} \frac{h^2 - 6h}{-h} == \lim_{h \to 0} \frac{h(h-6)}{-h} = \lim_{h \to 0} (6-h) = 6$$

Consider RHL at x = 3

$$= \lim_{x \to 3^+} f(x) = \lim_{h \to 0} f(3+h)$$

$$\lim_{x \to 3^+} f(x) = \lim_{h \to 0} f(3+h)$$

$$\lim_{h \to 0} \frac{(3+h)^2 - 9}{3 + h - 3} = \lim_{h \to 0} \frac{3^2 + h^2 + 6h - 9}{h} = \lim_{h \to 0} \frac{h^2 + 6h}{h} = \lim_{h \to 0} \frac{h(6+h)}{h} = \lim_{h \to 0} (6+h) = 6$$

We have f (3) = 6

$$\therefore \lim_{x \to 3^-} f(x) = \lim_{x \to 3^+} f(x) = f(3)$$

Hence f(x) is continuous at x = 3

$$4.\ f(x) = \begin{cases} \frac{x^2-1}{x-1};\ if\ x \neq 1 \\ 2;\ if\ x = 1 \end{cases}$$

Find whether f(x) is continuous at x = 1.

Solution:

Given

$$f(x) = \begin{cases} \frac{x^2-1}{x-1};\ if\ x \neq 1 \\ 2;\ if\ x = 1 \end{cases}$$

Consider LHL at x = 1

$$\lim_{x \to 1^-} f(x) = \lim_{h \to 0} f(1-h)$$

$$\lim_{h \to 0} \frac{(1-h)^2 - 1}{(1-h) - 1} = \lim_{h \to 0} \frac{1 + h^2 - 2h - 1}{1 - h - 1} = \lim_{h \to 0} \frac{h^2 - 2h}{-h} = \lim_{h \to 0} \frac{h(h-2)}{-h} = \lim_{h \to 0} (2-h) = 2$$

Consider RHL at x = 1

$$\lim_{x\to1^+} f(x) = \lim_{h\to0} f(1+h)$$

$$\lim_{h\to0} \frac{(1+h)^2-1}{(1+h)-1} = \lim_{h\to0} \frac{1+h^2+2h-1}{1+h-1} = \lim_{h\to0} \frac{h^2+2h}{h} = \lim_{h\to0} \frac{h(h+2)}{h} = \lim_{h\to0} (2+h) = 2$$

Given f (1) = 2

$$\therefore \lim_{x\to1^-} f(x) = \lim_{x\to1^+} f(x) = f(1)$$

Hence f(x) is continuous at x = 1

5. $$If\ f(x) = \begin{cases} \frac{sin3x}{x};\ when\ x \neq 0 \\ 1;\ when\ x = 0 \end{cases}$$

Find whether f(x) is continuous at x = 0.

Solution:

Given

$$f(x) = \begin{cases} \frac{sin3x}{x};\ when\ x \neq 0 \\ 1;\ when\ x = 0 \end{cases}$$

Consider LHL at x = 0

$$\lim_{x\to0^-} f(x) = \lim_{h\to0} f(0-h) = \lim_{h\to0} f(-h)$$

$$\lim_{h\to0} \frac{\sin(-3h)}{-h} = \lim_{h\to0} \frac{-\sin(3h)}{-h} = \lim_{h\to0} \frac{3\sin(3h)}{3h} = 3\lim_{h\to0} \frac{\sin(3h)}{3h} = 3\cdot1 = 3$$

Consider RHL at x = 0

$$\lim_{x\to0^+} f(x) = \lim_{h\to0} f(h)$$

$$\lim_{h\to0} \frac{\sin 3h}{h} = \lim_{h\to0} \frac{3\sin 3h}{3h} = 3\lim_{h\to0} \frac{\sin(3h)}{3h} = 3\cdot1 = 3$$

Given f (0) = 1

f (x) to be continuous at x = a

But here,

$$\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = f(a)$$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) \neq f(0)$$

Hence f (x) is discontinuous at x = 0

6. $If\ f(x) = \begin{cases} e^{\frac{1}{x}};\ when\ x \neq 0 \\ 1;\ when\ x = 0 \end{cases}$

Find whether f(x) is continuous at x = 0.

Solution:

Given

$$f(x) = \begin{pmatrix} e^{\frac{1}{x}}, if x \neq 0 \\ 1, if x = 0 \end{pmatrix}$$

Consider LHL at x = 0

$$\lim_{x\to 0^-} f(x) = \lim_{h\to 0} f(0-h) = \lim_{h\to 0} f(-h)$$

$$\lim_{h\to 0} e^{\frac{-1}{h}} = \lim_{h\to 0}\left(\frac{1}{e^{\frac{1}{h}}}\right) = \frac{1}{\lim_{h\to 0} e^{\frac{1}{h}}} = 0$$

Consider RHL at x = 0

$$\lim_{x\to 0^+} f(x) = \lim_{h\to 0} f(h)$$

$$\lim_{h\to 0} e^{\frac{1}{h}} = \infty$$

We have f (0) = 1

It is known that for a function f (x) to be continuous at x = a

$$\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = f(a)$$

But

$$\lim_{x\to 0^-} f(x) \neq \lim_{x\to 0^+} f(x)$$

Hence f(x) is discontinuous at x = 0

7. $Let\ f(x) = \begin{cases} \frac{1-cosx}{x^2}, & when\ x \neq 0 \\ 1, & whenx = 0. \end{cases}$ $Show\ that\ f(x)\ is\ discontinuous\ at\ x = 0$

Solution:

Given

$$f(x) = \begin{cases} \frac{1-cosx}{x^2}, & when\ x \neq 0 \\ 1, & whenx = 0. \end{cases}$$

Consider,

$$\lim_{x\to 0} f(x) = \lim_{x\to 0}\left(\frac{1-\ cosx}{x^2}\right)$$

$$\Rightarrow \lim_{x\to 0} f(x) = \lim_{x\to 0}\left(\frac{2\sin^2\frac{x}{2}}{x^2}\right)$$

$$\Rightarrow \lim_{x\to 0} f(x) = \lim_{x\to 0}\left(\frac{2\sin^2\frac{x}{2}}{4\left(\frac{x^2}{4}\right)}\right)$$

$$\Rightarrow \lim_{x\to 0} f(x) = \lim_{x\to 0}\left(\frac{2\left(\sin\frac{x}{2}\right)^2}{4\left(\frac{x}{2}\right)^2}\right)$$

$$\Rightarrow \lim_{x \to 0} f(x) = \frac{2}{4} \lim_{x \to 0} \left(\frac{sin \frac{x}{2}}{\frac{x}{2}} \right)^2$$

$$\Rightarrow \lim_{x \to 0} f(x) = \frac{1}{2} \cdot 1^2 = \frac{1}{2}$$

We have f (0) = 1

$$\lim_{x \to 0} f(x) \neq f(0)$$

Thus f (x) is discontinuous at x = 0

8. $Show\ that\ f(x) = \begin{cases} \frac{x-|x|}{2},\ when\ x \neq 0 \\ 2,\ whenx = 0. \end{cases}\ is\ discontinuous\ at\ x = 0$

Solution:

Given

$$f(x) = \begin{cases} \frac{x-|x|}{2},\ when\ x \neq 0 \\ 2,\ whenx = 0. \end{cases}$$

The given function can be written as

$$f(x) = \begin{cases} \frac{x-x}{2}, whenx > 0 \\ \frac{x+x}{2}, whenx < 0 \\ 2, whenx = 0 \end{cases}$$

$$f(x) = \begin{cases} 0, whenx > 0 \\ x, whenx < 0 \\ 2, whenx = 0 \end{cases}$$

Consider LHL at x = 0

$$\lim_{x \to 0^-} f(x) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} f(-h)$$

$$= \lim_{h \to 0} (-h) = 0$$

Consider LHL at x = 0

$$\lim_{x \to 0^-} f(x) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} f(-h)$$

$$= \lim_{h \to 0} (-h) = 0$$

Consider RHL at x = 0

$$\lim_{x \to 0^+} f(x) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} f(h)$$

$$\lim_{h \to 0} 0 = 0$$

And we have f (0) = 2

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) \neq f(0)$$

Hence, f(x) is discontinuous at x = 0

9. $\boldsymbol{Show\ that\ f(x) = \begin{cases} \frac{|x-a|}{x-a}, & when\ x \neq a \\ 1, & whenx = a. \end{cases}\ is\ discontinuous\ at\ x = a}$

Solution:

Given

$$f(x) = \begin{cases} \frac{|x-a|}{x-a}, & when\ x \neq a \\ 1, & whenx = a. \end{cases}$$

The given function can be written as

$$f(x) = \begin{cases} \frac{x-a}{x-a}, whenx > a \\ \frac{a-x}{x-a}, whenx < a \\ 1, whenx = a \end{cases}$$

$$\Rightarrow f(x) = \begin{cases} 1, whenx > a \\ -1, whenx < a \\ 1, whenx = a \end{cases}$$

$$\Rightarrow f(x)=\begin{pmatrix}1, when x \geq a\\ -1, when x < a\end{pmatrix}$$

Consider LHL at x = a

$$\lim_{x\to a^-} f(x)=\lim_{h\to 0} f(a-h)$$

$$=\lim_{h\to 0}(-1)=-1$$

Consider RHL at x = a

$$\lim_{x\to a^+} f(x)=\lim_{h\to 0} f(a+h)$$

$$\lim_{h\to 0}(1)=1$$

$$\lim_{x\to a^-} f(x)\neq \lim_{x\to a^+} f(x)$$

Thus f (x) is discontinuous at x = a

10. Discuss the continuity of the following functions at the indicated point(s):

(i) $f(x)=\begin{cases}|x|\cos\left(\frac{1}{x}\right), & x\neq 0\\ 0, & x=0\end{cases}$ $at x=0$

Solution:

Given

$$f(x)=\begin{cases}|x|\cos\left(\frac{1}{x}\right), & x\neq 0\\ 0, & x=0\end{cases}$$

Consider,

$$\lim_{x\to 0} f(x)=\lim_{x\to 0}|x|\cos\left(\frac{1}{x}\right)$$

$$\Rightarrow \lim_{x\to 0} f(x)=\lim_{x\to 0}|x|\lim_{x\to 0}\cos\left(\frac{1}{x}\right)$$

$$\Rightarrow \lim_{x\to 0} f(x) = 0 \times \lim_{x\to 0} \cos\left(\frac{1}{x}\right) = 0$$

$$\Rightarrow \lim_{x\to 0} f(x) = f(0)$$

Hence f(x) is continuous at x = 0

(ii) $f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases} at x = 0$

Solution:

Given

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Consider,

$$\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right) = \lim_{x\to 0} x^2 \lim_{x\to 0} \sin\left(\frac{1}{x}\right) = 0 \times \lim_{x\to 0} \sin\left(\frac{1}{x}\right) = 0$$

$$\Rightarrow \lim_{x\to 0} f(x) = f(0)$$

Hence f(x) is continuous at x = 0

(iii) $f(x) = \begin{cases} (x-a) \sin\left(\frac{1}{x-a}\right), & x \neq a \\ 0, & x = a \end{cases} at x = a$

Solution:

Given

$$f(x) = \begin{cases} (x-a) \sin\left(\frac{1}{x-a}\right), & x \neq a \\ 0, & x = a \end{cases}$$

Now substitute x – a = y in above equation then we get,

$$\lim_{x \to a} (x-a)\sin\left(\frac{1}{x-a}\right) = \lim_{y \to 0} y \sin\left(\frac{1}{y}\right)$$

$$= \lim_{y \to 0} y \lim_{y \to 0} \sin\left(\frac{1}{y}\right) = 0 \times \lim_{y \to 0} \sin\left(\frac{1}{y}\right) = 0$$

$$\Rightarrow \lim_{x \to a} f(x) = f(a) = 0$$

Hence f (x) is continuous at x = a

(iv) $f(x) = \begin{cases} \frac{e^x-1}{\log(1+2x)}, if & x \neq a \\ 7, if & x = 0 \end{cases} at x = 0$

Solution:

Given

$$f(x) = \begin{cases} \frac{e^x-1}{\log(1+2x)}, if & x \neq a \\ 7, if & x = 0 \end{cases}$$

Consider,

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{e^x-1}{\log(1+2x)}$$

$$\Rightarrow \lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{e^x-1}{\frac{2x\log(1+2x)}{2x}}$$

$$\Rightarrow \lim_{x \to 0} f(x) = \frac{1}{2} \lim_{x \to 0} \frac{\left(\frac{e^x-1}{x}\right)}{\left(\frac{\log(1+2x)}{2x}\right)}$$

$$\Rightarrow \lim_{x\to 0} f(x) = \frac{1}{2} \times \frac{\left(\lim_{x\to 0} \frac{e^x-1}{x}\right)}{\left(\lim_{x\to 0} \frac{\log(1+2x)}{2x}\right)} = \frac{1}{2} \times \frac{1}{1} = \frac{1}{2}$$

And we have f (0) = 7

$$\Rightarrow \lim_{x\to 0} f(x) \neq f(0)$$

Hence f (x) is discontinuous at x = 0

(v) $f(x) = \begin{cases} \frac{1-x^n}{1-x}, & x \neq 1 \\ n-1, & x = 1 \end{cases} n \in N \; at x = 1$

Solution:

Given

$$f(x) = \begin{cases} \frac{1-x^n}{1-x}, & x \neq 1 \\ n-1, & x = 1 \end{cases} n \in N$$

Clearly, f (1) = n – 1

LHL = $\lim_{h\to 0} f(1-h) = \lim_{h\to 0} \frac{1-(1-h)^n}{1-(1-h)} = \lim_{h\to 0} \frac{1-(1-h)^n}{h}$

Using binomial theorem we get

$$(1-h)^n = \sum_{k=0}^{n} \binom{n}{k} (-h)^k 1^{n-k}$$

$(1–h)^n = 1 – n h + \binom{n}{2} h^2 –$ ....

LHL =

$\lim_{h\to 0} \frac{1-1+nh-\binom{n}{2}h^2+\cdots \text{higher deg terms}}{h} = \lim_{h\to 0}\{n - \binom{n}{2}h + \binom{n}{3}h^2 - \cdots \text{higher deg terms}\}$

Putting h=0 we get,

LHL = n

RHL = $\lim\limits_{h\to 0} f(1+h) = \lim\limits_{h\to 0}\frac{1-(1+h)^n}{1-(1+h)} = \lim\limits_{h\to 0}\frac{1-(1+h)^n}{-h}$

Using binomial expansion as used above we get the following expression

Similarly,

RHL =

$\lim\limits_{h\to 0}\frac{1-1-nh-\binom{n}{2}h^2-\cdots \text{higher deg terms}}{-h} = \lim\limits_{h\to 0}\{n+\binom{n}{2}h+\binom{n}{3}h^2 - \cdots \text{higher deg terms}\}$

Putting h=0 we get,

RHL = n

Thus RHL = LHL ≠ f (1)

Hence f (x) is discontinuous at x=1

(vi) $f(x) = \begin{cases} \frac{|x^2-1|}{x-1}, for & x \neq 1 \\ 2, for & x = 1 \end{cases}$ $at x = 1$

Solution:

Given

$$f(x) = \begin{cases} \frac{|x^2-1|}{x-1}, for & x \neq 1 \\ 2, for & x = 1 \end{cases}$$

Clearly, f (1) = 2

LHL = $\lim\limits_{h\to 0} f(1-h) = \lim\limits_{h\to 0}\frac{|(1-h)^2-1|}{1-h-1} = \lim\limits_{h\to 0}\frac{|1+h^2-2h-1|}{-h} = \lim\limits_{h\to 0}\frac{|h(h-2)|}{-h}$

Since h is positive no which is very close to 0

∴ (h–2) is negative and hence h (h–2) is also negative.

|h (h–2)| = –h (h–2)

LHL = $\lim\limits_{h\to0}\frac{-h(h-2)}{-h} = \lim\limits_{h\to0}(h-2) = -2$

RHL = $\lim\limits_{h\to0} f(1+h) = \lim\limits_{h\to0}\frac{|(1+h)^2-1|}{1+h-1} = \lim\limits_{h\to0}\frac{|1+h^2+2h-1|}{h} = \lim\limits_{h\to0}\frac{|h(h+2)|}{h}$

Since h is a positive no which is very close to 0

(h+2) is positive and hence h (h–2) is also positive.

∴ |h (h+2)| = h (h+2)

∴ RHL = $\lim\limits_{h\to0}\frac{h(h+2)}{h} = \lim\limits_{h\to0}(h+2) = 2$

Clearly, LHL ≠ RHL

Hence f(x) is discontinuous at x=1

(vii) $f(x) = \begin{cases} \frac{2|x|+x^2}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases} at x = 0$

Solution:

Given

$$f(x) = \begin{cases} \frac{2|x|+x^2}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Clearly, f (0) = 0

LHL = $\lim\limits_{h\to0} f(0-h) = \lim\limits_{h\to0} f(-h) = \lim\limits_{h\to0}\frac{2|-h|+(-h)^2}{-h}$

$= \lim\limits_{h\to0}\frac{2h+h^2}{-h} = \lim\limits_{h\to0}(-2-h) = -2$

RHL = $\lim\limits_{h\to0} f(0+h) = \lim\limits_{h\to0} f(h) = \lim\limits_{h\to0}\frac{2|h|+(h)^2}{h}$

$= \lim\limits_{h\to0}\frac{2h+h^2}{h} = \lim\limits_{h\to0}(2+h) = 2$

Clearly, LHL ≠ RHL ≠ f (0)

Hence f(x) is discontinuous at x=0

$$(viii)\ f(x) = \begin{cases} |x-a| sin \frac{1}{x-a},\ when\ x \neq a \\ 0,\ when x = a. \end{cases}\ at\ x = a$$

Solution:
Given

$$f(x) = \begin{cases} |x-a| sin \frac{1}{x-a},\ when\ x \neq a \\ 0,\ when x = a. \end{cases}$$

Clearly, f (a) = 0

LHL = $\lim_{h\to 0} f(a-h) = \lim_{h\to 0} |(a-h-a)| \sin\left(\frac{1}{a-h-a}\right)$

$= \lim_{h\to 0} |-h| \sin\left(\frac{1}{-h}\right) = \lim_{h\to 0} h \sin\left(\frac{1}{h}\right) = 0$

RHL = $\lim_{h\to 0} f(a+h) = \lim_{h\to 0} |a+h-a| \sin\left(\frac{1}{a+h-a}\right) = \lim_{h\to 0} |h| \sin\left(\frac{1}{h}\right)$

$= \lim_{h\to 0} h\sin\left(\frac{1}{h}\right) = 0$

Since whatever is value of h, sin (1/h) is going to range from –1 to 1

As h→ 0, i.e. approximately 0

Clearly, LHL = RHL = f (a)

Hence f (x) is continuous at x = 0

$$11.\ Show\ that\ f(x) = \begin{cases} 1 + x^2,\ if\ 0 \leq x \leq 1 \\ 2 - x,\ if\ x > 1. \end{cases}\ is\ discontinuous\ at\ x = 1$$

Solution:
Given

$$f(x)=\begin{cases}1+x^2, if & 0\le x\le 1\\ 2-x, if & x>1\end{cases}$$

Consider LHL at x = 1

$$\lim_{x\to 1^-} f(x)=\lim_{h\to 0} f(1-h)$$

$$=\lim_{h\to 0}\left(1+(1-h)^2\right)=\lim_{h\to 0}\left(2+h^2-2h\right)=2$$

Now again consider RHL at x = 1

$$\lim_{x\to 1^+} f(x)=\lim_{h\to 0} f(1+h)$$

$$=\lim_{h\to 0}(2-(1+h))=\lim_{h\to 0}(1-h)=1$$

$$\lim_{x\to 1^-} f(x)\ne\lim_{x\to 1^+} f(x)$$

Hence f (x) is discontinuous at x = 1

$$12.\ Show\ that\ f(x)=\begin{cases}\frac{sin3x}{tan2x},\ if\ x<0\\ \frac{3}{2},\ if\ x=0\\ \frac{\log(1+3x)}{e^{2x}-1,\ if\ x>0}\end{cases}\quad is\ continuous\ at\ x=0$$

Solution:

Given

$$f(x)=\begin{cases}\frac{sin3x}{tan2x},\ if\ x<0\\ \frac{3}{2},\ if\ x=0\\ \frac{\log(1+3x)}{e^{2x}-1,\ if\ x>0}\end{cases}$$

Consider LHL at x = 0

$$\lim_{x \to 0^-} f(x) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} f(-h)$$

$$= \lim_{h \to 0} \left(\frac{\sin 3(-h)}{\tan 2(-h)} \right) = \lim_{h \to 0} \left(\frac{\sin 3h}{\tan 2h} \right) = \lim_{h \to 0} \left(\frac{\frac{3 \sin 3h}{3h}}{\frac{2 \tan 2h}{2h}} \right)$$

$$= \frac{\lim_{h \to 0} \left(\frac{3 \sin 3h}{3h} \right)}{\lim_{h \to 0} \left(\frac{2 \tan 2h}{2h} \right)} = \frac{3 \lim_{h \to 0} \left(\frac{\sin 3h}{3h} \right)}{2 \lim_{h \to 0} \left(\frac{\tan 2h}{2h} \right)} = \frac{3 \times 1}{2 \times 1} = \frac{3}{2}$$

Consider RHL at x = 0

$$\lim_{x \to 0^+} f(x) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} f(h)$$

$$= \lim_{h \to 0} \left(\frac{\log(1+3h)}{e^{2h}-1} \right) = \lim_{h \to 0} \left(\frac{3h \frac{\log(1+3h)}{3h}}{\frac{2h(e^{2h}-1)}{2h}} \right)$$

$$= \frac{3}{2} \lim_{h \to 0} \left(\frac{\frac{\log(1+3h)}{3h}}{\frac{(e^{2h}-1)}{2h}} \right) = \frac{3}{2} \frac{\lim_{h \to 0} \left(\frac{\log(1+3h)}{3h} \right)}{\lim_{h \to 0} \left(\frac{(e^{2h}-1)}{2h} \right)} = \frac{3 \times 1}{2 \times 1} = \frac{3}{2}$$

We have f(0) = 3/2

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0)$$

Thus f (x) is continuous at x = 0

13. Find the value of a for which the function f defined by

$$f(x) = \begin{cases} a \sin \frac{\pi}{2}(x+1), x \le 0 \\ \frac{\tan\ x - \sin\ x}{x^3}, x > 0 \end{cases} \quad is\ continuous\ at\ x = 0$$

Solution:

Given

$$f(x)=\begin{cases} a\sin\frac{\pi}{2}(x+1), x\le 0 \\ \frac{\tan x-\sin x}{x^3}, x>0 \end{cases}$$

Consider LHL at x = 0

$$\lim_{x\to 0^-} f(x)=\lim_{h\to 0} f(0-h)=\lim_{h\to 0} f(-h)=\lim_{h\to 0} a\sin\frac{\pi}{2}(-h+1)=a\sin\frac{\pi}{2}=a$$

Now again consider RHL at x = 0

$$\lim_{x\to 0^+} f(x)=\lim_{h\to 0} f(0+h)=\lim_{h\to 0} f(h)=\lim_{h\to 0}\frac{\tan h-\sin h}{h^3}$$

$$\Rightarrow \lim_{x\to 0^+} f(x)=\lim_{h\to 0}\frac{\frac{\sin h}{\cos h}-\sin h}{h^3}$$

$$\Rightarrow \lim_{x\to 0^+} f(x)=\lim_{h\to 0}\frac{\frac{\sin h}{\cos h}(1-\cos h)}{h^3}$$

$$\Rightarrow \lim_{x\to 0^+} f(x)=\lim_{h\to 0}\frac{(1-\cos h)\tan h}{h^3}$$

$$\Rightarrow \lim_{x\to 0^+} f(x)=\lim_{h\to 0}\frac{2\sin^2\frac{h}{2}\tan h}{4\frac{h^2}{4}\times h}$$

$$\Rightarrow \lim_{x\to 0^+} f(x)=\frac{2}{4}\lim_{h\to 0}\frac{\sin^2\frac{h}{2}\tan h}{\frac{h^2}{4}\times h}$$

$$\Rightarrow \lim_{x\to 0^+} f(x)=\frac{1}{2}\lim_{h\to 0}\left(\frac{\sin\frac{h}{2}}{\frac{h}{2}}\right)^2\lim_{h\to 0}\frac{\tan h}{h}$$

$$\Rightarrow \lim_{x \to 0^+} f(x) = \frac{1}{2} \times 1 \times 1$$

$$\Rightarrow \lim_{x \to 0^+} f(x) = \frac{1}{2}$$

If f (x) is continuous at x = 0, then

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x)$$

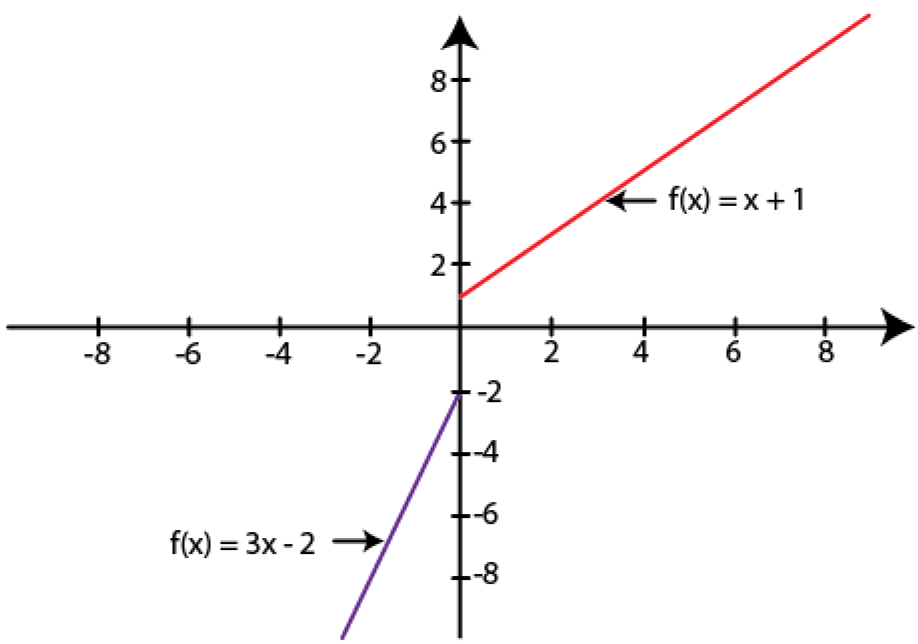
$$\Rightarrow a = \frac{1}{2}$$

14. Examine the continuity of the function

$$f(x) = \begin{cases} 3x - 2, x \leq 0 \\ x + 1, x > 0 \end{cases} \quad at\ x = 0$$

Also sketch the graph of this function.

Solution:



Given

$$f(x) = \begin{cases} 3x - 2, x \le 0 \\ x + 1, x > 0 \end{cases} \; at \; x = 0$$

The given function can be written as

$$f(x) = \begin{cases} 3x - 2, x < 0 \\ 3(0) - 2, x = 0 \\ x + 1, x > 0 \end{cases}$$

$$\Rightarrow f(x) = \begin{cases} 3x - 2, x < 0 \\ -2, x = 0 \\ x + 1, x > 0 \end{cases}$$

Consider LHL at x = 0

$$= \lim_{x \to 0^-} f(x) = \lim_{h \to 0} f(0 - h) = \lim_{h \to 0} f(-h)$$

$$\lim_{h \to 0} 3(-h) - 2 = -2$$

Now again consider RHL at x = 0

$$= \lim_{x \to 0^+} f(x) = \lim_{h \to 0} f(0 + h) = \lim_{h \to 0} f(h)$$

$$\lim_{h \to 0} (h + 1) = 1$$

$$\therefore \lim_{x \to 0^-} f(x) \ne \lim_{x \to 0^+} f(x)$$

Hence f (x) is discontinuous at x = 0

15. Discuss the continuity of the function

$$f(x) = \begin{cases} x, x > 0 \\ 1, x = 0 \\ -x, x < 0 \end{cases} \quad at \; the \; point \; x = 0$$

Solution:

Given

$$f(x)=\begin{cases} x, x>0 \\ 1, x=0 \\ -x, x<0 \end{cases} \quad at\ the\ point\ x=0$$

Consider LHL at x = 0

$$= \lim_{x\to 0^-} f(x) = \lim_{h\to 0} f(0-h) = \lim_{h\to 0} f(-h)$$

$$\lim_{h\to 0} -(-h) = 0$$

Consider RHL at x = 0

$$\lim_{x\to 0^+} f(x) = \lim_{h\to 0} f(0+h) = \lim_{h\to 0} f(h)$$

$$\lim_{h\to 0} (h) = 0$$

And we have f (0) = 1

$$\therefore \lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) \neq f(0)$$

Hence f (x) is discontinuous at x = 0

16. Discuss the continuity of the function

$$f(x)=\begin{cases} x, 0\le x<\frac{1}{2} \\ 12, x=\frac{1}{2} \\ 1-x, \frac{1}{2}<x\le 1 \end{cases} \quad at\ the\ point\ x=\frac{1}{2}$$

Solution:

Given

$$f(x)=\begin{cases} x, 0\le x<\frac{1}{2} \\ 12, x=\frac{1}{2} \\ 1-x, \frac{1}{2}<x\le 1 \end{cases}$$

Consider LHL at x = ½

$$\lim_{x \to \frac{1}{2}^-} f(x) = \lim_{h \to 0} f\left(\frac{1}{2} - h\right)$$

$$\lim_{h \to 0} \left(\frac{1}{2} - h\right) = \frac{1}{2}$$

Again consider RHL at x = ½

$$\lim_{x \to \frac{1}{2}^+} f(x) = \lim_{h \to 0} f\left(\frac{1}{2} + h\right)$$

$$\lim_{h \to 0} \left(1 - \left(\frac{1}{2} + h\right)\right) = \frac{1}{2}$$

We have f (1/2) = ½

$$\therefore \lim_{x \to \frac{1}{2}^-} f(x) = \lim_{x \to \frac{1}{2}^+} f(x) = f\left(\frac{1}{2}\right)$$

Hence f (x) is continuous at x = ½

17. Discuss the continuity of

$$f(x) = \begin{cases} 2x - 1, x < 0 \\ 2x + 1, x \geq 0 \end{cases} \; atx = 0$$

Solution:

Given

$$f(x) = \begin{cases} 2x - 1, x < 0 \\ 2x + 1, x \geq 0 \end{cases} \; atx = 0$$

Consider LHL at x = 0

$$\lim_{x \to 0^-} f(x) = 2(0) - 1 = -1$$

Again consider RHL at x = 0

$$\lim_{x\to 0^+} f(x) = 2(0) + 1 = 1$$

$$\Rightarrow \lim_{x\to 0^-} f(x) \neq \lim_{x\to 0+} f(x)$$

Hence f (x) is discontinuous at x = 0

18. For what value of k is the function

$$f(x) = \begin{cases} \frac{x^2-1}{x-1}, x \neq 1 \\ k, x = 1 \end{cases} \quad continuous\ at\ x = 1?$$

Solution:

Given

$$f(x) = \begin{cases} \frac{x^2-1}{x-1}, x \neq 1 \\ k, x = 1 \end{cases}$$

If f (x) is continuous at x = 1, then

$$\lim_{x\to 1} f(x) = f(1)$$

$$\lim_{x\to 1} \frac{x^2-1}{x-1} = k$$

$$\lim_{x\to 1} \frac{(x-1)(x+1)}{x-1} = k$$

$$\lim_{x\to 1} (x+1) = k$$

$$k = 2$$

19. Determine the value of the constant k so that the function

$$f(x)=\begin{cases}\frac{x^2-3x+2}{x-1}, x\neq 1\\ k, x=1\end{cases}\ continuous\ at\ x=1$$

Solution:

Given

$$f(x)=\begin{cases}\frac{x^2-3x+2}{x-1}, x\neq 1\\ k, x=1\end{cases}$$

If f (x) is continuous at x = 1, then

$$\lim_{x\to 1} f(x)=f(1)$$

$$\lim_{x\to 1}\frac{x^2-3x+2}{x-1}=k$$

$$\lim_{x\to 1}\frac{(x-2)(x-1)}{x-1}=k$$

$$\lim_{x\to 1}(x-2)=k$$

$$k=-1$$

20. For what value of k is the function

$$f(x)=\begin{cases}\frac{sin\ 5x}{3x},\ if\ x\neq 0\\ k,\ if\ x=0\end{cases}\ continuous\ at\ x=0?$$

Solution:

Given

$$f(x)=\begin{cases}\frac{sin\ 5x}{3x},\ if\ x\neq 0\\ k,\ if\ x=0\end{cases}$$

If f (x) is continuous at x = 0, then we have

$$\lim_{x \to 0} f(x) = f(0)$$

$$\lim_{x \to 0} \frac{\sin 5x}{3x} = k$$

$$\lim_{x \to 0} \frac{5 \sin 5x}{3 \times 5x} = k$$

$$\frac{5}{3} \lim_{x \to 0} \frac{\sin 5x}{5x} = k$$

$$\frac{5}{3} \times 1 = k$$

$$k = \frac{5}{3}$$

21. Determine the value of the constant k so that the function $f(x) = \begin{cases} kx^2, \ if \ x \leq 2 \\ 3, \ if \ x > 2 \end{cases}$ $is\ continuous\ at\ x = 2$

Solution:

Given

$$f(x) = \begin{cases} kx^2, \ if \ x \leq 2 \\ 3, \ if \ x > 2 \end{cases}$$

If f (x) is continuous at x = 2, then we have

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x) = f(2)$$

Now,

$$\lim_{x \to 2^-} f(x) = \lim_{h \to 0} f(2-h) = \lim_{h \to 0} k(2-h)^2 = 4k$$

$f(2) = 3$

From the above equation we can write as

$4k = 3$

$\Rightarrow k = \frac{3}{4}$

22. Determine the value of the constant k so that the function $f(x) = \begin{cases} \frac{sin\ 2x}{5x}, & if\ x \neq 0 \\ k, & if\ x = 0 \end{cases}$ $is\ continuous\ at\ x = 0$

Solution:

Given

$$f(x) = \begin{cases} \frac{sin\ 2x}{5x}, & if\ x \neq 0 \\ k, & if\ x = 0 \end{cases}$$

If f (x) is continuous at x = 0

$$\lim_{x \to 0} f(x) = f(0)$$

$$\lim_{x \to 0} \frac{\sin 2x}{5x} = k$$

$$\lim_{x \to 0} \frac{2 \sin 2x}{5 \times 2x} = k$$

$$\frac{2}{5} \lim_{x \to 0} \frac{\sin 2x}{2x} = k$$

$$\frac{2}{5} \times 1 = k$$

$$k = \frac{2}{5}$$

23. Find the values of a so that the function

$$f(x) = \begin{cases} ax+5, & if\ x \le 2 \\ x-1, & if\ x > 2 \end{cases} \quad is\ continuous\ at\ x = 2$$

Solution:

Given

$$f(x) = \begin{cases} ax+5, & if\ x \le 2 \\ x-1, & if\ x > 2 \end{cases}$$

Consider LHL at x = 2

$$\lim_{h \to 0} a(2-h)+5 = 2a+5$$

Now again consider

$$\lim_{x \to 2^+} f(x) = \lim_{h \to 0} f(2+h)$$

$$\lim_{h \to 0} (2+h-1)$$

= 1

$$f(2) = a(2)+5 = 2a+5$$

Since f (x) is continuous at x = 2 we have

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x) = f(2)$$

$$2a+5 = 1$$

$$2a = -4$$

$$a = -2$$

EXERCISE 9.2 PAGE NO: 9.34

1. *Prove that the function* $f(x)=\begin{cases}\frac{\sin x}{x}, x<0\\ x+1, x\geq 0\end{cases}$ *is everywhere continuous.*

Solution:

A real function f is said to be continuous at x = c, where c is any point in the domain of f if

$\lim_{h\to 0} f(c-h) = \lim_{h\to 0} f(c+h) = f(c)$

Where h is a very small positive number. i.e. left hand limit as x → c (LHL) = right hand limit as x → c (RHL) = value of function at x = c.

A function is continuous at x = c if

$\lim_{x\to c} f(x) = f(c)$

Here we have,

$f(x)=\begin{cases}\frac{\sin x}{x} & ,x<0\\ x+1 & ,x\geq 0\end{cases}$ ....equation 1

To prove it everywhere continuous we need to show that at every point in the domain of f(x) [domain is nothing but a set of real numbers for which function is defined]

$\lim_{x\to c} f(x) = f(c)$,where c is any random point from domain of f

Clearly from definition of f(x), f(x) is defined for all real numbers.

Now we need to check continuity for all real numbers.

Let c is any random number such that c < 0 [thus c being a random number, it can include all negative numbers]

f (c) = $\frac{\sin c}{c}$ [using equation 1]

$\lim_{x\to c} f(x) = \lim_{x\to c}\frac{\sin x}{x} = \frac{\sin c}{c}$

Clearly, $\lim_{x\to c} f(x) = f(c) = \frac{\sin c}{c}$

We can say that f(x) is continuous for all x < 0

Now, let m be any random number from the domain of f such that m > 0

Thus m being a random number, it can include all positive numbers] f (m) = m+1 [using equation 1]

$$\lim_{x\to m} f(x) = \lim_{x\to m} x + 1 = m + 1$$

Clearly, $\lim_{x\to c} f(x) = f(c) = m + 1$

Therefore we can say that f(x) is continuous for all x > 0

As zero is a point at which function is changing its nature so we need to check LHL, RHL separately

f (0) = 0 + 1 = 1 [using equation 1]

LHL = $\lim_{h\to 0} f(0 - h) = \lim_{h\to 0} \frac{\sin(-h)}{-h} = \lim_{h\to 0} \frac{\sin h}{h} = 1$

[∵ $\sin -\theta = -\sin\theta$ and $\lim_{h\to 0} \frac{\sin h}{h} = 1$]

RHL = $\lim_{h\to 0} f(0 + h) = \lim_{h\to 0} h + 1 = 1$

Thus LHL = RHL = f (0).

Therefore f (x) is continuous at x = 0

Hence, we proved that f is continuous for x < 0; x > 0 and x = 0

Thus f(x) is continuous everywhere.

Hence, proved.

2. *Discuss the the continuity of the function* $f(x) = \begin{cases} \frac{x}{|x|}, x \neq 0 \\ 0, x = 0 \end{cases}$

Solution:

A real function f is said to be continuous at x = c, where c is any point in the domain of f if

$$\lim_{h\to 0} f(c - h) = \lim_{h\to 0} f(c + h) = f(c)$$

Where h is a very small positive number. i.e. left hand limit as x → c (LHL) = right hand limit as x → c (RHL) = value of function at x = c.

A function is continuous at x = c if

$$\lim_{x\to c} f(x) = f(c)$$

Here we have,

$$f(x) = \begin{cases} \frac{x}{|x|} & , x \neq 0 \\ 0 & , x = 0 \end{cases}$$ ....equation 1

The function is defined for all real numbers, so we need to comment about its continuity for all numbers in its domain (domain = set of numbers for which f is defined)
Function is changing its nature (or expression) at x = 0, so we need to check its continuity at x = 0 first.
We know that from the definition of mod function we have

$$|x| = \begin{cases} -x, x < 0 \\ x, x \geq 0 \end{cases}$$

LHL = $\lim_{h\to0} f(0-h) = \lim_{h\to0} f(-h) = \frac{-h}{-(-h)} = \frac{-h}{h} = -1$

RHL = $\lim_{h\to0} f(0+h) = \lim_{h\to0} f(h) = \frac{h}{h} = 1$ [using equation 1 and mod function]

f (0) = 0 [using equation 1]
Clearly, LHL ≠ RHL ≠ f (0)
∴ Function is discontinuous at x = 0
Let c be any real number such that c > 0

∴ f (c) = $\frac{c}{|c|} = \frac{c}{c} = 1$ [using equation 1]

And, $\lim_{x\to c} f(x) = \lim_{x\to c} \frac{c}{|c|} = \lim_{x\to c} \frac{c}{c} = 1$

Thus, $\lim_{x\to c} f(x) = f(c)$

Therefore f (x) is continuous everywhere for x > 0.
Let c be any real number such that c < 0

Therefore f (c) = $\frac{c}{|c|} = \frac{c}{-c} = -1$

[Using equation 1 and idea of mod function]

And, $\lim_{x\to c} f(x) = \lim_{x\to c} \frac{c}{|c|} = \lim_{x\to c} \frac{c}{-c} = -1$

Thus, $\lim_{x\to c} f(x) = f(c)$

Therefore f (x) is continuous everywhere for x < 0.
Hence, we can conclude by stating that f (x) is continuous for all Real numbers except zero that is discontinuous at x = 0.

3. Find the points of discontinuity, if any, of the following functions:

(i) $f(x)=\begin{cases} x^3-x^2+2x-2, \ if\ x\neq 1 \\ 4, \ if\ x=1 \end{cases}$

Solution:

A real function f is said to be continuous at x = c, where c is any point in the domain of f if

$\lim_{h\to 0} f(c-h) = \lim_{h\to 0} f(c+h) = f(c)$

Where h is a very small positive number. i.e. left hand limit as x → c (LHL) = right hand limit as x → c (RHL) = value of function at x = c.

A function is continuous at x = c if

$\lim_{x\to c} f(x) = f(c)$

Here we have,

$f(x)=\begin{cases} x^3-x^2+2x-2 & ,\text{if } x\neq 1 \\ 4 & ,\text{if } x=1 \end{cases}$ ....equation 1

Function is defined for all real numbers so we need to comment about its continuity for all numbers in its domain (domain = set of numbers for which f is defined)

Function is changing its nature (or expression) at x = 1, so we need to check its continuity at x = 1.

Clearly, f (1) = 4 [using equation 1]

$\lim_{x\to 1} f(x) = \lim_{x\to 1}(x^3-x^2+2x-2) = 1^3-1^2+2*1-2=0$

Clearly, $\lim_{x\to c} f(x) \neq f(c)$

∴ f (x) is discontinuous at x = 1.

Let c be any real number such that c ≠ 0

f (c) = $c^3 - c^2 + 2c - 2$ [using equation 1]

$\lim_{x\to c} f(x) = \lim_{x\to c}(x^3-x^2+2x-2) = c^3 - c^2+2c-2$

Clearly, $\lim_{x\to c} f(x) = f(c)$

Therefore f(x) is continuous for all real x except x =1

(ii) $f(x)=\begin{cases} \frac{x^4-16}{x-2}, \ if\ x\neq 2 \\ 16, \ if\ x=2 \end{cases}$

Solution:

A real function f is said to be continuous at x = c, where c is any point in the domain of f if

$$\lim_{h\to 0} f(c-h) = \lim_{h\to 0} f(c+h) = f(c)$$

Where h is a very small positive number. i.e. left hand limit as x → c (LHL) = right hand limit as x → c (RHL) = value of function at x = c.

A function is continuous at x = c if

$$\lim_{x\to c} f(x) = f(c)$$

Here we have,

$$f(x) = \begin{cases} \dfrac{x^4-16}{x-2} = \dfrac{(x^2+4)(x-2)(x+2)}{(x-2)} = (x^2+4)(x+2) & ,\text{if } x \neq 2 \\ 16 & ,\text{if } x = 2 \end{cases}$$

...Equation 1

The function is defined for all real numbers, so we need to comment about its continuity for all numbers in its domain (domain = set of numbers for which f is defined)

Function is changing its nature (or expression) at x = 2, so we need to check its continuity at x = 2 first.

Clearly, f (2) = 16 [from equation 1]

$$\lim_{x\to 2} f(x) = \lim_{x\to 2} \frac{x^4-16}{x-2} = \lim_{x\to 2} \frac{(x^2+4)(x-2)(x+2)}{(x-2)} = \lim_{x\to 2} (x^2+4)(x+2)$$
$$= 16$$

Clearly, $\lim_{x\to c} f(x) = f(c)$

∴ f (x) is continuous at x = 2.

Let c be any real number such that c ≠ 0

f (c) = $(c^2+4)(c+2)$ [using equation 1]

$$\lim_{x\to c} f(x) = \lim_{x\to c} (x^2+4)(x+2) = (c^2+4)(c+2)$$

Clearly, $\lim_{x\to c} f(x) = f(c)$

Therefore f (x) is continuous for all real x

(iii) $f(x)=\begin{cases}\frac{\sin x}{x}, & if\ x<0\\ 2x+3, & x\geq 0\end{cases}$

Solution:

A real function f is said to be continuous at x = c, where c is any point in the domain of f if

$\lim_{h\to 0} f(c-h) = \lim_{h\to 0} f(c+h) = f(c)$

Where h is a very small positive number. i.e. left hand limit as x → c (LHL) = right hand limit as x → c (RHL) = value of function at x = c.

$\lim_{x\to c} f(x) = f(c)$

Here we have,

$f(x)=\begin{cases}\frac{\sin x}{x} & ,\text{if } x<0\\ 2x+3 & ,\text{if } x\geq 0\end{cases}$ ...Equation 1

The function is defined for all real numbers, so we need to comment about its continuity for all numbers in its domain (domain = set of numbers for which f is defined)

Let c is any random number such that c < 0 [thus c being a random number, it can include all negative numbers]

f (c) = $\frac{\sin c}{c}$ [using equation 1]

$\lim_{x\to c} f(x) = \lim_{x\to c}\frac{\sin x}{x} = \frac{\sin c}{c}$

Clearly, $\lim_{x\to c} f(x) = f(c) = \frac{\sin c}{c}$

We can say that f(x) is continuous for all x < 0

Now, let m be any random number from the domain of f such that m > 0

Thus m being a random number, it can include all positive numbers]

f (m) = 2m + 3 [from equation 1]

$\lim_{x\to m} f(x) = \lim_{x\to m} 2x+3 = 2m+3$

Clearly, $\lim_{x\to c} f(x) = f(c) = 2m+3$

We can say that f(x) is continuous for all x > 0

As zero is a point at which function is changing its nature so we need to check LHL, RHL separately f (0) = 2 × 0 + 3 = 3 [using equation 1]

LHL = $\lim_{h\to 0} f(0-h) = \lim_{h\to 0} \frac{\sin -h}{-h} = \lim_{h\to 0} \frac{\sin h}{h} = 1$

[∵ sin –θ = – sin θ and $\lim_{h\to 0} \frac{\sin h}{h} = 1$]

RHL = $\lim_{h\to 0} f(0+h) = \lim_{h\to 0} 2h+3 = 3$

Thus LHL ≠ RHL

∴ f (x) is discontinuous at x = 0

Hence, f is continuous for all x ≠ 0 but discontinuous at x = 0.

$$(iv)\ f(x) = \begin{cases} \frac{\sin 3x}{x},\ if\ x \neq 0 \\ 4,\ if\ x = 0 \end{cases}$$

Solution:

A real function f is said to be continuous at x = c, where c is any point in the domain of f if

$\lim_{h\to 0} f(c-h) = \lim_{h\to 0} f(c+h) = f(c)$

Where h is a very small positive number. i.e. left hand limit as x → c (LHL) = right hand limit as x → c (RHL) = value of function at x = c.

A function is continuous at x = c if

$\lim_{x\to c} f(x) = f(c)$

Here we have,

$$f(x) = \begin{cases} \frac{\sin 3x}{x} & ,\text{if } x \neq 0 \\ 4 & ,\text{if } x = 0 \end{cases}$$ ...Equation 1

The function is defined for all real numbers, so we need to comment about its continuity for all numbers in its domain (domain = set of numbers for which f is defined)

Let c is any random number such that c ≠ 0 [thus c being a random number, it can include all numbers except 0] f (c) = $\frac{\sin 3c}{c}$ [from equation 1]

$$\lim_{x\to c} f(x) = \lim_{x\to c}\frac{\sin 3x}{x} = \frac{\sin 3c}{c}$$

Clearly, $\lim_{x\to c} f(x) = f(c) = \frac{\sin 3c}{c}$

We can say that f(x) is continuous for all x ≠ 0

As zero is a point at which function is changing its nature, so we need to check the continuity here.

f (0) = 4 [using equation 1]

LHL = $\lim_{h\to 0} f(0-h) = \lim_{h\to 0}\frac{\sin -3h}{-h} = 3\lim_{h\to 0}\frac{\sin 3h}{3h} = 3$

[∵ sin –θ = – sin θ and $\lim_{x\to 0}\frac{\sin x}{x} = 1$]

RHL = $\lim_{h\to 0} f(0+h) = \lim_{h\to 0}\frac{\sin 3h}{h} = 3\lim_{h\to 0}\frac{\sin 3h}{3h} = 3$

Thus LHL = RHL ≠ f (0)

∴ f (x) is discontinuous at x = 0

Hence, f is continuous for all x ≠ 0 but discontinuous at x = 0.

$$(v)\ f(x) = \begin{cases} \frac{\sin x}{x} + \cos x, & if\ x \neq 0 \\ 5, & if\ x = 0 \end{cases}$$

Solution:

A real function f is said to be continuous at x = c, where c is any point in the domain of f if

$$\lim_{h\to 0} f(c-h) = \lim_{h\to 0} f(c+h) = f(c)$$

Where h is a very small positive number. i.e. left hand limit as x → c (LHL) = right hand limit as x → c (RHL) = value of function at x = c.

A function is continuous at x = c if

$$\lim_{x\to c} f(x) = f(c)$$

Here we have,

$$f(x) = \begin{cases} \frac{\sin x}{x} + \cos x & ,\text{if } x \neq 0 \\ 5 & ,\text{if } x = 0 \end{cases}$$ ...Equation 1

The function is defined for all real numbers, so we need to comment about its continuity for all numbers in its domain (domain = set of numbers for which f is defined)
Let c is any random number such that c ≠ 0 [thus c being a random number, it can include all numbers except 0]

$f(c) = \frac{\sin c}{c} + \cos c$ [using equation 1]

$$\lim_{x \to c} f(x) = \lim_{x \to c}\left(\frac{\sin x}{x} + \cos x\right) = \frac{\sin c}{c} + \cos c$$

Clearly, $\lim_{x \to c} f(x) = f(c)$

We can say that f(x) is continuous for all x ≠ 0
As zero is a point at which function is changing its nature, so we need to check the continuity here.
f (0) = 5 [using equation 1]

$$\lim_{x \to 0}\left(\frac{\sin x}{x} + \cos x\right) = \lim_{x \to 0}\frac{\sin x}{x} + \lim_{x \to 0}\cos x = 1 + \cos 0 = 2 \quad \left[\because \lim_{x \to 0}\frac{\sin x}{x} = 1\right]$$

Thus $\lim_{x \to c} f(x) \neq f(c)$

∴ f (x) is discontinuous at x = 0

Hence, f is continuous for all x ≠ 0 but discontinuous at x = 0.

$$(vi)\ f(x) = \begin{cases} \frac{x^4+x^3+2x^2}{\tan^{-1} x}, & if\ x \neq 0 \\ 10, & if\ x = 0 \end{cases}$$

Solution:
A real function f is said to be continuous at x = c, where c is any point in the domain of f if

$$\lim_{h \to 0} f(c-h) = \lim_{h \to 0} f(c+h) = f(c)$$

Where h is a very small positive number. i.e. left hand limit as x → c (LHL) = right hand limit as x → c (RHL) = value of function at x = c.
A function is continuous at x = c if

$$\lim_{x\to c} f(x) = f(c)$$

Here we have,

$$f(x) = \begin{cases} \frac{x^4+x^3+2x^2}{\tan^{-1} x} & ,\text{if } x \neq 0 \\ 10 & ,\text{if } x = 0 \end{cases}$$ ...Equation 1

The function is defined for all real numbers, so we need to comment about its continuity for all numbers in its domain (domain = set of numbers for which f is defined)
Let c is any random number such that c ≠ 0 [thus c being a random number, it can include all numbers except 0]

$f(c) = \frac{c^4+c^3+2c^2}{\tan^{-1} c}$ [from equation 1]

$$\lim_{x\to c} f(x) = \lim_{x\to c}\left(\frac{x^4+x^3+2x^2}{\tan^{-1} x}\right) = \frac{c^4+c^3+2c^2}{\tan^{-1} c}$$

Clearly, $\lim_{x\to c} f(x) = f(c)$

We can say that f(x) is continuous for all x ≠ 0
As zero is a point at which function is changing its nature so we need to check the continuity here.
f (0) = 10 [using equation 1]

$$\lim_{x\to 0} f(x) = \lim_{x\to 0}\left(\frac{x^4+x^3+2x^2}{\tan^{-1} x}\right)$$

or, $\lim_{x\to 0}\left(\frac{x^3+x^2+2x}{\frac{\tan^{-1}x}{x}}\right) = \frac{\lim_{x\to 0}(x^3+x^2+2x)}{\lim_{x\to 0}\frac{\tan^{-1}x}{x}} = \frac{0}{1} = 0$ [∵ using $\lim_{x\to 0}\frac{\tan^{-1} x}{x} = 1$]

Thus $\lim_{x\to c} f(x) \neq f(c)$

∴ f (x) is discontinuous at x = 0

Hence, f is continuous for all x ≠ 0 but discontinuous at x = 0

$$(vii)\ f(x) = \begin{cases} \frac{e^x-1}{\log_e(1+2x)}, & if\ x \neq 0 \\ 7, & if\ x = 0 \end{cases}$$

Solution:

A real function f is said to be continuous at x = c, where c is any point in the domain of f if

$$\lim_{h\to 0} f(c-h) = \lim_{h\to 0} f(c+h) = f(c)$$

Where h is a very small positive number. i.e. left hand limit as x → c (LHL) = right hand limit as x → c (RHL) = value of function at x = c.

A function is continuous at x = c if

$$\lim_{x\to c} f(x) = f(c)$$

Here we have,

$$f(x) = \begin{cases} \frac{e^x-1}{\log_e(1+2x)} & ,\text{if } x \neq 0 \\ 7 & ,\text{if } x = 0 \end{cases}$$

...Equation 1

Function is defined for all real numbers so we need to comment about its continuity for all numbers in its domain (domain = set of numbers for which f is defined)

Let c is any random number such that c ≠ 0 [thus c being random number, it is able to include all numbers except 0]

f (c) = $\frac{e^c-1}{\log_e(1+2c)}$ [using equation 1]

$$\lim_{x\to c} f(x) = \lim_{x\to c}\left(\frac{e^x-1}{\log_e(1+2x)}\right) = \frac{e^c-1}{\log_e(1+2c)}$$

Clearly, $\lim_{x\to c} f(x) = f(c)$

We can say that f(x) is continuous for all x ≠ 0

As x = 0 is a point at which function is changing its nature so we need to check the continuity here.

Since, f (0) = 7 [from equation 1]

$$\lim_{x\to 0} \frac{\log(1+x)}{x} = 1$$

$$\lim_{x\to 0} \frac{e^x-1}{x} = 1$$

Log (1+x) and e^x in its Taylor form.

From sandwich theorem numerator and denominator conditions also hold for this limit

$\lim\limits_{x \to 0} \frac{\log(1+2x)}{2x} = 1$

But, $\lim\limits_{x \to 0} \frac{\log(1+2x)}{x} \neq 1$ as denominator does not have 2x

$\lim\limits_{x \to 0} f(x)$

$= \lim\limits_{x \to 0} \frac{e^x - 1}{\log 1+2x}$ [Using logarithmic and exponential limit as explained above, we have:]

$= \frac{1}{2} \lim\limits_{x \to 0} \frac{\frac{(e^x-1)}{x}}{\frac{\log(1+2x)}{2x}} = \frac{1}{2}$

Thus, $\lim\limits_{x \to c} f(x) \neq f(c)$

∴ f (x) is discontinuous at x = 0

Hence, f is continuous for all x ≠ 0 but discontinuous at x = 0

$$(viii)\ f(x) = \begin{cases} |x-3|,\ if\ x \geq 1 \\ \frac{x^2}{4} - \frac{3x}{2} + \frac{13}{4},\ if\ x < 1 \end{cases}$$

Solution:

A real function f is said to be continuous at x = c, where c is any point in the domain of f if

$\lim\limits_{h \to 0} f(c-h) = \lim\limits_{h \to 0} f(c+h) = f(c)$

Where h is a very small positive number. i.e. left hand limit as x → c (LHL) = right hand limit as x → c (RHL) = value of function at x = c.

A function is continuous at x = c if

$\lim\limits_{x \to c} f(x) = f(c)$

Now, we can define it for variable x, if x ≥ 0 |x| = x

If x < 0 |x| = (–x)

$\therefore |x| = \begin{cases} -x, x < 0 \\ x, x \geq 0 \end{cases}$

Here we have,

$$f(x) = \begin{cases} |x-3| & ,\text{if } x \ge 1 \\ \frac{x^2}{4} - \frac{3x}{2} + \frac{13}{4} & ,\text{if } x < 1 \end{cases}$$

Applying the idea of mod function, f(x) can be rewritten as:

$$f(x) = \begin{cases} \frac{x^2}{4} - \frac{3x}{2} + \frac{13}{4}, \text{if } x < 1 \\ x - 3, \text{if } x \ge 3 \\ -(x-3), \text{if } 1 \le x < 3 \end{cases}$$ ......equation 1

Function is defined for all real numbers so we need to comment about its continuity for all numbers in its domain (domain = set of numbers for which f is defined)
Let c is any random number such that c < 1 [thus c being a random number, it can include all numbers less than 1]

$f(c) = \frac{c^2}{4} - \frac{3c}{2} + \frac{13}{4}$

$$\lim_{x\to c} f(x) = \lim_{x\to c}\left(\frac{x^2}{4} - \frac{3x}{2} + \frac{13}{4}\right) = \frac{c^2}{4} - \frac{3c}{2} + \frac{13}{4}$$

Clearly, $\lim_{x\to c} f(x) = f(c)$

We can say that f(x) is continuous for all x < 1
As x = 1 is a point at which function is changing its nature, so we need to check the continuity here.
f (1) = | 1 – 3 | = 2 [from equation 1]

$$\text{LHL} = \lim_{h\to 0} f(1-h) = \lim_{h\to 0}\left(\frac{(1-h)^2}{4} - \frac{3(1-h)}{2} + \frac{13}{4}\right) = \frac{1^2}{4} - \frac{3}{2} + \frac{13}{4} = 2$$

$$\text{RHL} = \lim_{h\to 0} f(1+h) = \lim_{h\to 0} |\,1+h-3\,| = |-2| = 2$$

Thus LHL = RHL = f (1)
∴ f (x) is continuous at x = 1
Now, again f(x) is changing its nature at x = 3, so we need to check continuity at x = 3
f (3) = 3 – 3 = 0 [using equation 1]

LHL = $\lim_{h\to 0} f(3-h) = \lim_{h\to 0} -(3-h-3) = 0$

RHL = $\lim_{h\to 0} f(3+h) = \lim_{h\to 0} 3+h-3 = 0$

Thus LHL = RHL = f (3)

∴ f (x) is continuous at x = 3

For x > 3; f(x) = x–3 whose plot is linear, so it is continuous for all x > 3

Similarly, for 1 < x < 3, f(x) = 3 – x whose plot is again a straight line and thus continuous for all point in this range.

Hence, f(x) is continuous for all real x.

$$(ix)\ f(x) = \begin{cases} |x|+3, & if\ x \le -3 \\ -2x, & if\ -3 < x < 3 \\ 6x+2, & if\ x > 3 \end{cases}$$

Solution:

A real function f is said to be continuous at x = c, where c is any point in the domain of f if

$\lim_{h\to 0} f(c-h) = \lim_{h\to 0} f(c+h) = f(c)$

Where h is a very small positive number. i.e. left hand limit as x → c (LHL) = right hand limit as x → c (RHL) = value of function at x = c.

A function is continuous at x = c if

A function is continuous at x = c if

$\lim_{x\to c} f(x) = f(c)$

Similarly, we can define it for variable x, if x ≥ 0 |x| = x

If x < 0 |x| = (–x)

∴ $|x| = \begin{cases} -x, x < 0 \\ x, x \ge 0 \end{cases}$

Here we have,

$$f(x) = \begin{cases} |x|+3 & ,\text{if } x \le -3 \\ -2x & ,\text{if } -3 < x < 3 \\ 6x+2 & ,\text{if } x \ge 3 \end{cases}$$

Applying the idea of mod function, f(x) can be rewritten as:

$$f(x) = \begin{cases} 3 - x & ,\text{if } x \le -3 \\ -2x & ,\text{if } -3 < x < 3 \\ 6x + 2 & ,\text{if } x \ge 3 \end{cases}$$ ......equation 1

Function is defined for all real numbers so we need to comment about its continuity for all numbers in its domain (domain = set of numbers for which f is defined)

Let c is any random number such that c < –3 [thus c being random number, it is able to include all numbers less than –3]

f (c) = 3 – c [from equation 1]

$$\lim_{x \to c} f(x) = \lim_{x \to c}(3 - x) = 3 - c$$

Clearly, $\lim_{x \to c} f(x) = f(c)$

We can say that f(x) is continuous for all x < –3

As x = –3 is a point at which function is changing its nature so we need to check the continuity here.

f (–3) = 3–(–3) = 6 [using equation 1]

LHL = $\lim_{h \to 0} f(-3 - h) = \lim_{h \to 0}(3 - (-3 - h)) = 6$

RHL = $\lim_{h \to 0} f(-3 + h) = \lim_{h \to 0} -2(-3 + h) = 6$

Thus LHL = RHL = f (–3)

∴ f (x) is continuous at x = –3

Let c is any random number such that –3 < m < 3 [thus c being random number, it is able to include all numbers between –3 and 3]

f (c) = -2m [using equation 1]

and, $\lim_{x \to m} f(x) = \lim_{x \to m}(-2x) = -2m$

Clearly, $\lim_{x \to c} f(x) = f(c)$

We can say that f(x) is continuous for all –3 < x < 3

Now, again f(x) is changing its nature at x = 3, so we need to check continuity at x = 3

f (3) = 6 × 3 + 2 = 20 [using equation 1]

LHL = $\lim_{h\to 0} f(3-h) = \lim_{h\to 0} -2*(3-h) = -6$

RHL = $\lim_{h\to 0} f(3+h) = \lim_{h\to 0} 6(3+h)+2 = 20$

Thus LHL ≠ RHL

∴ f (x) is discontinuous at x = 3

For x > 3; f(x) = 6x + 2 whose plot is linear, so it is continuous for all x > 3

Hence, f(x) is continuous for all real x except x = 3

There is only one point of discontinuity at x = 3

$$(x)\ f(x) = \begin{cases} x^{10}-1,\ if\ x \le 1 \\ x^2,\ if\ x > 1 \end{cases}$$

Solution:

A real function f is said to be continuous at x = c, where c is any point in the domain of f if

$\lim_{h\to 0} f(c-h) = \lim_{h\to 0} f(c+h) = f(c)$

Where h is a very small positive number. i.e. left hand limit as x → c (LHL) = right hand limit as x → c (RHL) = value of function at x = c.

A function is continuous at x = c if

$\lim_{x\to c} f(x) = f(c)$

Here we have,

$f(x) = \begin{cases} x^{10} & ,\text{if } x \le 1 \\ x^2 & ,\text{if } x > 1 \end{cases}$ ......equation 1

Function is defined for all real numbers so we need to comment about its continuity for all numbers in its domain (domain = set of numbers for which f is defined)

Let c is any random number such that c < 1 [thus c being random number, it is able to include all numbers less than 1]

f (c) = c^{10} [from equation 1]

$\lim_{x\to c} f(x) = \lim_{x\to c}(x^{10}) = c^{10}$

Clearly, $\lim_{x\to c} f(x) = f(c)$

We can say that f(x) is continuous for all x < 1

As x = 1 is a point at which function is changing its nature so we need to check the continuity here.

f (1) = 1^{10} = 1 [using equation 1]

LHL = $\lim_{h \to 0} f(1-h) = \lim_{h \to 0} (1-h)^{10} = 1$

RHL = $\lim_{h \to 0} f(1+h) = \lim_{h \to 0} (1+h)^{2} = 1$

Thus LHL = RHL = f (1)

∴ f (x) is continuous at x = 1

Let m is any random number such that m > 1 [thus m being random number, it is able to include all numbers greater than 1]

f (m) = m^2 [using equation 1]

and, $\lim_{x \to m} f(x) = \lim_{x \to m} (x^2) = m^2$

Clearly, $\lim_{x \to m} f(x) = f(m)$

We can say that f(x) is continuous for all m > 1

Hence, f(x) is continuous for all real x

There no point of discontinuity. It is everywhere continuous

$$(xi)\ f(x) = \begin{cases} 2x, \ if\ x < 0 \\ 0, \ if\ 0 \le x \le 1 \\ 4x, \ if\ x > 1 \end{cases}$$

Solution:

A real function f is said to be continuous at x = c, where c is any point in the domain of f if

$\lim_{h \to 0} f(c-h) = \lim_{h \to 0} f(c+h) = f(c)$

Where h is a very small positive number. i.e. left hand limit as x → c (LHL) = right hand limit as x → c (RHL) = value of function at x = c.

A function is continuous at x = c if

$\lim_{x\to c} f(x) = f(c)$

Here we have,

$$f(x) = \begin{cases} 2x & ,\text{if } x < 0 \\ 0 & ,\text{if } 0 \le x \le 1 \\ 4x & ,\text{if } x > 1 \end{cases} \quad \text{......equation 1}$$

The function is defined for all real numbers, so we need to comment about its continuity for all numbers in its domain (domain = set of numbers for which f is defined)

Let c is any random number such that c < 0 [thus c being a random number, it can include all numbers less than 0]

f (c) = 2c

$\lim_{x\to c} f(x) = \lim_{x\to c}(2x) = 2c$

Clearly, $\lim_{x\to c} f(x) = f(c)$

We can say that f(x) is continuous for all x < 0

As x = 0 is a point at which function is changing its nature, so we need to check the continuity here.

f (0) = 0 [using equation 1]

LHL = $\lim_{h\to 0} f(0-h) = \lim_{h\to 0} -2h = 0$

RHL = $\lim_{h\to 0} f(0+h) = \lim_{h\to 0} 0 = 0$

Thus LHL = RHL = f (0)

∴ f (x) is continuous at x = 0

Let m is any random number such that 0 < m < 1 [thus m being a random number, it can include all numbers greater than 0 and less than 1]

f (m) = 0 [using equation 1]

and, $\lim_{x\to m} f(x) = \lim_{x\to m}(0) = 0$

Clearly, $\lim_{x\to m} f(x) = f(m)$

We can say that f(x) is continuous for all 0 < x < 1

As x = 1 is again a point at which function is changing its nature, so we need to check the continuity here.

f (1) = 0

LHL = $\lim_{h\to 0} f(1-h) = \lim_{h\to 0} 0 = 0$

RHL = $\lim_{h\to 0} f(1+h) = \lim_{h\to 0} 4(1+h) = 4$

∴ f (x) is discontinuous at x = 1

Let k is any random number such that k > 1 [thus k being a random number, it can include all numbers greater than 1]

f (k) = 4k [using equation 1]

and, $\lim_{x\to k} f(x) = \lim_{x\to k} 4x = 4k$

Clearly, $\lim_{x\to k} f(x) = f(k)$

We can say that f(x) is continuous for all x > 1

Hence, f(x) is continuous for all real value of x, except x =1

There is a single point of discontinuity at x = 1

$$(xii)\ f(x) = \begin{cases} sin\ x - cos\ x, & if\ x \neq 0 \\ -1, & if\ x = 0 \end{cases}$$

Solution:

A real function f is said to be continuous at x = c, where c is any point in the domain of f if

$\lim_{h\to 0} f(c-h) = \lim_{h\to 0} f(c+h) = f(c)$

Where h is a very small positive number. i.e. left hand limit as x → c (LHL) = right hand limit as x → c (RHL) = value of function at x = c.

This is very precise, using our fundamental idea of limit from class 11 we can summarise it as, A function is continuous at x = c if

$\lim_{x\to c} f(x) = f(c)$

Here we have,

$$f(x) = \begin{cases} \sin x - \cos x & ,\text{if } x \neq 0 \\ -1 & ,\text{if } x = 0 \end{cases} \quad \text{...Equation 1}$$

Function is defined for all real numbers so we need to comment about its continuity for all numbers in its domain (domain = set of numbers for which f is defined)

Let c is any random number such that c ≠ 0 [thus c being a random number, it can include all numbers except 0] f (c) = sin c – cos c [using equation 1]

$$\lim_{x \to c} f(x) = \lim_{x \to c}(\sin x - \cos x) = \sin c - \cos c$$

Clearly, $\lim_{x \to c} f(x) = f(c)$

We can say that f(x) is continuous for all x ≠ 0

As zero is a point at which function is changing its nature, so we need to check the continuity here.

f (0) = –1 [using equation 1]

$$\lim_{x \to 0}(\sin x - \cos x) = \lim_{x \to 0} \sin x - \lim_{x \to 0} \cos x = 0 - \cos 0 = -1$$

Thus $\lim_{x \to c} f(x) = f(c)$

∴ f (x) is continuous at x = 0

Hence, f is continuous for all x.

f (x) is continuous everywhere.

No point of discontinuity.

$$(xiii)\ f(x) = \begin{cases} -2, & if\ x \le -1 \\ 2x, & if\ -1 < x < 1 \\ 2, & if\ x \ge 1 \end{cases}$$

Solution:

A real function f is said to be continuous at x = c, where c is any point in the domain of f if

$$\lim_{h \to 0} f(c-h) = \lim_{h \to 0} f(c+h) = f(c)$$

Where h is a very small positive number. i.e. left hand limit as x → c (LHL) = right hand limit as x → c (RHL) = value of function at x = c.

A function is continuous at x = c if

$$\lim_{x \to c} f(x) = f(c)$$

Here we have,

$$f(x) = \begin{cases} -2 & ,\text{if } x \le -1 \\ 2x & ,\text{if } -1 < x < 1 \\ 2 & ,\text{if } x \ge 1 \end{cases} \quad \text{..........equation 1}$$

Function is defined for all real numbers so we need to comment about its continuity for all numbers in its domain (domain = set of numbers for which f is defined)

For x < –1, f(x) is having a constant value, so the curve is going to be straight line parallel to x–axis.
So, it is everywhere continuous for x < –1.
Similarly for –1 < x < 1, plot on X–Y plane is a straight line passing through origin.
So, it is everywhere continuous for –1 < x < 1.
And similarly for x > 1, plot is going to be again a straight line parallel to x–axis
∴ it is also everywhere continuous for x > 1
As x = –1 is a point at which function is changing its nature so we need to check the continuity here.
f (–1) = –2

LHL = $\lim_{h\to 0} f(-1-h) = \lim_{h\to 0} -2 = -2$

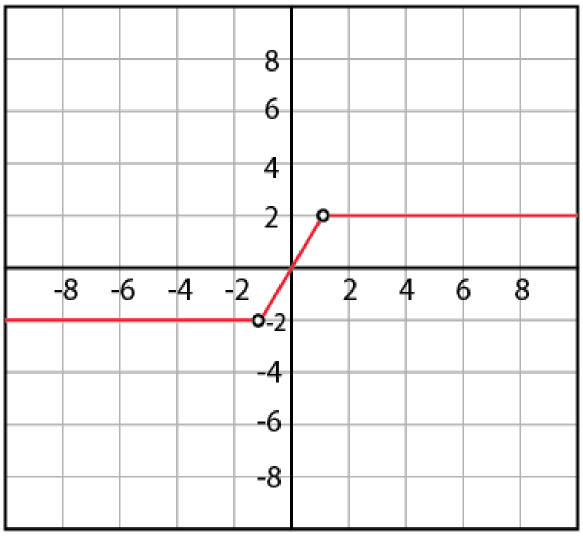
RHL = $\lim_{h\to 0} f(-1+h) = \lim_{h\to 0} 2(-1+h) = -2$

Thus LHL = RHL = f (–1)
∴ f (x) is continuous at x = –1
Also at x = 1 function is changing its nature so we need to check the continuity here too.
f (1) = 2 [using equation 1]

LHL = $\lim_{h\to 0} f(1-h) = \lim_{h\to 0} 2(1-h) = 2$

RHL = $\lim_{h\to 0} f(1+h) = \lim_{h\to 0} 2 = 2$

Thus LHL = RHL = f (1)
∴ f (x) is continuous at x = 1
Thus, f(x) is continuous everywhere and there is no point of discontinuity.



4. In the following, determine the value(s) of constant(s) involved in the definition so that the given function is continuous:

$$(i)\ f(x)=\begin{cases}\frac{\sin 2x}{5x} & if\ x\neq 0\\ 3k, & if\ x=0\end{cases}$$

Solution:

A real function f is said to be continuous at x = c, where c is any point in the domain of f if

$$\lim_{h\to 0} f(c-h) = \lim_{h\to 0} f(c+h) = f(c)$$

Where h is a very small positive number. i.e. left hand limit as x → c (LHL) = right hand limit as x → c (RHL) = value of function at x = c.

A function is continuous at x = c if

$$\lim_{x\to c} f(x) = f(c)$$

Here we have,

$$f(x)=\begin{cases}\frac{\sin 2x}{5x} & ,\text{if } x\neq 0\\ 3k & ,\text{if } x=0\end{cases} \quad \text{Equation 1}$$

Function is defined for all real numbers and we need to find the value of k so that it is continuous everywhere in its domain (domain = set of numbers for which f is defined)

As, for x ≠ 0 it is just a combination of trigonometric and linear polynomial both of which are continuous everywhere.

As x = 0 is only point at which function is changing its nature so it needs to be continuous here.

f (0) = 3k [using equation 1]

$$\lim_{x\to 0}\frac{\sin 2x}{5x} = \frac{1}{5}\lim_{x\to 0} 2 * \frac{\sin 2x}{2x} = \frac{2}{5}\lim_{x\to 0}\frac{\sin 2x}{2x} = \frac{2}{5}\left[\because \lim_{x\to 0}\frac{\sin x}{x} = 1 - \text{sandwich theorem}\right]$$

∵ f (x) is continuous everywhere [given in question]

$$\therefore \lim_{x\to c} f(x) = f(c)$$

$\therefore 3k = \frac{2}{5}$

$\therefore k = \frac{2}{15}$

$$(ii)\ f(x) = \begin{cases} kx + 5\ if\ x \le 2 \\ x - 1,\ if\ x > 2 \end{cases}$$

Solution:

A real function f is said to be continuous at x = c, where c is any point in the domain of f if

$$\lim_{h\to 0} f(c-h) = \lim_{h\to 0} f(c+h) = f(c)$$

Where h is a very small positive number. i.e. left hand limit as x → c (LHL) = right hand limit as x → c (RHL) = value of function at x = c.

A function is continuous at x = c if

$$\lim_{x\to c} f(x) = f(c)$$

Here we have,

$$f(x) = \begin{cases} kx + 5 & ,\text{if } x \le 2 \\ x - 1 & ,\text{if } x > 2 \end{cases}$$ ....equation 1

To find the value of constants always try to check continuity at the values of x for which f(x) is changing its expression.

As most of the time discontinuities are here only, if we make the function continuous here, it will automatically become continuous everywhere.

From equation 1, it is clear that f(x) is changing its expression at x = 2

Given, f (x) is continuous everywhere

$\therefore \lim\limits_{x\to 2} f(x) = f(2)$

$\lim\limits_{h\to 0} f(2-h) = \lim\limits_{h\to 0} f(2+h) = f(2)$

$\lim\limits_{h\to 0} f(2+h) = f(2)$ [Considering RHL as RHL will give expression independent of k]

$\lim\limits_{h\to 0} 2+h-1 = 2k+5$ [Using equation 1]

$\therefore 2k + 5 = 1$

$2k = -4$

$k = \frac{-4}{2} = -2$

$(iii)\ f(x) = \begin{cases} k(x^2+3x)\ if\ x<0 \\ \cos 2x,\ if\ x \geq 0 \end{cases}$

Solution:

A real function f is said to be continuous at x = c, where c is any point in the domain of f if

$\lim\limits_{h\to 0} f(c-h) = \lim\limits_{h\to 0} f(c+h) = f(c)$

Where h is a very small positive number. i.e. left hand limit as x → c (LHL) = right hand limit as x → c (RHL) = value of function at x = c.

This is very precise, using our fundamental idea of limit from class 11 we can summarise it as, A function is continuous at x = c if

$\lim\limits_{x\to c} f(x) = f(c)$

Here we have,

$f(x) = \begin{cases} k(x^2+3x) & ,\text{if } x<0 \\ \cos 2x & ,\text{if } x \geq 0 \end{cases}$ ....equation 1

To find the value of constants always try to check continuity at the values of x for which f(x) is changing its expression.

As most of the time discontinuities are here only, if we make the function continuous here, it will automatically become continuous everywhere

From equation 1, it is clear that f(x) is changing its expression at x = 0
Given, f (x) is continuous everywhere

$$\therefore \lim_{x\to 0} f(x) = f(0)$$

$$\lim_{h\to 0} f(0-h) = \lim_{h\to 0} f(0+h) = f(0)$$

$$\lim_{h\to 0} f(-h) = f(0)$$

$$\lim_{h\to 0} k\{ (-h)^2 + 3(-h) \} = \cos 0$$ [Using equation 1]

$\because k * 0 = 1$

As above equality never holds true for any value of k
k = not defined
No such value of k is possible for which f(x) is continuous everywhere.
f (x) will always have a discontinuity at x = 0

$$(iv)\ f(x) = \begin{cases} 2\ if\ x \le 3 \\ ax + b,\ if\ 3 < x < 5 \\ 9,\ if\ x \ge 5 \end{cases}$$

Solution:

A real function f is said to be continuous at x = c, where c is any point in the domain of f if

$$\lim_{h\to 0} f(c-h) = \lim_{h\to 0} f(c+h) = f(c)$$

Where h is a very small positive number. i.e. left hand limit as x → c (LHL) = right hand limit as x → c (RHL) = value of function at x = c.

This is very precise, using our fundamental idea of limit from class 11 we can summarise it as, A function is continuous at x = c if

$$\lim_{x\to c} f(x) = f(c)$$

Here we have,

$$f(x) = \begin{cases} 2 & ,\text{if} \quad x \le 3 \\ ax + b & ,\text{if} \quad 3 < x < 5 \\ 9 & ,\text{if} \quad x \ge 5 \end{cases}$$ ....equation 1

To find the value of constants always try to check continuity at the values of x for which f(x) is changing its expression.

As most of the time discontinuities are here only, if we make the function continuous here, it will automatically become continuous everywhere
From equation 1, it is clear that f(x) is changing its expression at x = 3
Given, f(x) is continuous everywhere

$\therefore \lim_{x\to 3} f(x) = f(3)$

$\lim_{h\to 0} f(3-h) = \lim_{h\to 0} f(3+h) = f(3)$

$\lim_{h\to 0} f(3+h) = f(3)$

$\lim_{h\to 0}\{a(3+h)+b\} = 2$ [Using equation 1]

$\therefore$ 3a + b = 2 ………………Equation 2
Also from equation 1, it is clear that f(x) is also changing its expression at x = 5
Given, f(x) is continuous everywhere

$\therefore \lim_{x\to 5} f(x) = f(3)$

$\lim_{h\to 0} f(5-h) = \lim_{h\to 0} f(5+h) = f(5)$

$\lim_{h\to 0} f(5-h) = f(5)$

$\lim_{h\to 0}\{a(5-h)+b\} = 9$ [Using equation 1]

$\therefore$ 5a + b = 9 ………………Equation 3

As, b = 9 – 5a

Putting value of b in equation 2:

3a + 9 – 5a = 2

2a = 7

a = $\frac{7}{2}$

$\therefore$ b = 9 – 5($\frac{7}{2}$) $= -\frac{17}{2}$

$\therefore$ a = $\frac{7}{2}$ and b = $-\frac{17}{2}$

$$(v)\ f(x) = \begin{cases} 4\ if\ x \leq -1 \\ ax^2 + b,\ if\ -1 < x < 0 \\ cos\ x,\ if\ x \geq 0 \end{cases}$$

Solution:

A real function f is said to be continuous at x = c, where c is any point in the domain of f if

$$\lim_{h \to 0} f(c-h) = \lim_{h \to 0} f(c+h) = f(c)$$

Where h is a very small positive number. i.e. left hand limit as x → c (LHL) = right hand limit as x → c (RHL) = value of function at x = c.

A function is continuous at x = c if

$$\lim_{x \to c} f(x) = f(c)$$

We have,

$$f(x) = \begin{cases} 4 & ,\text{if} \quad x \leq -1 \\ ax^2 + b & ,\text{if} \quad -1 < x < 0 \\ \cos x & ,\text{if} \quad x \geq 0 \end{cases}$$ ....equation 1

To find the value of constants always try to check continuity at the values of x for which f(x) is changing its expression.

As most of the time discontinuities are here only, if we make the function continuous here, it will automatically become continuous everywhere

From equation 1, it is clear that f(x) is changing its expression at x = –1

Given, f(x) is continuous everywhere

$$\therefore \lim_{x \to -1} f(x) = f(-1)$$

$$\lim_{h \to 0} f(-1-h) = \lim_{h \to 0} f(-1+h) = f(-1)$$

$$\lim_{h \to 0} f(-1+h) = f(-1)$$

$$\lim_{h \to 0} \{a(-1+h)^2 + b\} = 4$$ [Using equation 1]

∴ a + b = 4 ....Equation 2

Also from equation 1, it is clear that f(x) is also changing its expression at x = 0

Given, f (x) is continuous everywhere

$\therefore \lim_{x\to 0} f(x) = f(0)$

$\lim_{h\to 0} f(0-h) = \lim_{h\to 0} f(0+h) = f(0)$

$\lim_{h\to 0} f(-h) = f(0)$

$\lim_{h\to 0}\{a(-h)^2 + b\} = \cos 0 = 1$ [Using equation 1]

∴ b = 1 ……………….Equation 3

Putting value of b in equation 2:

a + 1 = 4

a = 3

∴ a = 3 and b = 1

$$(vi)\ f(x) = \begin{cases} \frac{\sqrt{1+px}-\sqrt{1-px}}{x}, & if\ -1 \le x < 0 \\ \frac{2x+1}{x-2}, & if\ 0 \le x \le 1 \end{cases}$$

Solution:

A real function f is said to be continuous at x = c, where c is any point in the domain of f if

$\lim_{h\to 0} f(c-h) = \lim_{h\to 0} f(c+h) = f(c)$

Where h is a very small positive number. i.e. left hand limit as x → c (LHL) = right hand limit as x → c (RHL) = value of function at x = c.

A function is continuous at x = c if

$\lim_{x\to c} f(x) = f(c)$

Here we have,

$$f(x) = \begin{cases} \frac{\sqrt{1+px}-\sqrt{1-px}}{x} & ,\text{if } -1 \le x < 0 \\ \frac{2x+1}{x-2} & ,\text{if } 0 \le x \le 1 \end{cases}$$ ……………..equation 1

To find the value of constants always try to check continuity at the values of x for which f(x) is changing its expression.

As most of the time discontinuities are here only, if we make the function continuous here, it will automatically become continuous everywhere
From equation 1, it is clear that f(x) is changing its expression at x = 0
Given, f(x) is continuous everywhere

$$\therefore \lim_{x\to 0} f(x) = f(0)$$

$$\lim_{h\to 0} f(0-h) = \lim_{h\to 0} f(0+h) = f(0)$$

$$\lim_{h\to 0} f(-h) = f(0)$$

$$\lim_{h\to 0}\left\{\frac{\sqrt{1-ph}-\sqrt{1+ph}}{-h}\right\} = \frac{2*0+1}{0-2} = -\frac{1}{2}$$ [Using equation 1]

$$\lim_{h\to 0}\left\{\left(\frac{\sqrt{1-ph}-\sqrt{1+ph}}{-h}\right)\left(\frac{\sqrt{1-ph}+\sqrt{1+ph}}{\sqrt{1-ph}+\sqrt{1+ph}}\right)\right\} = -\frac{1}{2}$$

$$\lim_{h\to 0}\left\{\left(\frac{1-ph-1-ph}{-h}\right)\left(\frac{1}{\sqrt{1-ph}+\sqrt{1+ph}}\right)\right\} = -\frac{1}{2}$$

$$\lim_{h\to 0}\left\{\frac{-2ph}{(-h)(\sqrt{1-ph}+\sqrt{1+ph})}\right\} = -\frac{1}{2}$$

$$\lim_{h\to 0}\left\{\frac{2p}{(\sqrt{1-ph}+\sqrt{1+ph})}\right\} = -\frac{1}{2}$$

$$\frac{2p}{2} = p = -\frac{1}{2}$$

$$\therefore p = \frac{-1}{2}$$

$$(vii)\ f(x) = \begin{cases} 5, & if\ x \le 2 \\ ax+b, & if\ 2 < x < 10 \\ 21, & if\ x \ge 10 \end{cases}$$

Solution:
A real function f is said to be continuous at x = c, where c is any point in the domain of f if

$$\lim_{h\to 0} f(c-h) = \lim_{h\to 0} f(c+h) = f(c)$$

Where h is a very small positive number. i.e. left hand limit as x → c (LHL) = right hand limit as x → c (RHL) = value of function at x = c.

A function is continuous at x = c if

$$\lim_{x\to c} f(x) = f(c)$$

Here we have,

$$f(x) = \begin{cases} 5 & ,\text{if} \quad x \le 2 \\ ax+b & ,\text{if} \quad 2 < x < 10 \\ 21 & ,\text{if} \quad x \ge 10 \end{cases}$$ …………………..equation 1

To find the value of constants always try to check continuity at the values of x for which f(x) is changing its expression.

As most of the time discontinuities are here only, if we make the function continuous here, it will automatically become continuous everywhere

From equation 1, it is clear that f(x) is changing its expression at x = 2

Given, f(x) is continuous everywhere

$$\therefore \lim_{x\to 2} f(x) = f(2)$$

$$\lim_{h\to 0} f(2-h) = \lim_{h\to 0} f(2+h) = f(2)$$

$$\lim_{h\to 0} f(2+h) = f(2)$$

$$\lim_{h\to 0}\{a(2+h)+b\} = 5$$ [Using equation 1]

∴ 2a + b = 5 ………………Equation 2

Also from equation 1, it is clear that f(x) is also changing its expression at x = 10

Given, f(x) is continuous everywhere

$$\lim_{h\to 0} f(10-h) = \lim_{h\to 0} f(10+h) = f(10)$$

$$\lim_{h\to 0} f(10-h) = f(10)$$

$$\lim_{h\to 0}\{a(10-h)+b\} = 21$$ [Using equation 1]

∴ 10a + b = 21 ………………Equation 3

As, b = 21 – 10a

Putting value of b in equation 2, we get

$2a + 21 - 10a = 5$

$8a = 16$

$a = \frac{16}{8} = 2$

$\therefore b = 21 - 10 \times 2 = 1$

$\therefore a = 2$ and $b = 1$

$$(viii)\ f(x) = \begin{cases} \frac{k\cos x}{\pi - 2x}, & x < \frac{\pi}{2} \\ 3, & x = \frac{\pi}{2} \\ \frac{3\tan 2x}{2x - \pi}, & x > \frac{\pi}{2} \end{cases}$$

Solution:

A real function f is said to be continuous at x = c, where c is any point in the domain of f if

$$\lim_{h\to 0} f(c-h) = \lim_{h\to 0} f(c+h) = f(c)$$

Where h is a very small positive number. i.e. left hand limit as x → c (LHL) = right hand limit as x → c (RHL) = value of function at x = c.

A function is continuous at x = c if

$$\lim_{x\to c} f(x) = f(c)$$

Here we have,

$$f(x) = \begin{cases} \frac{k\cos x}{\pi - 2x}, & x < \frac{\pi}{2} \\ 3, & x = \frac{\pi}{2} \\ \frac{3\tan 2x}{2x - \pi}, & x > \frac{\pi}{2} \end{cases} \text{.................equation 1}$$

To find the value of constants always try to check continuity at the values of x for which f(x) is changing its expression.

As most of the time discontinuities are here only, if we make the function continuous here, it will automatically become continuous everywhere

From equation 1, it is clear that f(x) is changing its expression at x = π/2

Given, f(x) is continuous everywhere

$\therefore \lim_{x\to\pi/2} f(x) = f(\pi/2)$

$\lim_{h\to 0} f(\pi/2 - h) = \lim_{h\to 0} f(\pi/2 + h) = f(\pi/2)$

$\lim_{h\to 0} f(\pi/2 - h) = f(\pi/2)$

$\lim_{h\to 0}\left\{\frac{k\cos\left(\frac{\pi}{2}-h\right)}{\pi-2\left(\frac{\pi}{2}-h\right)}\right\} = 3$ [Using equation 1]

$\lim_{h\to 0}\left\{\frac{k\sin h}{2h}\right\} = \frac{k}{2}\lim_{h\to 0}\left\{\frac{\sin h}{h}\right\} = \frac{k}{2} = 3$

$[\because \lim_{x\to 0}\frac{\sin x}{x} = 1$ (sandwich theorem)]

$\therefore k = 3 \times 2 = 6$

$\therefore k = 6$

5. *The function* $f(x) = \begin{cases} \frac{x^2}{a}, \text{ if } 0 \le x < 1 \\ a, \text{ if } 1 \le x < \sqrt{2} \\ \frac{2b^2-4b}{x^2}, \text{ if } \sqrt{2} \le x < \infty \end{cases}$

Is continuous on [0, ∞). Find the most suitable values of a and b.

Solution:

A real function f is said to be continuous at x = c, where c is any point in the domain of f if

$\lim_{h\to 0} f(c - h) = \lim_{h\to 0} f(c + h) = f(c)$

Where h is a very small positive number. i.e. left hand limit as x → c (LHL) = right hand limit as x → c (RHL) = value of function at x = c.

A function is continuous at x = c if

$\lim_{x\to c} f(x) = f(c)$

Here we have,

$$f(x) = \begin{cases} \frac{x^2}{a} & ,\text{if} \quad 0 \le x < 1 \\ a & ,\text{if} \quad 1 \le x < \sqrt{2} \\ \frac{2b^2-4b}{x^2} & ,\text{if} \quad \sqrt{2} \le x < \infty \end{cases}$$ ……………………equation 1

The function is defined for [0, ∞) and we need to find the value of a and b so that it is continuous everywhere in its domain (domain = set of numbers for which f is defined)
To find the value of constants always try to check continuity at the values of x for which f(x) is changing its expression.
As most of the time discontinuities are here only, if we make the function continuous here, it will automatically become continuous everywhere
From equation 1, it is clear that f(x) is changing its expression at x = 1
Given, f (x) is continuous everywhere

$$\therefore \lim_{x\to 1} f(x) = f(1)$$

$$\lim_{h\to 0} f(1-h) = \lim_{h\to 0} f(1+h) = f(1)$$

$$\lim_{h\to 0} f(1-h) = f(1)$$

$$\lim_{h\to 0} \left\{\frac{(1-h)^2}{a}\right\} = a$$ [Using equation 1]

$$\therefore \frac{1}{a} = a => a^2 = 1$$

$\therefore$ a = ± 1 …………… equation 2

Also from equation 1, it is clear that f(x) is also changing its expression at x = √2
Given, f (x) is continuous everywhere

$$\therefore \lim_{x\to \sqrt{2}} f(x) = f(\sqrt{2})$$

$$\lim_{h\to 0} f(\sqrt{2}-h) = \lim_{h\to 0} f(\sqrt{2}+h) = f(\sqrt{2})$$

$$\lim_{h\to 0} f(\sqrt{2}-h) = f(\sqrt{2})$$

$$\lim_{h\to 0} a = a = \frac{2b^2-4b}{(\sqrt{2})^2} = b^2 - 2b$$ [Using equation 1]

$\therefore$ $b^2 - 2b = a$ ……………….Equation 3
From equation 2, a = –1
$b^2 - 2b = -1$
$\Rightarrow b^2 - 2b + 1 = 0$
$\Rightarrow (b - 1)^2 = 0$
$\therefore$ b = 1 when a = –1
Putting a = 1 in equation 3:
$b^2 - 2b = 1$
$\Rightarrow b^2 - 2b - 1 = 0$
$\Rightarrow b = \frac{-(-2)\pm\sqrt{(-2)^2-4(-1)}}{2} = \frac{2\pm\sqrt{8}}{2} = 1 \pm \sqrt{2}$
Thus,
For a = –1; b = 1
For a = 1; b = 1 ± √2

6. Find the values of a and b so that the function f (x) defined by

$$f(x) = \begin{cases} x + a\sqrt{2}\sin x, \; if \; 0 \le x < \frac{\pi}{4} \\ 2x\cot x + b, \; if \; \frac{\pi}{4} \le x < \frac{\pi}{2} \\ a\cos 2x - b\sin x, \; if \; \frac{\pi}{2} \le x < \pi \end{cases} \quad becomes\ continuous\ on\ [0, \pi]$$

Solution:

A real function f is said to be continuous at x = c, where c is any point in the domain of f if

$$\lim_{h\to 0} f(c-h) = \lim_{h\to 0} f(c+h) = f(c)$$

Where h is a very small positive number. i.e. left hand limit as x → c (LHL) = right hand limit as x → c (RHL) = value of function at x = c.

A function is continuous at x = c if

$$\lim_{x\to c} f(x) = f(c)$$

Here we have,

$$f(x) = \begin{cases} x + a\sqrt{2}\sin x & ,\text{if} \quad 0 \le x < \pi/4 \\ 2x\cot x + b & ,\text{if} \quad \pi/4 \le x < \pi/2 \\ a\cos 2x - b\sin x & ,\text{if} \quad \pi/2 \le x \le \pi \end{cases}$$ …………………..equation 1

Function is defined for [0, π] and we need to find the value of a and b so that it is continuous everywhere in its domain (domain = set of numbers for which f is defined)

To find the value of constants always try to check continuity at the values of x for which f(x) is changing its expression.
As most of the time discontinuities are here only, if we make the function continuous here, it will automatically become continuous everywhere
From equation 1, it is clear that f(x) is changing its expression at x = π/4
Given, f (x) is continuous everywhere

$$\lim_{x\to\pi/4} f(x) = f(\pi/4)$$

$$\lim_{x\to\pi/4^-} f(x) = \lim_{h\to 0} f(\pi/4 - h) = \lim_{h\to 0}\left\{\left(\frac{\pi}{4} - h\right) + a\sqrt{2}\sin\left(\frac{\pi}{4} - h\right)\right\} = \frac{\pi}{4} + a\sqrt{2}\sin\frac{\pi}{4} = a + \frac{\pi}{4}$$

$$\lim_{x\to\pi/4^+} f(x) = \lim_{h\to 0} f(\pi/4 + h) = \lim_{h\to 0} 2(\pi/4 + h)\cot\left(\frac{\pi}{4} + h\right) + b = \frac{\pi}{2}\cot\frac{\pi}{4} + b = \frac{\pi}{2} + b$$

Since f(x) is continuous at $x = \frac{\pi}{4}$, we have

$$\lim_{x\to\pi/4^-} f(x) = \lim_{x\to\pi/4^+} f(x)$$

$$\Rightarrow a + \frac{\pi}{4} + \frac{\pi}{2} + b$$

$$\Rightarrow a - b = \frac{\pi}{4} \quad ....(1)$$

$$\lim_{x\to\pi/2} f(x) = f(\pi/2)$$

$$\lim_{x\to\pi/2^-} f(x) = \lim_{h\to 0} f(\pi/2 - h) = \lim_{h\to 0} 2\left(\frac{\pi}{2} - h\right)\cot\left(\frac{\pi}{2} - h\right) + b = b$$

$$\lim_{x\to\pi/2^+} f(x) = \lim_{h\to 0} f(\pi/2 + h) = \lim_{h\to 0} a\cos\left(2\left(\frac{\pi}{2} + h\right)\right) - b\sin\left(\frac{\pi}{2} + h\right) = -(a + b)$$

Since f(x) is continuous at $x = \frac{\pi}{2}$, we have

$$\lim_{x\to\pi/2^-} f(x) = \lim_{x\to\pi/2^+} f(x)$$

b = - a - b
b = -a/2 ......(2)
Solving equation (1) and (2), we get

$$a = \frac{\pi}{6} \text{ and } b = \frac{-\pi}{12}$$

7. *The function* $f(x)$ *is defined by* $f(x) = \begin{cases} x^2 + ax + b, \ 0 \le x < 2 \\ 3x + 2, \ 2 \le x \le 4 \\ 2ax + 5b, \ 4 < x \le 8 \end{cases}$

If f is continuous on [0, 8], find the values of a and b.

Solution:

A real function f is said to be continuous at x = c, where c is any point in the domain of f if

$$\lim_{h \to 0} f(c-h) = \lim_{h \to 0} f(c+h) = f(c)$$

Where h is a very small positive number. i.e. left hand limit as x → c (LHL) = right hand limit as x → c (RHL) = value of function at x = c.

A function is continuous at x = c if

$$\lim_{x \to c} f(x) = f(c)$$

Here we have,

$$f(x) = \begin{cases} x^2 + ax + b & , \quad 0 \le x < 2 \\ 3x + 2 & , \quad 2 \le x \le 4 \\ 2ax + 5b & , \quad 4 < x \le 8 \end{cases}$$ ....equation 1

Function is defined for [0, 8] and we need to find the value of a and b so that it is continuous everywhere in its domain (domain = set of numbers for which f is defined)

To find the value of constants always try to check continuity at the values of x for which f(x) is changing its expression.

As most of the time discontinuities are here only, if we make the function continuous here, it will automatically become continuous everywhere

From equation 1, it is clear that f(x) is changing its expression at x = 2

Given, f (x) is continuous everywhere

$$\therefore \lim_{x \to 2} f(x) = f(2)$$

$$\lim_{h \to 0} f(2-h) = \lim_{h \to 0} f(2+h) = f(2)$$

$$\lim_{h \to 0} f(2-h) = f(2)$$

$$\lim_{h \to 0} \{(2-h)^2 + a\,(2-h) + b\} = 3 * 2 + 2 = 10$$ [Using equation 1]

4+2a + b = 8
∴ 2a + b = 4
∴ b = 4 – 2a …………… equation 2
Also from equation 1, it is clear that f(x) is also changing its expression at x = 4
Given, f (x) is continuous everywhere

$$\therefore \lim_{x\to 4} f(x) = f(4)$$

$$\lim_{h\to 0} f(4-h) = \lim_{h\to 0} f(4+h) = f(4)$$

$$\lim_{h\to 0} f(4+h) = f(4)$$

$$\lim_{h\to 0} 2a(4+h) + 5b = 3 \times 4 + 2$$

∴ 8a + 5b = 14 ……………….Equation 3
Putting value of a from equation 2 to equation 3
∴ 8a + 5(4–2a) = 14
⇒ 2a = 6
∴ a = 6/2
= 3
∴ b = 4 – 2×3 = –2
Thus, a = 3 and b = –2

8. If $f(x) = \frac{\tan(\frac{\pi}{4} - x)}{\cot 2x}$ for x ≠ π/4, find the value which can be assigned to f (x) at x = π/4 so that the function f (x) becomes continuous everywhere in [0, π/2].

Solution:
A real function f is said to be continuous at x = c, where c is any point in the domain of f if

$$\lim_{h\to 0} f(c-h) = \lim_{h\to 0} f(c+h) = f(c)$$

Where h is a very small positive number. i.e. left hand limit as x → c (LHL) = right hand limit as x → c (RHL) = value of function at x = c.
A function is continuous at x = c if

$$\lim_{x\to c} f(x) = f(c)$$

Function is defined for [0, π] and we need to find the value of f(x) so that it is continuous everywhere in its domain (domain = set of numbers for which f is defined)

As we have expression for x ≠ π/4, which is continuous everywhere in [0, π], so
If we make it continuous at x = π/4 it is continuous everywhere in its domain.

Given $f(x) = \frac{\tan\left(\frac{\pi}{4}-x\right)}{\cot 2x}$ $for\ x \neq \pi/4$ ....equation 1

Let f(x) is continuous for x = π/4

$\therefore \lim_{x\to\pi/4} f(x) = f(\pi/4)$

$\therefore f\left(\frac{\pi}{4}\right) = \lim_{x\to\pi/4} f(x)$

$= \lim_{x\to\pi/4} \frac{\tan\left(\frac{\pi}{4}-x\right)}{\cot 2x} = \lim_{x\to\pi/4} \frac{\tan\left(\frac{\pi}{4}-x\right)}{\tan\left(\frac{\pi}{2}-2x\right)}$ [∵ tan (π/2–θ) = cot θ]

Multiplying and dividing by π/4–x and π/2–2x to apply sandwich theorem, we get

$$= \lim_{x\to\pi/4} \frac{\frac{\tan\left(\frac{\pi}{4}-x\right)}{\frac{\pi}{4}-x}}{\frac{\tan\left(\frac{\pi}{2}-2x\right)}{\frac{\pi}{2}-2x}} * \frac{\frac{\pi}{4}-x}{\frac{\pi}{2}-2x}$$

We know that from sandwich theorem we have $\lim_{x\to 0} \frac{\tan x}{x} = 1$

$$= \frac{\lim_{x\to\pi/4} \frac{\tan\left(\frac{\pi}{4}-x\right)}{\frac{\pi}{4}-x}}{\lim_{x\to\pi/4} \frac{\tan\left(\frac{\pi}{2}-2x\right)}{\frac{\pi}{2}-2x}} * \frac{1}{2} * \lim_{x\to\frac{\pi}{4}} \frac{\pi-4x}{\pi-4x} = \frac{1}{2}$$

Therefore value that can be assigned to f(x) at x = π/4 is $\frac{1}{2}$

9. $Discuss\ the\ continuity\ of\ the\ function\ f(x) = \begin{cases} 2x-1, & if\ x < 2 \\ \frac{3x}{2}, & if\ x \geq 2 \end{cases}$

Solution:

A real function f is said to be continuous at x = c, where c is any point in the domain of f if

$\lim_{h\to 0} f(c-h) = \lim_{h\to 0} f(c+h) = f(c)$

Where h is a very small positive number. i.e. left hand limit as x → c (LHL) = right hand limit as x → c (RHL) = value of function at x = c.
A function is continuous at x = c if

$$\lim_{x\to c} f(x) = f(c)$$

Here we have,

$$f(x) = \begin{cases} 2x-1 & \text{,if } x < 2 \\ \frac{3x}{2} & \text{,if } x \geq 2 \end{cases}$$ ....equation 1

Function is changing its nature (or expression) at x = 2, so we need to check its continuity at x = 2 first.

LHL = $\lim_{h\to 0} f(2-h) = \lim_{h\to 0} 2(2-h) - 1 = 4 - 1 = 3$

RHL = $\lim_{h\to 0} f(2+h) = \lim_{h\to 0} \frac{3*(2+h)}{2} = \frac{3*2}{2} = 3$

F (2) = $\frac{3*2}{2} = 3$

Clearly, LHL = RHL = f (2)
∴ Function is continuous at x = 2
Let c be any real number such that c > 2
And, $\lim_{x\to c} f(x) = \lim_{x\to c} \frac{3x}{2} = \frac{3c}{2}$

Thus, $\lim_{x\to c} f(x) = f(c)$

∴ f (x) is continuous everywhere for x > 2.
Let m be any real number such that m < 2
∴ f (m) = 2m - 1 [using equation 1]
And, $\lim_{x\to m} f(x) = \lim_{x\to m} 2m - 1 = 2m - 1$

Thus, $\lim_{x\to m} f(x) = f(m)$

∴ f (x) is continuous everywhere for x < 2.
Hence, we can conclude by stating that f(x) is continuous for all Real numbers