EXERCISE 6.1 PAGE NO: 6.10

1. Write the minors and cofactors of each element of the first column of the following matrices and hence evaluate the determinant in each case:

$$(i) A = \begin{bmatrix} 5 & 20 \\ 0 & -1 \end{bmatrix}$$

$$(ii) A = \begin{bmatrix} -1 & 4 \\ 2 & 3 \end{bmatrix}$$

$$(iii) A = \begin{bmatrix} 1 & -3 & 2 \\ 4 & -1 & 2 \\ 3 & 5 & 2 \end{bmatrix}$$

$$(iv) A = \begin{bmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{bmatrix}$$

$$(v) A = \begin{bmatrix} 0 & 2 & 6 \\ 1 & 5 & 0 \\ 3 & 7 & 1 \end{bmatrix}$$

$$(vi) A = \begin{bmatrix} a & h & g \\ h & b & f \\ f & f & c \end{bmatrix}$$

$$(vii) A = \begin{bmatrix} 2 & -1 & 0 & 1 \\ -3 & 0 & 1 & -2 \\ 1 & 1 & -1 & 1 \\ 2 & -1 & 5 & 0 \end{bmatrix}$$

Solution:

(i) Let M_{ij} and C_{ij} represents the minor and co–factor of an element, where i and j represent the row and column.The minor of the matrix can be obtained for a particular element by removing the row and column where the element is present. Then finding the absolute value of the matrix newly formed.

Also, $C_{ij} = (-1)^{i+j} \times M_{ij}$

Given,

$$A = \begin{bmatrix} 5 & 20 \\ 0 & -1 \end{bmatrix}$$

From the given matrix we have,

$M_{11} = -1$

$M_{21} = 20$

$C_{11} = (-1)^{1+1} \times M_{11}$
$= 1 \times -1$
$= -1$
$C_{21} = (-1)^{2+1} \times M_{21}$
$= 20 \times -1$
$= -20$
Now expanding along the first column we get
$|A| = a_{11} \times C_{11} + a_{21} \times C_{21}$
$= 5 \times (-1) + 0 \times (-20)$
$= -5$

(ii) Let M_{ij} and C_{ij} represents the minor and co–factor of an element, where i and j represent the row and column. The minor of matrix can be obtained for particular element by removing the row and column where the element is present. Then finding the absolute value of the matrix newly formed.
Also, $C_{ij} = (-1)^{i+j} \times M_{ij}$
Given
$$A = \begin{bmatrix} -1 & 4 \\ 2 & 3 \end{bmatrix}$$
From the above matrix we have
$M_{11} = 3$
$M_{21} = 4$
$C_{11} = (-1)^{1+1} \times M_{11}$
$= 1 \times 3$
$= 3$
$C_{21} = (-1)^{2+1} \times 4$
$= -1 \times 4$
$= -4$
Now expanding along the first column we get
$|A| = a_{11} \times C_{11} + a_{21} \times C_{21}$
$= -1 \times 3 + 2 \times (-4)$
$= -11$

(iii) Let M_{ij} and C_{ij} represents the minor and co–factor of an element, where i and j represent the row and column. The minor of the matrix can be obtained for a particular element by removing the row and column where the element is present. Then finding the absolute value of the matrix newly formed.

Also, $C_{ij} = (-1)^{i+j} \times M_{ij}$

Given,

$$A = \begin{bmatrix} 1 & -3 & 2 \\ 4 & -1 & 2 \\ 3 & 5 & 2 \end{bmatrix}$$

From given matrix we have,

$$\Rightarrow M_{11} = \begin{bmatrix} -1 & 2 \\ 5 & 2 \end{bmatrix}$$

$M_{11} = -1\times2 - 5\times2$

$M_{11} = -12$

$$\Rightarrow M_{21} = \begin{bmatrix} -3 & 2 \\ 5 & 2 \end{bmatrix}$$

$M_{21} = -3\times2 - 5\times2$

$M_{21} = -16$

$$\Rightarrow M_{31} = \begin{bmatrix} -3 & 2 \\ -1 & 2 \end{bmatrix}$$

$M_{31} = -3 \times 2 - (-1) \times 2$

$M_{31} = -4$

$C_{11} = (-1)^{1+1} \times M_{11}$

$= 1 \times -12$

$= -12$

$C_{21} = (-1)^{2+1} \times M_{21}$

$= -1 \times -16$

$= 16$

$C_{31} = (-1)^{3+1} \times M_{31}$

$= 1 \times -4$

$= -4$

Now expanding along the first column we get

$|A| = a_{11} \times C_{11} + a_{21}\times C_{21} + a_{31}\times C_{31}$

$= 1\times (-12) + 4 \times 16 + 3\times (-4)$

$= -12 + 64 -12$

$= 40$

(iv) Let M_{ij} and C_{ij} represents the minor and co–factor of an element, where i and j represent the row and column. The minor of the matrix can be obtained for a particular element by removing the row and column where the element is present. Then finding the absolute value of the matrix newly formed.

Also, $C_{ij} = (-1)^{i+j} \times M_{ij}$

Given,

$$A = \begin{bmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{bmatrix}$$

$$\Rightarrow M_{11} = \begin{bmatrix} b & ca \\ c & ab \end{bmatrix}$$

$M_{11} = b \times ab - c \times ca$

$M_{11} = ab^2 - ac^2$

$$\Rightarrow M_{21} = \begin{bmatrix} a & bc \\ c & ab \end{bmatrix}$$

$M_{21} = a \times ab - c \times bc$

$M_{21} = a^2b - c^2b$

$$\Rightarrow M_{31} = \begin{bmatrix} a & bc \\ b & ca \end{bmatrix}$$

$M_{31} = a \times c\,a - b \times bc$

$M_{31} = a^2c - b^2c$

$C_{11} = (-1)^{1+1} \times M_{11}$

$= 1 \times (ab^2 - ac^2)$

$= ab^2 - ac^2$

$C_{21} = (-1)^{2+1} \times M_{21}$

$= -1 \times (a^2b - c^2b)$

$= c^2b - a^2b$

$C_{31} = (-1)^{3+1} \times M_{31}$

$= 1 \times (a^2c - b^2c)$

$= a^2c - b^2c$

Now expanding along the first column we get

$|A| = a_{11} \times C_{11} + a_{21}\times C_{21}+ a_{31}\times C_{31}$

$= 1\times (ab^2 - ac^2) + 1 \times (c^2b - a^2b) + 1\times (a^2c - b^2c)$

$= ab^2 - ac^2 + c^2b - a^2b + a^2c - b^2c$

(v) Let M_{ij} and C_{ij} represents the minor and co–factor of an element, where i and j represent the row and column. The minor of matrix can be obtained for particular element by removing the row and column where the element is present. Then finding the absolute value of the matrix newly formed.

Also, $C_{ij} = (-1)^{i+j} \times M_{ij}$

Given,

$$A = \begin{bmatrix} 0 & 2 & 6 \\ 1 & 5 & 0 \\ 3 & 7 & 1 \end{bmatrix}$$

From the above matrix we have,

$$\Rightarrow M_{11} = \begin{bmatrix} 5 & 0 \\ 7 & 1 \end{bmatrix}$$

$M_{11} = 5\times1 - 7\times0$

$M_{11} = 5$

$$\Rightarrow M_{21} = \begin{bmatrix} 2 & 6 \\ 7 & 1 \end{bmatrix}$$

$M_{21} = 2\times1 - 7\times6$

$M_{21} = -40$

$$\Rightarrow M_{31} = \begin{bmatrix} 2 & 6 \\ 5 & 0 \end{bmatrix}$$

$M_{31} = 2\times0 - 5\times6$

$M_{31} = -30$

$C_{11} = (-1)^{1+1} \times M_{11}$

$= 1 \times 5$

$= 5$

$C_{21} = (-1)^{2+1} \times M_{21}$

$= -1 \times -40$

$= 40$

$C_{31} = (-1)^{3+1} \times M_{31}$

$= 1 \times -30$

$= -30$

Now expanding along the first column we get

$|A| = a_{11} \times C_{11} + a_{21}\times C_{21}+ a_{31}\times C_{31}$
$= 0\times 5 + 1 \times 40 + 3\times (-30)$
$= 0 + 40 - 90$
$= 50$

(vi) Let M_{ij} and C_{ij} represents the minor and co–factor of an element, where i and j represent the row and column. The minor of matrix can be obtained for particular element by removing the row and column where the element is present. Then finding the absolute value of the matrix newly formed.
Also, $C_{ij} = (-1)^{i+j} \times M_{ij}$
Given,

$$A = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$$

From the given matrices we have,

$$\Rightarrow M_{11} = \begin{bmatrix} b & f \\ f & c \end{bmatrix}$$

$M_{11} = b \times c - f \times f$

$M_{11} = bc - f^2$

$$\Rightarrow M_{21} = \begin{bmatrix} h & g \\ f & c \end{bmatrix}$$

$M_{21} = h \times c - f \times g$

$M_{21} = hc - fg$

$$\Rightarrow M_{31} = \begin{bmatrix} h & g \\ b & f \end{bmatrix}$$

$M_{31} = h \times f - b \times g$
$M_{31} = hf - bg$
$C_{11} = (-1)^{1+1} \times M_{11}$
$= 1 \times (bc - f^2)$
$= bc - f^2$
$C_{21} = (-1)^{2+1} \times M_{21}$
$= -1 \times (hc - fg)$
$= fg - hc$

$C_{31} = (-1)^{3+1} \times M_{31}$
$= 1 \times (hf - bg)$
$= hf - bg$
Now expanding along the first column we get
$|A| = a_{11} \times C_{11} + a_{21} \times C_{21} + a_{31} \times C_{31}$
$= a\times (bc - f^2) + h\times (fg - hc) + g\times (hf - bg)$
$= abc - af^2 + hgf - h^2c + ghf - bg^2$

(vii) Let M_{ij} and C_{ij} represents the minor and co–factor of an element, where i and j represent the row and column. The minor of matrix can be obtained for particular element by removing the row and column where the element is present. Then finding the absolute value of the matrix newly formed.
Also, $C_{ij} = (-1)^{i+j} \times M_{ij}$
Given,

$$A = \begin{bmatrix} 2 & -1 & 0 & 1 \\ -3 & 0 & 1 & -2 \\ 1 & 1 & -1 & 1 \\ 2 & -1 & 5 & 0 \end{bmatrix}$$

From the given matrix we have,

$$\Rightarrow M_{11} = \begin{bmatrix} 0 & 1 & -2 \\ 1 & -1 & 1 \\ -1 & 5 & 0 \end{bmatrix}$$

$M_{11} = 0(-1 \times 0 - 5 \times 1) - 1(1 \times 0 - (-1) \times 1) + (-2)(1 \times 5 - (-1) \times (-1))$

$M_{11} = -9$

$$\Rightarrow M_{21} = \begin{bmatrix} -1 & 0 & 1 \\ 1 & -1 & 1 \\ -1 & 5 & 0 \end{bmatrix}$$

$M_{21} = -1(-1 \times 0 - 5 \times 1) - 0(1 \times 0 - (-1) \times 1) + 1(1 \times 5 - (-1) \times (-1))$

$M_{21} = 9$

$$\Rightarrow M_{31} = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & -2 \\ -1 & 5 & 0 \end{bmatrix}$$

$M_{31} = -1(1 \times 0 - 5 \times (-2)) - 0(0 \times 0 - (-1) \times (-2)) + 1(0 \times 5 - (-1) \times 1)$
$M_{31} = -9$

$\Rightarrow M_{41} = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & -2 \\ 1 & -1 & 1 \end{bmatrix}$

$M_{41} = -1(1\times1 - (-1) \times (-2)) - 0(0 \times 1 - 1 \times (-2)) + 1(0 \times (-1) - 1 \times 1)$

$M_{41} = 0$

$C_{11} = (-1)^{1+1} \times M_{11}$

$= 1 \times (-9)$

$= -9$

$C_{21} = (-1)^{2+1} \times M_{21}$

$= -1 \times 9$

$= -9$

$C_{31} = (-1)^{3+1} \times M_{31}$

$= 1 \times -9$

$= -9$

$C_{41} = (-1)^{4+1} \times M_{41}$

$= -1 \times 0$

$= 0$

Now expanding along the first column we get

$|A| = a_{11} \times C_{11} + a_{21}\times C_{21} + a_{31}\times C_{31} + a_{41}\times C_{41}$

$= 2 \times (-9) + (-3) \times -9 + 1 \times (-9) + 2 \times 0$

$= -18 + 27 - 9$

$= 0$

2. Evaluate the following determinants:

$(i) \begin{vmatrix} x & -7 \\ x & 5x+1 \end{vmatrix}$

$(ii) \begin{vmatrix} \cos\theta & -\sin\theta \\ sin\theta & \cos\theta \end{vmatrix}$

$(iii) \begin{vmatrix} \cos 15^0 & \sin 15^0 \\ sin75^0 & \cos 75^0 \end{vmatrix}$

$(iv) \begin{vmatrix} a+ib & c+id \\ -c+id & a-ib \end{vmatrix}$

Solution:

(i) Given

$\begin{vmatrix} x & -7 \\ x & 5x+1 \end{vmatrix}$

⇒ $|A| = x(5x + 1) - (-7)x$
$|A| = 5x^2 + 8x$

(ii) Given
$$\begin{vmatrix} \cos\theta & -\sin\theta \\ sin\theta & \cos\theta \end{vmatrix}$$
⇒ $|A| = \cos\theta \times \cos\theta - (-\sin\theta) \times \sin\theta$
$|A| = \cos^2\theta + \sin^2\theta$
We know that $\cos^2\theta + \sin^2\theta = 1$
$|A| = 1$

(iii) Given
$$\begin{vmatrix} \cos 15^0 & \sin 15^0 \\ sin75^0 & \cos 75^0 \end{vmatrix}$$
⇒ $|A| = \cos 15° \times \cos 75° + \sin 15° \times \sin 75°$
We know that cos (A – B) = cos A cos B + Sin A sin B
By substituting this we get, $|A| = \cos(75 - 15)°$
$|A| = \cos 60°$
$|A| = 0.5$

(iv) Given
$$\begin{vmatrix} a+ib & c+id \\ -c+id & a-ib \end{vmatrix}$$
⇒ $|A| = (a + ib)(a - ib) - (c + id)(-c + id)$
$= (a + ib)(a - ib) + (c + id)(c - id)$
$= a^2 - i^2 b^2 + c^2 - i^2 d^2$
We know that $i^2 = -1$
$= a^2 - (-1) b^2 + c^2 - (-1) d^2$
$= a^2 + b^2 + c^2 + d^2$

3. Evaluate:
$$\begin{vmatrix} 2 & 3 & 7 \\ 13 & 17 & 5 \\ 15 & 20 & 12 \end{vmatrix}^2$$

Solution:

Since |AB|= |A||B|

$$|A| = \begin{vmatrix} 2 & 3 & 7 \\ 13 & 17 & 5 \\ 15 & 20 & 12 \end{vmatrix}$$

$$|A| = 2\begin{vmatrix} 17 & 5 \\ 20 & 12 \end{vmatrix} - 3\begin{vmatrix} 13 & 5 \\ 15 & 12 \end{vmatrix} + 7\begin{vmatrix} 13 & 17 \\ 15 & 20 \end{vmatrix}$$

= 2(17 × 12 – 5 × 20) – 3(13 × 12 – 5 × 15) + 7(13 × 20 – 15 × 17)
= 2 (204 – 100) – 3 (156 – 75) + 7 (260 – 255)
= 2×104 – 3×81 + 7×5
= 208 – 243 +35
= 0
Now $|A|^2 = |A| \times |A|$
$|A|^2 = 0$

4. Show that

$$\begin{vmatrix} sin\ 10^0 & -cos\ 10^0 \\ sin\ 80^0 & cos\ 80^0 \end{vmatrix}$$

Solution:
Given

$$\begin{vmatrix} sin\ 10^0 & -cos\ 10^0 \\ sin\ 80^0 & cos\ 80^0 \end{vmatrix}$$

Let the given determinant as A
Using sin (A+B) = sin A × cos B + cos A × sin B
⇒ |A| = sin 10° × cos 80° + cos 10° x sin 80°
|A| = sin (10 + 80)°
|A| = sin90°
|A| = 1
Hence Proved

5. Evaluate $\begin{vmatrix} 2 & 3 & -5 \\ 7 & 1 & -2 \\ -3 & 4 & 1 \end{vmatrix}$ **by two methods.**

Solution:
Given,

$$|A| = \begin{vmatrix} 2 & 3 & -5 \\ 7 & 1 & -2 \\ -3 & 4 & 1 \end{vmatrix}$$

Expanding along the first row

$$|A| = 2\begin{vmatrix} 1 & -2 \\ 4 & 1 \end{vmatrix} - 3\begin{vmatrix} 7 & -2 \\ -3 & 1 \end{vmatrix} - 5\begin{vmatrix} 7 & 1 \\ -3 & 4 \end{vmatrix}$$

= 2(1 × 1 – 4 × (–2)) – 3(7 × 1 – (–2) × (–3)) – 5(7 × 4 – 1 × (–3))
= 2(1 + 8) – 3(7 – 6) – 5(28 + 3)
= 2 × 9 – 3 × 1 – 5 × 31
= 18 – 3 – 155
= –140
Now by expanding along the second column

$$|A| = 2\begin{vmatrix} 1 & -2 \\ 4 & 1 \end{vmatrix} - 7\begin{vmatrix} 3 & -5 \\ 4 & 1 \end{vmatrix} - 3\begin{vmatrix} 3 & -5 \\ 1 & -2 \end{vmatrix}$$

= 2(1 × 1 – 4 × (–2)) – 7(3 × 1 – 4 × (–5)) – 3(3 × (–2) – 1 × (–5))
= 2 (1 + 8) – 7 (3 + 20) – 3 (–6 + 5)
= 2 × 9 – 7 × 23 – 3 × (–1)
= 18 – 161 +3
= –140

$$6.\ Evaluate:\ \Delta = \begin{vmatrix} 0 & sin\alpha & -cos\alpha \\ -sin\alpha & 0 & sin\beta \\ cos\alpha & -sin\beta & 0 \end{vmatrix}$$

Solution:
Given

$$\Delta = \begin{vmatrix} 0 & \sin\alpha & -\cos\alpha \\ -\sin\alpha & 0 & \sin\beta \\ \cos\alpha & -\sin\beta & 0 \end{vmatrix}$$

Expanding along the first row

$$|A| = 0\begin{vmatrix} 0 & \sin\beta \\ -\sin\beta & 0 \end{vmatrix} - \sin\alpha\begin{vmatrix} -\sin\alpha & \sin\beta \\ \cos\alpha & 0 \end{vmatrix} - \cos\alpha\begin{vmatrix} -\sin\alpha & 0 \\ \cos\alpha & -\sin\beta \end{vmatrix}$$

⇒ |A| = 0 (0 – sinβ (–sinβ)) –sinα (–sinα × 0 – sinβ cosα) – cosα ((–sinα) (–sinβ) – 0 × cosα)
|A| = 0 + sinα sinβ cosα – cosα sinα sinβ
|A| = 0

EXERCISE 6.2 PAGE NO: 6.57

1. Evaluate the following determinant:

$$(i)\begin{vmatrix}1 & 3 & 5\\ 2 & 6 & 10\\ 31 & 11 & 38\end{vmatrix}$$

$$(ii)\begin{vmatrix}67 & 19 & 21\\ 39 & 13 & 14\\ 81 & 24 & 26\end{vmatrix}$$

$$(iii)\begin{vmatrix}a & h & g\\ h & b & f\\ g & f & c\end{vmatrix}$$

$$(iv)\begin{vmatrix}1 & -3 & 2\\ 4 & -1 & 2\\ 3 & 5 & 2\end{vmatrix}$$

$$(v)\begin{vmatrix}1 & 4 & 9\\ 4 & 9 & 16\\ 9 & 16 & 25\end{vmatrix}$$

$$(vi)\begin{vmatrix}6 & 3 & -2\\ 2 & -1 & 2\\ -10 & 5 & 2\end{vmatrix}$$

$$(vii)\begin{vmatrix}1 & 3 & 9 & 27\\ 3 & 9 & 27 & 1\\ 9 & 27 & 1 & 3\\ 27 & 1 & 3 & 9\end{vmatrix}$$

$$(viii)\begin{vmatrix}102 & 18 & 36\\ 1 & 3 & 4\\ 17 & 3 & 6\end{vmatrix}$$

Solution:

(i) Given

$$\begin{vmatrix}1 & 3 & 5\\ 2 & 6 & 10\\ 31 & 11 & 38\end{vmatrix}$$

Let, $\Delta = \begin{vmatrix} 1 & 3 & 5 \\ 2 & 6 & 10 \\ 31 & 11 & 38 \end{vmatrix} = 2\begin{vmatrix} 1 & 3 & 5 \\ 1 & 3 & 5 \\ 31 & 11 & 38 \end{vmatrix}$

Now by applying, $R_2 \to R_2 - R_1$, we get,

$$\Rightarrow \Delta = 2\begin{vmatrix} 1 & 3 & 5 \\ 0 & 0 & 0 \\ 31 & 11 & 38 \end{vmatrix} = 0$$

So, $\Delta = 0$

(ii) Given

$$\begin{vmatrix} 67 & 19 & 21 \\ 39 & 13 & 14 \\ 81 & 24 & 26 \end{vmatrix}$$

Let, $\Delta = \begin{vmatrix} 67 & 19 & 21 \\ 39 & 13 & 14 \\ 81 & 24 & 26 \end{vmatrix}$

By applying column operation $C_1 \to C_1 - 4\,C_3$, we get,

$$\Rightarrow \Delta = \begin{vmatrix} 4 & 19 & 21 \\ -3 & 13 & 14 \\ -3 & 24 & 26 \end{vmatrix}$$

Again by applying row operation, $R_1 \to R_1 + R_2$ and $R_3 \to R_3 - R_2$, we get

$$\Rightarrow \Delta = \begin{vmatrix} 1 & 32 & 35 \\ -3 & 13 & 14 \\ 0 & 11 & 12 \end{vmatrix}$$

Now, applying $R_2 \to R_2 + 3\,R_1$, we get,

$$\Rightarrow \Delta = \begin{vmatrix} 1 & 32 & 35 \\ 0 & 109 & 119 \\ 0 & 11 & 12 \end{vmatrix}$$

= 1[(109) (12) – (119) (11)]

= 1308 – 1309

= – 1

So, $\Delta = -1$

(iii) Given,

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

$$\text{Let}, \Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

$= a\,(bc - f^2) - h\,(hc - fg) + g\,(hf - bg)$
$= abc - af^2 - ch^2 + fgh + fgh - bg^2$
$= abc + 2fgh - af^2 - bg^2 - ch^2$
So, $\Delta = abc + 2fgh - af^2 - bg^2 - ch^2$

(iv) Given

$$= \begin{vmatrix} 1 & -3 & 2 \\ 4 & -1 & 2 \\ 3 & 5 & 2 \end{vmatrix}$$

(Ctrl)

$$\text{Let}, \Delta = \begin{vmatrix} 1 & -3 & 2 \\ 4 & -1 & 2 \\ 3 & 5 & 2 \end{vmatrix}$$

By taking 2 as common we get,

$$\Rightarrow \Delta = 2\begin{vmatrix} 1 & -3 & 1 \\ 4 & -1 & 1 \\ 3 & 5 & 1 \end{vmatrix}$$

Now by applying, row operation $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we get

$$\Rightarrow \Delta = 2\begin{vmatrix} 1 & -3 & 1 \\ 3 & 2 & 0 \\ 2 & 8 & 0 \end{vmatrix}$$

$= 2[1(24 - 4)] = 40$
So, $\Delta = 40$

(v) Given

$$\begin{vmatrix} 1 & 4 & 9 \\ 4 & 9 & 16 \\ 9 & 16 & 25 \end{vmatrix}$$

Let, $\Delta = \begin{vmatrix} 1 & 4 & 9 \\ 4 & 9 & 16 \\ 9 & 16 & 25 \end{vmatrix}$

By applying column operation $C_3 \to C_3 - C_2$, we get,

$$\Rightarrow \Delta = \begin{vmatrix} 1 & 4 & 5 \\ 4 & 9 & 7 \\ 9 & 16 & 9 \end{vmatrix}$$

Again by applying column operation $C_2 \to C_2 + C_1$, we get,

$$\Rightarrow \Delta = \begin{vmatrix} 1 & 5 & 5 \\ 4 & 13 & 7 \\ 9 & 25 & 9 \end{vmatrix}$$

Now by applying $C_2 \to C_2 - 5C_1$ and $C_3 \to C_3 - 5C_1$ we get,

$$\Rightarrow \Delta = \begin{vmatrix} 1 & 0 & 0 \\ 4 & -7 & -13 \\ 9 & -20 & -36 \end{vmatrix}$$

$= 1[(-7)(-36) - (-20)(-13)]$
$= 252 - 260$
$= -8$
So, $\Delta = -8$

(vi) Given,

$$\begin{vmatrix} 6 & -3 & 2 \\ 2 & -1 & 2 \\ -10 & 5 & 2 \end{vmatrix}$$

Let, $\Delta = \begin{vmatrix} 6 & -3 & 2 \\ 2 & -1 & 2 \\ -10 & 5 & 2 \end{vmatrix}$

Applying row operations, $R_1 \to R_1 - 3R_2$ and $R_3 \to R_3 + 5R_2$ we get,

$$\Rightarrow \Delta = \begin{vmatrix} 0 & 0 & -4 \\ 2 & -1 & 2 \\ 0 & 0 & 12 \end{vmatrix} = 0$$

So, $\Delta = 0$

(vii) Given

$$\begin{vmatrix} 1 & 3 & 9 & 27 \\ 3 & 9 & 27 & 1 \\ 9 & 27 & 1 & 3 \\ 27 & 1 & 3 & 9 \end{vmatrix}$$

$$\text{Let}, \Delta = \begin{vmatrix} 1 & 3 & 9 & 27 \\ 3 & 9 & 27 & 1 \\ 9 & 27 & 1 & 3 \\ 27 & 1 & 3 & 9 \end{vmatrix}$$

$$\Rightarrow \Delta = \begin{vmatrix} 1 & 3 & 3^2 & 3^3 \\ 3 & 3^2 & 3^3 & 1 \\ 3^2 & 3^3 & 1 & 3 \\ 3^3 & 1 & 3 & 3^2 \end{vmatrix}$$

Applying $C_1 \to C_1 + C_2 + C_3 + C_4$, we get,

$$\Rightarrow \Delta = \begin{vmatrix} 1+3+3^2+3^3 & 3 & 3^2 & 3^3 \\ 1+3+3^2+3^3 & 3^2 & 3^3 & 1 \\ 1+3+3^2+3^3 & 3^3 & 1 & 3 \\ 1+3+3^2+3^3 & 1 & 3 & 3^2 \end{vmatrix}$$

$$\Rightarrow \Delta = (1+3+3^2+3^3)\begin{vmatrix} 1 & 3 & 3^2 & 3^3 \\ 1 & 3^2 & 3^3 & 1 \\ 1 & 3^3 & 1 & 3 \\ 1 & 1 & 3 & 3^2 \end{vmatrix}$$

Now, applying $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$, $R_4 \to R_4 - R_1$, we get

$$\Rightarrow \Delta = (1+3+3^2+3^3)\begin{vmatrix} 1 & 3 & 3^2 & 3^3 \\ 0 & 3^2-3 & 3^3-3^2 & 1-3^3 \\ 0 & 3^3-3 & 1-3^2 & 3-3^3 \\ 0 & 1-3 & 3-3^2 & 3^2-3^3 \end{vmatrix}$$

$$\Rightarrow \Delta = (1+3+3^2+3^3)\begin{vmatrix} 6 & 18 & -26 \\ 24 & -8 & -24 \\ -2 & -6 & -18 \end{vmatrix}$$

$$\Rightarrow \Delta = (1+3+3^2+3^3)2^3\begin{vmatrix} 3 & -9 & 13 \\ 12 & 4 & 12 \\ -1 & 3 & 9 \end{vmatrix}$$

Now, applying $R_1 \to R_1 + 3R_3$

$$\Rightarrow \Delta = (1 + 3 + 3^2 + 3^3)2^3 \begin{vmatrix} 0 & 0 & 40 \\ 12 & 4 & 12 \\ -1 & 3 & 9 \end{vmatrix}$$

$$= (1 + 3 + 3^2 + 3^3)2^3 [40(36 - (-4))]$$

= (40) (8) (40) (40) = 512000

So, Δ = 512000

(viii) Given,

$$\begin{vmatrix} 102 & 18 & 36 \\ 1 & 3 & 4 \\ 17 & 3 & 6 \end{vmatrix}$$

$$\text{Let, } \Delta = \begin{vmatrix} 102 & 18 & 36 \\ 1 & 3 & 4 \\ 17 & 3 & 6 \end{vmatrix}$$

$$\Rightarrow \Delta = 6 \begin{vmatrix} 17 & 3 & 6 \\ 1 & 3 & 4 \\ 17 & 3 & 6 \end{vmatrix}$$

Applying $R_3 \to R_3 - R_1$, we get,

$$\Rightarrow \Delta = 6 \begin{vmatrix} 17 & 3 & 6 \\ 1 & 3 & 4 \\ 0 & 0 & 0 \end{vmatrix} = 0$$

So, $\Delta = 0$

2. Without expanding, show that the value of each of the following determinants is zero:

$$(i) \begin{vmatrix} 8 & 2 & 7 \\ 12 & 3 & 5 \\ 16 & 4 & 3 \end{vmatrix}$$

$$(ii) \begin{vmatrix} 6 & 3 & -2 \\ 2 & -1 & 2 \\ -10 & 5 & 2 \end{vmatrix}$$

$$(iii)\ \begin{vmatrix} 2 & 3 & 7 \\ 13 & 17 & 5 \\ 15 & 20 & 12 \end{vmatrix}$$

$$(iv)\ \begin{vmatrix} \frac{1}{a} & a^2 & bc \\ \frac{1}{b} & b^2 & ac \\ \frac{1}{c} & c^2 & ab \end{vmatrix}$$

$$(v)\ \begin{vmatrix} a+b & 2a+b & 3a+b \\ 2a+b & 3a+b & 4a+b \\ 4a+b & 5a+b & 6a+b \end{vmatrix}$$

$$(vi)\ \begin{vmatrix} 1 & a & a^2-bc \\ 1 & b & b^2-ac \\ 1 & c & c^2-ab \end{vmatrix}$$

$$(vii)\ \begin{vmatrix} 49 & 1 & 6 \\ 39 & 7 & 4 \\ 26 & 2 & 3 \end{vmatrix}$$

$$(viii)\ \begin{vmatrix} 0 & x & y \\ -x & 0 & z \\ -y & -z & 0 \end{vmatrix}$$

$$(ix)\ \begin{vmatrix} 1 & 43 & 6 \\ 7 & 35 & 4 \\ 3 & 17 & 2 \end{vmatrix}$$

$$(x)\ \begin{vmatrix} 1^2 & 2^2 & 3^2 & 4^2 \\ 2^2 & 3^2 & 4^2 & 5^2 \\ 3^2 & 4^2 & 5^2 & 6^2 \\ 4^2 & 5^2 & 6^2 & 7^2 \end{vmatrix}$$

$$(xi)\ \begin{vmatrix} a & b & c \\ a+2x & b+2y & c+2z \\ x & y & z \end{vmatrix}$$

$$(xii)\ \begin{vmatrix} (2^x+2^{-x})^2 & (2^x-2^{-x})^2 & 1 \\ (3^x+3^{-x})^2 & (3^x-3^{-x})^2 & 1 \\ (4^x+4^{-x})^2 & (4^x-4^{-x})^2 & 1 \end{vmatrix}$$

$$(xiii)\ \begin{vmatrix} sin\alpha & cos\alpha & cos(\alpha+\delta) \\ sin\beta & cos\beta & cos(\beta+\delta) \\ sin\gamma & cos\gamma & cos(\gamma+\delta) \end{vmatrix}$$

$$(xiv)\begin{vmatrix} sin^2 23^o & sin^2 67^o & cos180^o \\ -sin^2 67^o & -sin^2 23^o & cos^2 180^o \\ cos180^o & sin^2 23^o & sin^2 67^o \end{vmatrix}$$

$$(xv)\begin{vmatrix} cos(x+y) & -sin(x+y) & cos2y \\ sinx & cosx & siny \\ -cosx & sinx & -cosy \end{vmatrix}$$

$$(xvi)\begin{vmatrix} \sqrt{23}+\sqrt{3} & \sqrt{5} & \sqrt{5} \\ \sqrt{15}+\sqrt{46} & 5 & \sqrt{10} \\ 3+\sqrt{115} & \sqrt{15} & 5 \end{vmatrix}$$

$$(xvii)\begin{vmatrix} sin^2 A & cotA & 1 \\ sin^2 B & cotB & 1 \\ sin^2 C & cotC & 1 \end{vmatrix}, where\ A, B, C\ are\ the\ angles\ of\ \Delta ABC$$

Solution:

(i) Given,

$$\begin{vmatrix} 8 & 2 & 7 \\ 12 & 3 & 5 \\ 16 & 4 & 3 \end{vmatrix}$$

$$\text{Let, } \Delta = \begin{vmatrix} 8 & 2 & 7 \\ 12 & 3 & 5 \\ 16 & 4 & 3 \end{vmatrix}$$

Now by applying row operation $R_3 \to R_3 - R_2$, we get

$$\Rightarrow \Delta = \begin{vmatrix} 8 & 2 & 7 \\ 12 & 3 & 5 \\ 4 & 1 & -2 \end{vmatrix}$$

Again apply row operations $R_2 \to R_2 - R_1$, we get

$$\Rightarrow \Delta = \begin{vmatrix} 8 & 2 & 7 \\ 4 & 1 & -2 \\ 4 & 1 & -2 \end{vmatrix}$$

As, $R_2 = R_3$, therefore the value of the determinant is zero.

(ii) Given,

$$\begin{vmatrix} 6 & -3 & 2 \\ 2 & -1 & 2 \\ -10 & 5 & 2 \end{vmatrix}$$

$$\text{Let, } \Delta = \begin{vmatrix} 6 & -3 & 2 \\ 2 & -1 & 2 \\ -10 & 5 & 2 \end{vmatrix}$$

Taking (– 2) common from C_1 in above matrix we get,

$$\Rightarrow \Delta = \begin{vmatrix} -3 & -3 & 2 \\ -1 & -1 & 2 \\ 5 & 5 & 2 \end{vmatrix}$$

As, $C_1 = C_2$, hence the value of the determinant is zero.

(iii) Given,

$$\begin{vmatrix} 2 & 3 & 7 \\ 13 & 17 & 5 \\ 15 & 20 & 12 \end{vmatrix}$$

$$\text{Let, } \Delta = \begin{vmatrix} 2 & 3 & 7 \\ 13 & 17 & 5 \\ 15 & 20 & 12 \end{vmatrix}$$

Now by applying column operation $C_3 \rightarrow C_3 - C_2$, we get

$$\Rightarrow \Delta = \begin{vmatrix} 2 & 3 & 7 \\ 13 & 17 & 5 \\ 2 & 3 & 7 \end{vmatrix}$$

As, $R_1 = R_3$, so value so determinant is zero.

(iv) Given,

$$\begin{vmatrix} 1/a & a^2 & bc \\ 1/b & b^2 & ac \\ 1/c & c^2 & ab \end{vmatrix}$$

$$\text{Let, } \Delta = \begin{vmatrix} 1/a & a^2 & bc \\ 1/b & b^2 & ac \\ 1/c & c^2 & ab \end{vmatrix}$$

Multiplying R_1, R_2 and R_3 with a, b and c respectively we get,

$$\Rightarrow \Delta = \begin{vmatrix} 1 & a^3 & abc \\ 1 & b^3 & abc \\ 1 & c^3 & abc \end{vmatrix}$$

Now by taking, abc common from C_3 gives,

$$\Rightarrow \Delta = \begin{vmatrix} 1 & a^3 & 1 \\ 1 & b^3 & 1 \\ 1 & c^3 & 1 \end{vmatrix}$$

As, $C_1 = C_3$ hence the value of determinant is zero.

(v) Given,

$$\begin{vmatrix} a+b & 2a+b & 3a+b \\ 2a+b & 3a+b & 4a+b \\ 4a+b & 5a+b & 6a+b \end{vmatrix}$$

$$\text{Let, } \Delta = \begin{vmatrix} a+b & 2a+b & 3a+b \\ 2a+b & 3a+b & 4a+b \\ 4a+b & 5a+b & 6a+b \end{vmatrix}$$

Now by applying column operation $C_3 \rightarrow C_3 - C_2$, we get,

$$\Rightarrow \Delta = \begin{vmatrix} a+b & 2a+b & a \\ 2a+b & 3a+b & a \\ 4a+b & 5a+b & a \end{vmatrix}$$

Again applying column operation $C_2 \rightarrow C_2 - C_1$ gives,

$$\Rightarrow \Delta = \begin{vmatrix} a+b & a & a \\ 2a+b & a & a \\ 4a+b & a & a \end{vmatrix}$$

As, $C_2 = C_3$, so the value of the determinant is zero.

(vi) Given,

$$\begin{vmatrix} 1 & a & a^2-bc \\ 1 & b & b^2-ac \\ 1 & c & c^2-ab \end{vmatrix}$$

$$\text{Let, } \Delta = \begin{vmatrix} 1 & a & a^2-bc \\ 1 & b & b^2-ac \\ 1 & c & c^2-ab \end{vmatrix}$$

$$\Rightarrow \Delta = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} - \begin{vmatrix} 1 & a & bc \\ 1 & b & ac \\ 1 & c & ab \end{vmatrix}$$

Applying $R_2 \rightarrow R_2 - R_1$ and $R_3 \rightarrow R_3 - R_1$, we get,

$$\Rightarrow \Delta = \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix} - \begin{vmatrix} 1 & a & bc \\ 0 & b-a & (a-b)c \\ 0 & c-a & (a-c)b \end{vmatrix}$$

Taking $(b - a)$ and $(c - a)$ common from R_2 and R_3 respectively,

$$\Rightarrow \Delta = (b-a)(c-a)\begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & 1 & c+a \end{vmatrix} - (b-a)(c-a)\begin{vmatrix} 1 & a & bc \\ 0 & 1 & -c \\ 0 & 1 & -b \end{vmatrix}$$

$= [(b - a)(c - a)][(c + a) - (b + a) - (-b + c)]$

$= [(b - a)(c - a)][c + a + b - a - b - c]$

$= [(b - a)(c - a)][0] = 0$

(vii) Given,

$$\begin{vmatrix} 49 & 1 & 6 \\ 39 & 7 & 4 \\ 26 & 2 & 3 \end{vmatrix}$$

$$\text{Let, } \Delta = \begin{vmatrix} 49 & 1 & 6 \\ 39 & 7 & 4 \\ 26 & 2 & 3 \end{vmatrix}$$

Now by applying column operation, $C_1 \rightarrow C_1 - 8C_3$ we get

$$\Rightarrow \Delta = \begin{vmatrix} 1 & 1 & 6 \\ 7 & 7 & 4 \\ 2 & 2 & 3 \end{vmatrix}$$

As, $C_1 = C_2$ hence, the determinant is zero.

(viii) Given,

$$\begin{vmatrix} 0 & x & y \\ -x & 0 & z \\ -y & -z & 0 \end{vmatrix}$$

Let, $\Delta = \begin{vmatrix} 0 & x & y \\ -x & 0 & z \\ -y & -z & 0 \end{vmatrix}$

Multiplying C_1, C_2 and C_3 with z, y and x respectively we get,

$$\Rightarrow \Delta = \left(\frac{1}{xyz}\right)\begin{vmatrix} 0 & xy & yx \\ -xz & 0 & zx \\ -yz & -zy & 0 \end{vmatrix}$$

Now, taking y, x and z common from R_1, R_2 and R_3 gives,

$$\Rightarrow \Delta = \left(\frac{1}{xyz}\right)\begin{vmatrix} 0 & x & x \\ -z & 0 & z \\ -y & -y & 0 \end{vmatrix}$$

Applying $C_2 \to C_2 - C_3$ gives,

$$\Rightarrow \Delta = \left(\frac{1}{xyz}\right)\begin{vmatrix} 0 & 0 & x \\ -z & -z & z \\ -y & -y & 0 \end{vmatrix}$$

As, $C_1 = C_2$, therefore determinant is zero.

(ix) Given,

$$\begin{vmatrix} 1 & 43 & 6 \\ 7 & 35 & 4 \\ 3 & 17 & 2 \end{vmatrix}$$

Let, $\Delta = \begin{vmatrix} 1 & 43 & 6 \\ 7 & 35 & 4 \\ 3 & 17 & 2 \end{vmatrix}$

Applying $C_2 \to C_2 - 7C_3$, we get

$$\Rightarrow \Delta = \begin{vmatrix} 1 & 1 & 6 \\ 7 & 7 & 4 \\ 3 & 3 & 2 \end{vmatrix}$$

As, $C_1 = C_2$, hence determinant is zero

(x) Given,

$$\begin{vmatrix} 1^2 & 2^2 & 3^2 & 4^2 \\ 2^2 & 3^2 & 4^2 & 5^2 \\ 3^2 & 4^2 & 5^2 & 6^2 \\ 4^2 & 5^2 & 6^2 & 7^2 \end{vmatrix}$$

$$\text{Let}, \Delta = \begin{vmatrix} 1^2 & 2^2 & 3^2 & 4^2 \\ 2^2 & 3^2 & 4^2 & 5^2 \\ 3^2 & 4^2 & 5^2 & 6^2 \\ 4^2 & 5^2 & 6^2 & 7^2 \end{vmatrix}$$

Now we have to apply the column operation $C_3 \to C_3 - C_2$, and $C_4 \to C_4 - C_1$, then we get,

$$\Rightarrow \Delta = \begin{vmatrix} 1^2 & 2^2 & 3^2 - 2^2 & 4^2 - 1^2 \\ 2^2 & 3^2 & 4^2 - 3^2 & 5^2 - 2^2 \\ 3^2 & 4^2 & 5^2 - 4^2 & 6^2 - 3^2 \\ 4^2 & 5^2 & 6^2 - 5^2 & 7^2 - 4^2 \end{vmatrix}$$

$$\Rightarrow \Delta = \begin{vmatrix} 1^2 & 2^2 & 5 & 15 \\ 2^2 & 3^2 & 7 & 21 \\ 3^2 & 4^2 & 9 & 27 \\ 4^2 & 5^2 & 11 & 33 \end{vmatrix}$$

Taking 3 common from C_4 we get,

$$\Rightarrow \Delta = 3\begin{vmatrix} 1^2 & 2^2 & 5 & 5 \\ 2^2 & 3^2 & 7 & 7 \\ 3^2 & 4^2 & 9 & 9 \\ 4^2 & 5^2 & 11 & 11 \end{vmatrix}$$

As, C3 = C4 so, the determinant is zero.

(xi) Given,

$$\begin{vmatrix} a & b & c \\ a + 2x & b + 2y & c + 2z \\ x & y & z \end{vmatrix}$$

$$\text{Let, } \Delta = \begin{vmatrix} a & b & c \\ a+2x & b+2y & c+2z \\ x & y & z \end{vmatrix}$$

Now by applying, $C_2 \rightarrow C_2 + C_1$ and $C_3 \rightarrow C_3 + C_1$, we get

$$\Rightarrow \Delta = \begin{vmatrix} a & b & c \\ 2a+2x & 2b+2y & 2c+2z \\ a+x & b+y & c+z \end{vmatrix}$$

Taking 2 common from R_2 we get,

$$\Rightarrow \Delta = 2\begin{vmatrix} a & b & c \\ a+x & b+y & c+z \\ a+x & b+y & c+z \end{vmatrix}$$

As, $R_2 = R_3$, hence value of determinant is zero.

(xii) Given,

$$\begin{vmatrix} (2^x+2^{-x})^2 & (2^x-2^{-x})^2 & 1 \\ (3^x+3^{-x})^2 & (3^x-3^{-x})^2 & 1 \\ (4^x+4^{-x})^2 & (4^x-4^{-x})^2 & 1 \end{vmatrix}$$

$$\text{Let, } \Delta = \begin{vmatrix} (2^x+2^{-x})^2 & (2^x-2^{-x})^2 & 1 \\ (3^x+3^{-x})^2 & (3^x-3^{-x})^2 & 1 \\ (4^x+4^{-x})^2 & (4^x-4^{-x})^2 & 1 \end{vmatrix}$$

$$\Rightarrow \Delta = \begin{vmatrix} 2^{2x}+2^{-2x}+2 & 2^{2x}+2^{-2x}-2 & 1 \\ 3^{2x}+3^{-2x}+2 & 3^{2x}+3^{-2x}-2 & 1 \\ 4^{2x}+4^{-2x}+2 & 4^{2x}+4^{-2x}-2 & 1 \end{vmatrix}$$

By applying, column operation $C_1 \rightarrow C_1 - C_2$, we get

$$\Rightarrow \Delta = \begin{vmatrix} 4 & 2^{2x}+2^{-2x}-2 & 1 \\ 4 & 3^{2x}+3^{-2x}-2 & 1 \\ 4 & 4^{2x}+4^{-2x}-2 & 1 \end{vmatrix}$$

$$\Rightarrow \Delta = 4\begin{vmatrix} 1 & 2^{2x}+2^{-2x}-2 & 1 \\ 1 & 3^{2x}+3^{-2x}-2 & 1 \\ 1 & 4^{2x}+4^{-2x}-2 & 1 \end{vmatrix}$$

As $C_1 = C_3$ hence determinant is zero.

(xiii) Given,

$$\begin{vmatrix} \sin\alpha & \cos\alpha & \cos(\alpha+\delta) \\ \sin\beta & \cos\beta & \cos(\beta+\delta) \\ \sin\gamma & \cos\gamma & \cos(\gamma+\delta) \end{vmatrix}$$

$$\text{Let, } \Delta = \begin{vmatrix} \sin\alpha & \cos\alpha & \cos(\alpha+\delta) \\ \sin\beta & \cos\beta & \cos(\beta+\delta) \\ \sin\gamma & \cos\gamma & \cos(\gamma+\delta) \end{vmatrix}$$

Multiplying C_1 with $\sin\delta$, C_2 with $\cos\delta$, we get

$$\Rightarrow \Delta = \frac{1}{\sin\delta\cos\delta}\begin{vmatrix} \sin\alpha\sin\delta & \cos\alpha\cos\delta & \cos(\alpha+\delta) \\ \sin\beta\sin\delta & \cos\beta\cos\delta & \cos(\beta+\delta) \\ \sin\gamma\sin\delta & \cos\gamma\cos\delta & \cos(\gamma+\delta) \end{vmatrix}$$

Now, by applying column operation, $C_2 \to C_2 - C_1$, we get,

$$\Rightarrow \Delta = \frac{1}{\sin\delta\cos\delta}\begin{vmatrix} \sin\alpha\sin\delta & \cos\alpha\cos\delta - \sin\alpha\sin\delta & \cos(\alpha+\delta) \\ \sin\beta\sin\delta & \cos\beta\cos\delta - \sin\beta\sin\delta & \cos(\beta+\delta) \\ \sin\gamma\sin\delta & \cos\gamma\cos\delta - \sin\gamma\sin\delta & \cos(\gamma+\delta) \end{vmatrix}$$

$$\Rightarrow \Delta = \frac{1}{\sin\delta\cos\delta}\begin{vmatrix} \sin\alpha\sin\delta & \cos(\alpha+\delta) & \cos(\alpha+\delta) \\ \sin\beta\sin\delta & \cos(\beta+\delta) & \cos(\beta+\delta) \\ \sin\gamma\sin\delta & \cos(\gamma+\delta) & \cos(\gamma+\delta) \end{vmatrix}$$

As $C_2 = C_3$ hence determinant is zero.

(xiv) Given,

$$\begin{vmatrix} \sin^2 23^\circ & \sin^2 67^\circ & \cos 180^\circ \\ -\sin^2 67^\circ & -\sin^2 23^\circ & \cos^2 180^\circ \\ \cos 180^\circ & \sin^2 23^\circ & \sin^2 67^\circ \end{vmatrix}$$

$$\text{Let, } \Delta = \begin{vmatrix} \sin^2 23^\circ & \sin^2 67^\circ & \cos 180^\circ \\ -\sin^2 67^\circ & -\sin^2 23^\circ & \cos^2 180^\circ \\ \cos 180^\circ & \sin^2 23^\circ & \sin^2 67^\circ \end{vmatrix}$$

Applying $C_1 \to C_1 + C_2$, we get

$$\Rightarrow \Delta = \begin{vmatrix} \sin^2 23^\circ + \sin^2 67^\circ & \sin^2 67^\circ & \cos 180^\circ \\ -\sin^2 67^\circ - \sin^2 23^\circ & -\sin^2 23^\circ & \cos^2 180^\circ \\ \cos 180^\circ + \sin^2 23^\circ & \sin^2 23^\circ & \sin^2 67^\circ \end{vmatrix}$$

Using, sin (90 – A) = cos A, $\sin^2 A + \cos^2 A = 1$, and $\cos 180^\circ = -1$,

$$\Rightarrow \Delta = \begin{vmatrix} \sin^2 23^\circ + \cos^2 23^\circ & \sin^2 67^\circ & \cos 180^\circ \\ -(\sin^2 67^\circ + \cos^2 67^\circ) & -\sin^2 23^\circ & \cos^2 180^\circ \\ -(1 - \sin^2 23^\circ) & \sin^2 23^\circ & \sin^2 67^\circ \end{vmatrix}$$

$$\Rightarrow \Delta = \begin{vmatrix} 1 & \sin^2 67^\circ & -1 \\ -1 & -\sin^2 23^\circ & 1 \\ -\cos^2 23^\circ & \sin^2 23^\circ & \cos^2 23^\circ \end{vmatrix}$$

Taking, (– 1) common from C_1, we get

$$\Rightarrow \Delta = -\begin{vmatrix} -1 & \sin^2 67^\circ & -1 \\ 1 & -\sin^2 23^\circ & 1 \\ \cos^2 23^\circ & \sin^2 23^\circ & \cos^2 23^\circ \end{vmatrix}$$

Therefore, as $C_1 = C_3$ determinant is zero.

(xv) Given,

$$\begin{vmatrix} \cos(x+y) & -\sin(x+y) & \cos 2y \\ \sin x & \cos x & \sin y \\ -\cos x & \sin x & -\cos y \end{vmatrix}$$

$$\text{Let, } \Delta = \begin{vmatrix} \cos(x+y) & -\sin(x+y) & \cos 2y \\ \sin x & \cos x & \sin y \\ -\cos x & \sin x & -\cos y \end{vmatrix}$$

Multiplying R_2 with sin y and R_3 with cos y we get,

$$\Rightarrow \Delta = \frac{1}{\sin y \cos y}\begin{vmatrix} \cos(x+y) & -\sin(x+y) & \cos 2y \\ \sin x \sin y & \cos x \sin y & \sin^2 y \\ -\cos x \cos y & \sin x \cos y & -\cos^2 y \end{vmatrix}$$

Now, by applying row operation $R_2 \rightarrow R_2 + R_3$, we get,

$$= \frac{1}{\sin y \cos y}\begin{vmatrix} \cos(x+y) & -\sin(x+y) & \cos 2y \\ \sin x \sin y - \cos x \cos y & \cos x \sin y + \sin x \cos y & \sin^2 y - \cos^2 y \\ -\cos x \cos y & \sin x \cos y & -\cos^2 y \end{vmatrix}$$

Taking (– 1) common from R_2, we get

$$= \frac{-1}{\sin y \cos y}\begin{vmatrix} \cos(x+y) & -\sin(x+y) & \cos 2y \\ -\sin x \sin y + \cos x \cos y & -(\cos x \sin y + \sin x \cos y) & -\sin^2 y + \cos^2 y \\ -\cos x \cos y & \sin x \cos y & -\cos^2 y \end{vmatrix}$$

$$\Rightarrow \Delta = \frac{-1}{\sin y \cos y}\begin{vmatrix} \cos(x+y) & -\sin(x+y) & \cos 2y \\ \cos(x+y) & -\sin(x+y) & \cos 2y \\ -\cos x \cos y & \sin x \cos y & -\cos^2 y \end{vmatrix}$$

As $R_1 = R_2$ hence determinant is zero.

(xvi) Given,

$$\begin{vmatrix} \sqrt{23}+\sqrt{3} & \sqrt{5} & \sqrt{5} \\ \sqrt{15}+\sqrt{46} & 5 & \sqrt{10} \\ 3+\sqrt{115} & \sqrt{15} & 5 \end{vmatrix}$$

$$\text{Let, } \Delta = \begin{vmatrix} \sqrt{23}+\sqrt{3} & \sqrt{5} & \sqrt{5} \\ \sqrt{15}+\sqrt{46} & 5 & \sqrt{10} \\ 3+\sqrt{115} & \sqrt{15} & 5 \end{vmatrix}$$

Multiplying C_2 with √3 and C_3 with √23 we get,

$$\Rightarrow \Delta = \begin{vmatrix} \sqrt{23}+\sqrt{3} & \sqrt{15} & \sqrt{115} \\ \sqrt{15}+\sqrt{46} & 5\sqrt{3} & \sqrt{230} \\ 3+\sqrt{115} & \sqrt{45} & 5\sqrt{23} \end{vmatrix}$$

$$\Rightarrow \Delta = \begin{vmatrix} \sqrt{23}+\sqrt{3} & \sqrt{5}(\sqrt{3}) & \sqrt{5}(\sqrt{23}) \\ \sqrt{15}+\sqrt{46} & \sqrt{5}(\sqrt{15}) & \sqrt{5}(\sqrt{46}) \\ 3+\sqrt{115} & \sqrt{5}(3) & \sqrt{5}(\sqrt{115}) \end{vmatrix}$$

Now taking √5 common from C_2 and C_3 we get,

$$\Rightarrow \Delta = \sqrt{5}\sqrt{5}\begin{vmatrix} \sqrt{23}+\sqrt{3} & (\sqrt{3}) & (\sqrt{23}) \\ \sqrt{15}+\sqrt{46} & (\sqrt{15}) & (\sqrt{46}) \\ 3+\sqrt{115} & (3) & (\sqrt{115}) \end{vmatrix}$$

Applying $C_2 \to C_2 + C_3$

$$\Rightarrow \Delta = 5\begin{vmatrix} \sqrt{23} + \sqrt{3} & \sqrt{23} + \sqrt{3} & (\sqrt{23}) \\ \sqrt{15} + \sqrt{46} & \sqrt{15} + \sqrt{46} & (\sqrt{46}) \\ 3 + \sqrt{115} & 3 + \sqrt{115} & (\sqrt{115}) \end{vmatrix}$$

As $C_1 = C_2$ hence determinant is zero.

(xvii) Given,

$$\begin{vmatrix} \sin^2 A & \cot A & 1 \\ \sin^2 B & \cot B & 1 \\ \sin^2 C & \cot C & 1 \end{vmatrix}$$

$$\text{Let}, \Delta = \begin{vmatrix} \sin^2 A & \cot A & 1 \\ \sin^2 B & \cot B & 1 \\ \sin^2 C & \cot C & 1 \end{vmatrix}$$

Now,

$\Delta = \sin^2 A (\cot B - \cot C) - \cot A (\sin^2 B - \sin^2 C) + 1 (\sin^2 B \cot C - \cot B \sin^2 C$

As A, B and C are angles of a triangle,

$A + B + C = 180°$

$\Delta = \sin^2 A \cot B - \sin^2 A \cot C - \cot A \sin^2 B + \cot A \sin^2 C + \sin^2 B \cot C - \cot B \sin^2 C$

By using formulae, we get

$$\frac{sinA}{a} = \frac{sinB}{b} = \frac{sinC}{c} = k$$

$$\cos A = \frac{b^2 + c^2 - a^2}{2\,bc}, \cos B = \frac{a^2 + c^2 - b^2}{2\,a\,c}, \cos C = \frac{a^2 + b^2 - c^2}{2\,a\,b}$$

$\Delta = 0$

Hence proved.

Evaluate the following (3 – 9):

3. $\begin{vmatrix} a & b+c & a^2 \\ b & c+a & b^2 \\ c & a+b & c^2 \end{vmatrix}$

Solution:

Given,

$$\begin{vmatrix} a & b+c & a^2 \\ b & c+a & b^2 \\ c & a+b & c^2 \end{vmatrix}$$

$$\text{Let, } \Delta = \begin{vmatrix} a & b+c & a^2 \\ b & c+a & b^2 \\ c & a+b & c^2 \end{vmatrix}$$

Now by applying column operation $C_2 \to C_2 + C_1$

$$\Rightarrow \Delta = \begin{vmatrix} a & b+c+a & a^2 \\ b & c+a+b & b^2 \\ c & a+b+c & c^2 \end{vmatrix}$$

Taking, (a + b + c) common,

$$\Rightarrow \Delta = (a+b+c)\begin{vmatrix} a & 1 & a^2 \\ b & 1 & b^2 \\ c & 1 & c^2 \end{vmatrix}$$

Again by applying row operation $R_2 \to R_2 - R_1$, and $R_3 \to R_3 - R_1$

$$\Rightarrow \Delta = (a+b+c)\begin{vmatrix} a & 1 & a^2 \\ b-a & 0 & b^2-a^2 \\ c-a & 0 & c^2-a^2 \end{vmatrix}$$

Taking, (b – c) and (c – a) common,

$$\Rightarrow \Delta = (a+b+c)(b-a)(c-a)\begin{vmatrix} a & 1 & a^2 \\ 1 & 0 & b+a \\ 1 & 0 & c+a \end{vmatrix}$$

= (a + b + c) (b – a) (c – a) (b – c)

So, $\Delta = (a + b + c)(b - a)(c - a)(b - c)$

4. $\begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix}$

Solution:

Given,

$$\begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix}$$

$$\text{Let, } \Delta = \begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix}$$

Now by applying row operation, $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$ we get,

$$\Rightarrow \Delta = \begin{vmatrix} 1 & a & bc \\ 0 & b-a & ca-bc \\ 0 & c-a & ab-bc \end{vmatrix}$$

$$= \begin{vmatrix} 1 & a & bc \\ 0 & b-a & c(a-b) \\ 0 & c-a & b(a-c) \end{vmatrix}$$

Taking $(a - b)$ and $(a - c)$ common we get,

$$\Rightarrow \Delta = (a-b)(a-c)\begin{vmatrix} 1 & a & bc \\ 0 & -1 & c \\ 0 & -1 & b \end{vmatrix}$$

$= (a - b)(c - a)(b - c)$

So, $\Delta = (a - b)(b - c)(c - a)$

5. $\begin{vmatrix} x+\lambda & x & x \\ x & x+\lambda & x \\ x & x & x+\lambda \end{vmatrix}$

Solution:

Given,

$$\begin{vmatrix} x+\lambda & x & x \\ x & x+\lambda & x \\ x & x & x+\lambda \end{vmatrix}$$

Let, $\Delta = \begin{vmatrix} x+\lambda & x & x \\ x & x+\lambda & x \\ x & x & x+\lambda \end{vmatrix}$

Applying, $C_1 \to C_1 + C_2 + C_3$, we have,

$$\Rightarrow \Delta = \begin{vmatrix} 3x+\lambda & x & x \\ 3x+\lambda & x+\lambda & x \\ 3x+\lambda & x & x+\lambda \end{vmatrix}$$

Taking, $(3x + \lambda)$ common, we get

$$\Rightarrow \Delta = (3x+\lambda)\begin{vmatrix} 1 & x & x \\ 1 & x+\lambda & x \\ 1 & x & x+\lambda \end{vmatrix}$$

Applying, $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$, we get,

$$\Rightarrow \Delta = (3x+\lambda)\begin{vmatrix} 1 & x & x \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{vmatrix}$$

$= \lambda^2 (3x + \lambda)$

So, $\Delta = \lambda^2 (3x + \lambda)$

6. $\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}$

Solution:

Given,

$$\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}$$

$$\text{Let}, \Delta = \begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}$$

Now we have to apply column operation, $C_1 \rightarrow C_1 + C_2 + C_3$, we get,

$$\Rightarrow \Delta = \begin{vmatrix} a+b+c & b & c \\ a+b+c & a & b \\ a+b+c & c & a \end{vmatrix}$$

Taking, (a + b + c) we get,

$$\Rightarrow \Delta = (a+b+c)\begin{vmatrix} 1 & b & c \\ 1 & a & b \\ 1 & c & a \end{vmatrix}$$

Now by applying row operation, $R_2 \rightarrow R_2 - R_1$, $R_3 \rightarrow R_3 - R_1$, we get,

$$\Rightarrow \Delta = (a+b+c)\begin{vmatrix} 1 & b & c \\ 0 & a-b & b-c \\ 0 & c-b & a-c \end{vmatrix}$$

$= (a + b + c) [(a - b) (a - c) - (b - c) (c - b)]$

$= (a + b + c) [a^2 - ac - ab + bc + b^2 + c^2 - 2bc]$

$= (a + b + c) [a^2 + b^2 + c^2 - ac - ab - bc]$

So, $\Delta = (a + b + c) [a^2 + b^2 + c^2 - ac - ab - bc]$

7. $$\begin{vmatrix} x & 1 & 1 \\ 1 & x & 1 \\ 1 & 1 & x \end{vmatrix}$$

Solution:

Given,

$$\begin{vmatrix} x & 1 & 1 \\ 1 & x & 1 \\ 1 & 1 & x \end{vmatrix}$$

$$\text{Let}, \Delta = \begin{vmatrix} x & 1 & 1 \\ 1 & x & 1 \\ 1 & 1 & x \end{vmatrix}$$

Now by applying column operation, $C_1 \rightarrow C_1 + C_2 + C_3$, we get,

$$\Rightarrow \Delta = \begin{vmatrix} 2+x & 1 & 1 \\ 2+x & x & 1 \\ 2+x & 1 & x \end{vmatrix}$$

$$\Rightarrow \Delta = (2+x)\begin{vmatrix} 1 & 1 & 1 \\ 1 & x & 1 \\ 1 & 1 & x \end{vmatrix}$$

Again by applying row operation, $R_2 \rightarrow R_2 - R_1$, $R_3 \rightarrow R_3 - R_1$, we get,

$$\Rightarrow \Delta = (2+x)\begin{vmatrix} 1 & 1 & 1 \\ 0 & x-1 & 0 \\ 0 & 0 & x-1 \end{vmatrix}$$

$= (2 + x)(x - 1)^2$

So, $\Delta = (2 + x)(x - 1)^2$

8. $\begin{vmatrix} 0 & xy^2 & xz^2 \\ x^2y & 0 & yz^2 \\ xz^2 & zy^2 & 0 \end{vmatrix}$

Solution:

Given,

$$\begin{vmatrix} 0 & xy^2 & xz^2 \\ x^2y & 0 & yz^2 \\ x^2z & zy^2 & 0 \end{vmatrix}$$

$$\text{Let}, \Delta = \begin{vmatrix} 0 & xy^2 & xz^2 \\ x^2y & 0 & yz^2 \\ x^2z & zy^2 & 0 \end{vmatrix}$$

On simplification we get,

$= 0(0 - y^3z^3) - xy^2(0 - x^2yz^3) + xz^2(x^2y^3z - 0)$

$= 0 + x^3y^3z^3 + x^3y^3z^3$

$= 2x^3y^3z^3$

So, $\Delta = 2x^3y^3z^3$

9. $$\begin{vmatrix} a+x & y & z \\ x & a+y & z \\ x & y & a+z \end{vmatrix}$$

Solution:

Given,

$$\begin{vmatrix} a+x & y & z \\ x & a+y & z \\ x & y & a+z \end{vmatrix}$$

$$\text{Let}, \Delta = \begin{vmatrix} a+x & y & z \\ x & a+y & z \\ x & y & a+z \end{vmatrix}$$

Now by applying row operation we get $R_1 \rightarrow R_1 - R_2$ and $R_3 \rightarrow R_3 - R_2$

$$\Rightarrow \Delta = \begin{vmatrix} a & -a & 0 \\ x & a+y & z \\ 0 & -a & a \end{vmatrix}$$

Again by applying column operation, $C_2 \rightarrow C_2 - C_1$

$$\Rightarrow \Delta = \begin{vmatrix} a & 0 & 0 \\ x & a+x+y & z \\ 0 & -a & a \end{vmatrix}$$

= a [a (a + x + y) + az] + 0 + 0

= a^2 (a + x + y + z)

So, Δ = a^2 (a + x + y + z)

$$10.\ \mathbf{If}\ \Delta = \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix},\ \Delta_1 = \begin{vmatrix} 1 & 1 & 1 \\ yz & zx & xy \\ x & y & z \end{vmatrix},\ \mathit{then\ prove\ that}\ \Delta + \Delta_1 = 0$$

Solution:

$$\text{Let}, \Delta = \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} + \begin{vmatrix} 1 & 1 & 1 \\ yz & zx & xy \\ x & y & z \end{vmatrix}$$

As |A| = $|A|^T$

$$\Rightarrow \Delta = \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} + \begin{vmatrix} 1 & yz & x \\ 1 & zx & y \\ 1 & xy & z \end{vmatrix}$$

If any two rows or columns of the determinant are interchanged, then determinant changes its sign

$$\Rightarrow \Delta = \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} - \begin{vmatrix} 1 & x & yz \\ 1 & y & zx \\ 1 & z & xy \end{vmatrix}$$

$$\Rightarrow \Delta = \begin{vmatrix} 0 & 0 & x^2 - yz \\ 0 & 0 & y^2 - zx \\ 0 & 0 & z^2 - xy \end{vmatrix} = 0$$

So, Δ = 0

Hence the proof

Prove the following identities (11 – 45):

11. $\begin{vmatrix} a & b & c \\ a-b & b-c & c-a \\ b+c & c+a & a+b \end{vmatrix} = a^3+b^3+c^3-3abc$

Solution:

Given,

$$\begin{vmatrix} a & b & c \\ a-b & b-c & c-a \\ b+c & c+a & a+b \end{vmatrix}$$

L.H.S = $\begin{vmatrix} a & b & c \\ a-b & b-c & c-a \\ b+c & c+a & a+b \end{vmatrix}$

Apply $C_1 \to C_1 + C_2 + C_3$

$$= \begin{vmatrix} a+b+c & b & c \\ 0 & b-c & c-a \\ 2(a+b+c) & c+a & a+b \end{vmatrix}$$

Taking (a + b + c) common from C_1 we get,

$$= (a+b+c)\begin{vmatrix} 1 & b & c \\ 0 & b-c & c-a \\ 2 & c+a & a+b \end{vmatrix}$$

Applying, $R_3 \to R_3 - 2R_1$

$$= (a+b+c)\begin{vmatrix} 1 & b & c \\ 0 & b-c & c-a \\ 0 & c+a-2b & a+b-2c \end{vmatrix}$$

= (a + b + c) [(b – c) (a + b – 2c) – (c – a) (c + a – 2b)]

= $a^3 + b^3 + c^3 - 3abc$

As, L.H.S = R.H.S

Hence, the proof.

12. $\begin{vmatrix} b+c & a-b & a \\ c+a & b-c & b \\ a+b & c-a & c \end{vmatrix} = 3abc - a^3 - b^3 - c^3$

Solution:

Consider,

L.H.S = $\begin{vmatrix} b+c & a-b & a \\ c+a & b-c & b \\ a+b & c-a & c \end{vmatrix}$

As $|A| = |A|^T$

So, $\begin{vmatrix} b+c & c+a & a+b \\ a-b & b-c & c-a \\ a & b & c \end{vmatrix}$

If any two rows or columns of the determinant are interchanged, then determinant changes its sign

$-\begin{vmatrix} a & b & c \\ a-b & b-c & c-a \\ b+c & c+a & a+b \end{vmatrix}$

Apply $C_1 \rightarrow C_1 + C_2 + C_3$

$= -\begin{vmatrix} a+b+c & b & c \\ 0 & b-c & c-a \\ 2(a+b+c) & c+a & a+b \end{vmatrix}$

Taking (a + b + c) common from C_1 we get,

$= -(a+b+c)\begin{vmatrix} 1 & b & c \\ 0 & b-c & c-a \\ 2 & c+a & a+b \end{vmatrix}$

Applying, $R_3 \rightarrow R_3 - 2R_1$

$$= -(a + b + c)\begin{vmatrix} 1 & b & c \\ 0 & b-c & c-a \\ 0 & c+a-2b & a+b-2c \end{vmatrix}$$

= – (a + b + c) [(b – c) (a + b – 2c) – (c – a) (c + a – 2b)]

= 3abc – a^3 – b^3 – c^3

Therefore, L.H.S = R.H.S,

Hence the proof.

13. $\begin{vmatrix} a+b & b+c & c+a \\ b+c & c+a & a+b \\ c+a & a+b & b+c \end{vmatrix} = 2\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$

Solution:

Given,

$$\begin{vmatrix} a+b & b+c & c+a \\ b+c & c+a & a+b \\ c+a & a+b & b+c \end{vmatrix} = 2\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$

L.H.S = $\begin{vmatrix} a+b & b+c & c+a \\ b+c & c+a & a+b \\ c+a & a+b & b+c \end{vmatrix}$

Now by applying, $C_1 \to C_1 + C_2 + C_3$

$$= \begin{vmatrix} 2(a+b+c) & b+c & c+a \\ 2(a+b+c) & c+a & a+b \\ 2(a+b+c) & a+b & b+c \end{vmatrix}$$

$$= 2\begin{vmatrix} (a+b+c) & b+c & c+a \\ (a+b+c) & c+a & a+b \\ (a+b+c) & a+b & b+c \end{vmatrix}$$

Again apply, $C_2 \to C_2 - C_1$, and $C_3 \to C_3 - C_1$, we have

$$= 2\begin{vmatrix} (a+b+c) & -a & -b \\ (a+b+c) & -b & -c \\ (a+b+c) & -c & -a \end{vmatrix}$$

$$= 2\begin{vmatrix} (a+b+c) & a & b \\ (a+b+c) & b & c \\ (a+b+c) & c & a \end{vmatrix}$$

By expanding, we get

$$= 2\left(\begin{vmatrix} c & a & b \\ a & b & c \\ b & c & a \end{vmatrix} + \begin{vmatrix} a & a & b \\ b & b & c \\ c & c & a \end{vmatrix} + \begin{vmatrix} b & a & b \\ c & b & c \\ a & c & a \end{vmatrix}\right)$$

As in second and third determinant both have same column and its value is zero

Therefore,

$$= 2\begin{vmatrix} c & a & b \\ a & b & c \\ b & c & a \end{vmatrix}$$

$$= 2\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = \text{R. H. S}$$

Hence, the proof.

14. $\begin{vmatrix} a+b+2c & a & b \\ c & b+c+2a & b \\ c & a & c+a+2b \end{vmatrix} = 2(a+b+c)^3$

Solution:

Consider,

$$\text{L. H. S} = \begin{vmatrix} a+b+2c & a & b \\ c & b+c+2a & b \\ c & a & c+a+2b \end{vmatrix},$$

R.H.S = $2(a + b + c)^2$

Applying $C_1 \to C_1 + C_2 + C_3$, we have

$$= \begin{vmatrix} 2(a+b+c) & a & b \\ 2(a+b+c) & b+c+2a & b \\ 2(a+b+c) & a & c+a+2b \end{vmatrix}$$

Taking, 2(a + b + c) common we get,

$$= 2(a+b+C) \begin{vmatrix} 1 & a & b \\ 1 & b+c+2a & b \\ 1 & a & c+a+2b \end{vmatrix}$$

Now, applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we get,

$$= 2(a+b+C) \begin{vmatrix} 1 & a & b \\ 0 & b+c+a & 0 \\ 0 & 0 & c+a+b \end{vmatrix}$$

Thus, we have

L.H.S = $2(a + b + c)\ [1(a + b + c)^2]$

= $2(a + b + c)^3$ = R.H.S

15. $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c)^3$

Solution:

Consider,

L.H.S = $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$

Now by applying, $R_1 \to R_1 + R_2 + R_3$, we get,

$$= \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

Taking (a + b + c) common we get,

$$= (a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

Applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$, we get,

$$= (a+b+c)\begin{vmatrix} 1 & 0 & 0 \\ 2b & -b-c-a & 0 \\ 2c & 0 & -c-a-b \end{vmatrix}$$

$$= (a+b+c)\begin{vmatrix} 1 & 0 & 0 \\ 2b & b+c+a & 0 \\ 2c & 0 & b+c+a \end{vmatrix}$$

$= (a + b + c)^3 =$ R.H.S

Hence, proved.

16. $\begin{vmatrix} 1 & b+c & b^2+c^2 \\ 1 & c+a & c^2+a^2 \\ 1 & a+b & a^2+b^2 \end{vmatrix} = (a-b)(b-c)(c-a)$

Solution:

Consider,

$$\text{L.H.S} = \begin{vmatrix} 1 & b+c & b^2+c^2 \\ 1 & c+a & c^2+a^2 \\ 1 & a+b & a^2+b^2 \end{vmatrix}$$

Now by applying, $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we get,

$$= \begin{vmatrix} 1 & b+c & b^2+c^2 \\ 0 & a-b & a^2-b^2 \\ 0 & a-c & a^2-c^2 \end{vmatrix}$$

$$= (a-b)(a-c)\begin{vmatrix} 1 & b+c & b^2+c^2 \\ 0 & 1 & a+b \\ 0 & 1 & a+c \end{vmatrix}$$

Again by applying $R_3 \to R_3 - R_2$, we get,

$$= (a-b)(a-c)\begin{vmatrix} 1 & b+c & b^2+c^2 \\ 0 & 1 & a+b \\ 0 & 0 & c-a \end{vmatrix}$$

= (a – b) (a – c) (b – c) = R.H.S

Hence, the proof.

17. $\begin{vmatrix} a & a+b & a+2b \\ a+2b & a & a+b \\ a+b & a+2b & a \end{vmatrix} = 9(a+b)b^2$

Solution:

Consider,

$$\text{L.H.S} = \begin{vmatrix} a & a+b & a+2b \\ a+2b & a & a+b \\ a+b & a+2b & a \end{vmatrix}$$

Applying $R_1 \to R_1 + R_2 + R_3$, we get,

$$= \begin{vmatrix} 3a+3b & 3a+3b & 3a+3b \\ a+2b & a & a+b \\ a+b & a+2b & a \end{vmatrix}$$

Taking, (3a + 2b) common we get,

$$= (3a+3b)\begin{vmatrix} 1 & 1 & 1 \\ a+2b & a & a+b \\ a+b & a+2b & a \end{vmatrix}$$

Applying, $C_1 \to C_1 - C_2$ and $C_3 \to C_3 - C_2$, we get,

$$= (3a + 3b)\begin{vmatrix} 0 & 1 & 0 \\ 2b & a & b \\ -b & a + 2b & -2b \end{vmatrix}$$

$$= (3a + 3b)b^2\begin{vmatrix} 0 & 1 & 0 \\ 2 & a & 1 \\ -1 & a + 2b & -2 \end{vmatrix}$$

$= 3(a + b)\ b^2\ (3) = 9(a + b)\ b^2$

= R.H.S

Hence, the proof.

18. $\begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$

Solution:

Consider,

$$\text{L.H.S} = \begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix}$$

Now by applying, $R_1 \to a\ R_1$, $R_2 \to b\ R_2$, $R_3 \to c\ R_3$

We get,

$$= \left(\frac{1}{abc}\right)\begin{vmatrix} a & a^2 & abc \\ b & b^2 & cab \\ c & c^2 & abc \end{vmatrix}$$

$$= \left(\frac{abc}{abc}\right)\begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix}$$

$$= -\begin{vmatrix} a & 1 & a^2 \\ b & 1 & b^2 \\ c & 1 & c^2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

Hence, the proof.

19. $\begin{vmatrix} z & x & y \\ z^2 & x^2 & y^2 \\ z^4 & x^4 & y^4 \end{vmatrix} = \begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ x^4 & y^4 & z^4 \end{vmatrix} = \begin{vmatrix} x^2 & y^2 & z^2 \\ x^4 & y^4 & z^4 \\ x & y & z \end{vmatrix} = xyz(x-y)(y-z)(z-x)(x+y+z)$

Solution:

Given,

$$\begin{vmatrix} z & x & y \\ z^2 & x^2 & y^2 \\ z^4 & x^4 & y^4 \end{vmatrix} = \begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ x^4 & y^4 & z^4 \end{vmatrix} = \begin{vmatrix} x^2 & y^2 & z^2 \\ x^4 & y^4 & z^4 \\ x & y & z \end{vmatrix}$$
$$= xyz(x-y)(y-z)(z-x)(x+y+z)$$

Consider,

$$\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ x^4 & y^4 & z^4 \end{vmatrix}$$

By taking xyz common

$$= xyz \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^3 & y^3 & z^3 \end{vmatrix}$$

$$= xyz \begin{vmatrix} 0 & 1 & 0 \\ x-y & y & z-y \\ x^3-y^3 & y^3 & z^3-y^3 \end{vmatrix}$$

$$= xyz(x-y)(z-y) \begin{vmatrix} 0 & 1 & 0 \\ 1 & y & 1 \\ x^2+y^2+xy & y^3 & z^2+y^2+zy \end{vmatrix}$$

= – xyz(x – y) (z – y) [$z^2 + y^2 + zy - x^2 - y^2 - xy$]

= – xyz(x – y) (z – y) [(z – x) (z + x0 + y (z – x)]

= – xyz(x – y) (z – y) (z – x) (x + y + z)

= R.H.S

Hence, the proof.

20. $\begin{vmatrix} (b+c)^2 & a^2 & bc \\ (c+a)^2 & b^2 & ca \\ (a+b)^2 & c^4 & ab \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)(a^2+b^2+c^2)$

Solution:

Consider,

$$L.H.S = \begin{vmatrix} (b+c)^2 & a^2 & bc \\ (c+a)^2 & b^2 & ca \\ (a+b)^2 & c^2 & ab \end{vmatrix}$$

Applying, $C_1 \to C_1 + C_2 - 2C_3$

$$= \begin{vmatrix} (b+c)^2 - a^2 - 2bc & a^2 & bc \\ (c+a)^2 - b^2 - 2ca & b^2 & ca \\ (a+b)^2 - c^2 - 2ab & c^2 & ab \end{vmatrix}$$

$$= \begin{vmatrix} a^2 + b^2 + c^2 & a^2 & bc \\ a^2 + b^2 + c^2 & b^2 & ca \\ a^2 + b^2 + c^2 & c^2 & ab \end{vmatrix}$$

Taking ($a^2 + b^2 + c^2$), common, we get,

$$= (a^2 + b^2 + c^2)\begin{vmatrix} 1 & a^2 & bc \\ 1 & b^2 & ca \\ 1 & c^2 & ab \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we get,

$$= (a^2 + b^2 + c^2)\begin{vmatrix} 1 & a^2 & bc \\ 0 & b^2 - a^2 & ca - bc \\ 0 & c^2 - a^2 & ab - bc \end{vmatrix}$$

$$= (a^2 + b^2 + c^2)(b - a)(c - a)\begin{vmatrix} 1 & a^2 & bc \\ 0 & b + a & -c \\ 0 & c + a & -b \end{vmatrix}$$

= $(a^2 + b^2 + c^2)$ (b – a) (c – a) [(b + a) (– b) – (– c) (c + a)]

= $(a^2 + b^2 + c^2)$ (a – b) (c – a) (b – c) (a + b + c)

= R.H.S

Hence, the proof.

21. $\begin{vmatrix} (a+1)(a+2) & a+2 & 1 \\ (a+2)(a+3) & a+3 & 1 \\ (a+3)(a+4) & a+4 & 1 \end{vmatrix} = -2$

Solution:

Consider,

$$L.H.S = \begin{vmatrix} (a + 1)(a + 2) & a + 2 & 1 \\ (a + 2)(a + 3) & a + 3 & 1 \\ (a + 3)(a + 4) & a + 4 & 1 \end{vmatrix}$$

Now by applying row operation, $R_3 \rightarrow R_3 - R_2$

$$= \begin{vmatrix} (a + 1)(a + 2) & a + 2 & 1 \\ (a + 2)(a + 3) & a + 3 & 1 \\ (a + 3)2 & 1 & 0 \end{vmatrix}$$

Again by applying, $R_2 \rightarrow R_2 - R_1$

$$= \begin{vmatrix} (a + 1)(a + 2) & a + 2 & 1 \\ (a + 2)2 & 1 & 0 \\ (a + 3)2 & 1 & 0 \end{vmatrix}$$

= [(2a + 4) (1) – (1) (2a + 6)]

= – 2

= R.H.S

Hence, the proof.

22. $$\begin{vmatrix} a^2 & a^2-(b-c)^2 & bc \\ b^2 & b^2-(c-a)^2 & ca \\ c^2 & c^2-(a-b)^2 & ab \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)(a^2+b^2+c^2)$$

Solution:

Consider,

$$L.H.S = \begin{vmatrix} a^2 & a^2-(b-c)^2 & bc \\ b^2 & b^2-(c-a)^2 & ca \\ c^2 & c-(a-b)^2 & ab \end{vmatrix}$$

Applying, $C_2 \to C_2 - 2C_1 - 2C_3$, we get,

$$= \begin{vmatrix} a^2 & a^2-(b-c)^2-2a^2-2bc & bc \\ b^2 & b^2-(c-a)^2a^2-(b-c)^2-2b^2-2ca & ca \\ c^2 & c-(a-b)^2a^2-(b-c)^2-2c^2-2ab & ab \end{vmatrix}$$

$$= \begin{vmatrix} a^2 & -(a^2+b^2+c^2) & bc \\ b^2 & -(a^2+b^2+c^2) & ca \\ c^2 & -(a^2+b^2+c^2) & ab \end{vmatrix}$$

Taking, $-(a^2+b^2+c^2)$ common from C_2 we get,

$$= -(a^2+b^2+c^2)\begin{vmatrix} a^2 & 1 & bc \\ b^2 & 1 & ca \\ c^2 & 1 & ab \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we get

$$= -(a^2+b^2+c^2)\begin{vmatrix} a^2 & 1 & bc \\ b^2-a^2 & 0 & ca-bc \\ c^2-a^2 & 0 & ab-bc \end{vmatrix}$$

$$= -(a^2 + b^2 + c^2)(a-b)(c-a)\begin{vmatrix} a^2 & 1 & bc \\ -(b+a) & 0 & c \\ c+a & 0 & -b \end{vmatrix}$$

$= -(a^2 + b^2 + c^2)(a-b)(c-a)[(-(b+a))(-b) - (c)(c+a)]$

$= (a-b)(b-c)(c-a)(a+b+c)(a^2+b^2+c^2)$

= R.H.S

Hence, the proof.

23. $\begin{vmatrix} 1 & a^2+bc & a^3 \\ 1 & b^2+ca & b^3 \\ 1 & c^2+ab & c^3 \end{vmatrix} = -(a-b)(b-c)(c-a)(a^2+b^2+c^2)$

Solution:

Consider,

$$L.H.S = \begin{vmatrix} 1 & a^2+bc & a^3 \\ 1 & b^2+ca & b^3 \\ 1 & c^2+ab & c^3 \end{vmatrix}$$

Applying, $R_2 \to R_2 - R_1$, and $R_3 \to R_3 - R_1$

$$= \begin{vmatrix} 1 & a^2+bc & a^3 \\ 0 & b^2+ca-a^2-bc & b^3-a^3 \\ 0 & c^2+ab-a^2-bc & c^3-a^3 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & a^2+bc & a^3 \\ 0 & b^2-a^2-c(b-a) & b^3-a^3 \\ 0 & c^2-a^2+b(c-a) & c^3-a^3 \end{vmatrix}$$

$$= (b-a)(c-a)\begin{vmatrix} 1 & a^2+bc & a^3 \\ 0 & b+a-c & b^2+a^2+ab \\ 0 & c+a+b & c^2+a^2+ac \end{vmatrix}$$

$= (b-a)(c-a)[((b+a-c))(c^2+a^2+ac) - (b^2+a^2+ab)(c^2+a^2+ac)]$

$= -(a-b)(c-a)(b-c)(a^2+b^2+c^2)$

= R.H.S

Hence, the proof.

24. $\begin{vmatrix} a^2 & bc & ac+c^2 \\ a^2+ab & b^2 & ac \\ ab & b^2+bc & c^2 \end{vmatrix} = 4a^2b^2c^2$

Solution:

Consider,

$$L.H.S = \begin{vmatrix} a^2 & bc & ac + c^2 \\ a^2 + ab & b^2 & ac \\ ab & b^2 + bc & c^2 \end{vmatrix}$$

Taking, a, b, c common from C_1, C_2, C_3 respectively we get,

$$= abc\begin{vmatrix} a & c & a + c \\ a + b & b & a \\ b & b + c & c \end{vmatrix}$$

Applying, $C_1 \to C_1 + C_2 + C_3$, we get,

$$= abc\begin{vmatrix} 2(a + c) & c & a + c \\ 2(a + b) & b & a \\ 2(b + c) & b + c & c \end{vmatrix}$$

$$= 2abc\begin{vmatrix} (a + c) & c & a + c \\ (a + b) & b & a \\ (b + c) & b + c & c \end{vmatrix}$$

Applying, $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$, we get,

$$= 2abc\begin{vmatrix} (a + c) & -a & 0 \\ (a + b) & -a & -b \\ (b + c) & 0 & -b \end{vmatrix}$$

Applying, $C_1 \to C_1 + C_2 + C_3$, we get,

$$= 2abc\begin{vmatrix} c & -a & 0 \\ 0 & -a & -b \\ c & 0 & -b \end{vmatrix}$$

Taking c, a, b common from C_1, C_2, C_3 respectively, we get,

$$= 2a^2b^2c^2\begin{vmatrix} 1 & -1 & 0 \\ 0 & -1 & -1 \\ 1 & 0 & -1 \end{vmatrix}$$

Applying, $R_3 \to R_3 - R_1$, we have

$$= 2a^2b^2c^2\begin{vmatrix} 1 & -1 & 0 \\ 0 & -1 & -1 \\ 0 & 1 & -1 \end{vmatrix}$$

$= 2a^2b^2c^2 (2)$

$= 4a^2b^2c^2$ = R.H.S

Hence, proved.

25. $\begin{vmatrix} x+4 & x & x \\ x & x+4 & x \\ x & x & x+4 \end{vmatrix} = 16(3x+4)$

Solution:

Consider,

$$\text{L.H.S} = \begin{vmatrix} x+4 & x & x \\ x & x+4 & x \\ x & x & x+4 \end{vmatrix}$$

Applying, $C_1 \to C_1 + C_2 + C_3$, we get,

$$= \begin{vmatrix} 3x+4 & x & x \\ 3x+4 & x+4 & x \\ 3x+4 & x & x+4 \end{vmatrix}$$

Taking (3x + 4) common we get,

$$= (3x + 4)\begin{vmatrix} 1 & x & x \\ 1 & x+4 & x \\ 1 & x & x+4 \end{vmatrix}$$

Now by applying, $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we get,

$$= (3x + 4)\begin{vmatrix} 1 & x & x \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{vmatrix}$$

= 16 (3x + 4)

Hence the proof.

26. $\begin{vmatrix} 1 & 1+p & 1+p+q \\ 2 & 3+2p & 4+3p+2q \\ 3 & 6+3p & 10+6p+3q \end{vmatrix} = 1$

Solution:

Let $\Delta = \begin{vmatrix} 1 & 1+p & 1+p+q \\ 2 & 3+2p & 4+3p+2q \\ 3 & 6+3p & 10+6p+3q \end{vmatrix}$

We know that the value of a determinant remains same if we apply the operation $R_i \to R_i + kR_j$ or $C_i \to C_i + kC_j$.

Applying $C_2 \to C_2 - pC_1$, we get

$$\Delta = \begin{vmatrix} 1 & 1+p-p(1) & 1+p+q \\ 2 & 3+2p-p(2) & 4+3p+2q \\ 3 & 6+3p-p(3) & 10+6p+3q \end{vmatrix}$$

$$\Rightarrow \Delta = \begin{vmatrix} 1 & 1 & 1+p+q \\ 2 & 3 & 4+3p+2q \\ 3 & 6 & 10+6p+3q \end{vmatrix}$$

Applying $C_3 \to C_3 - qC_1$, we get

$$\Delta = \begin{vmatrix} 1 & 1 & 1+p+q-q(1) \\ 2 & 3 & 4+3p+2q-q(2) \\ 3 & 6 & 10+6p+3q-q(3) \end{vmatrix}$$

$$\Rightarrow \Delta = \begin{vmatrix} 1 & 1 & 1+p \\ 2 & 3 & 4+3p \\ 3 & 6 & 10+6p \end{vmatrix}$$

Applying $C_3 \rightarrow C_3 - pC_2$, we get

$$\Delta = \begin{vmatrix} 1 & 1 & 1+p-p(1) \\ 2 & 3 & 4+3p-p(3) \\ 3 & 6 & 10+6p-p(6) \end{vmatrix}$$

$$\Rightarrow \Delta = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 3 & 6 & 10 \end{vmatrix}$$

Applying $C_2 \rightarrow C_2 - C_1$, we get

$$\Delta = \begin{vmatrix} 1 & 1-1 & 1 \\ 2 & 3-2 & 4 \\ 3 & 6-3 & 10 \end{vmatrix}$$

$$\Rightarrow \Delta = \begin{vmatrix} 1 & 0 & 1 \\ 2 & 1 & 4 \\ 3 & 3 & 10 \end{vmatrix}$$

Applying $C_3 \rightarrow C_3 - C_1$, we get

$$\Delta = \begin{vmatrix} 1 & 0 & 1-1 \\ 2 & 1 & 4-2 \\ 3 & 3 & 10-3 \end{vmatrix}$$

$$\Rightarrow \Delta = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 3 & 6 & 10 \end{vmatrix}$$

$$\Rightarrow \Delta = \begin{vmatrix} 1 & 0 & 0 \\ 2 & 1 & 2 \\ 3 & 3 & 7 \end{vmatrix}$$

Expanding the determinant along R_1, we have

$\Delta = 1[(1)(7) - (3)(2)] - 0 + 0$

∴ $\Delta = 7 - 6 = 1$

Thus, $\begin{vmatrix} 1 & 1+p & 1+p+q \\ 2 & 3+2p & 4+3p+2q \\ 3 & 6+3p & 10+6p+3q \end{vmatrix} = 1$

Hence the proof.

EXERCISE 6.3 PAGE NO: 6.71

1. Find the area of the triangle with vertices at the points:
(i) (3, 8), (-4, 2) and (5, -1)
(ii) (2, 7), (1, 1) and (10, 8)
(iii) (-1, -8), (-2, -3) and (3, 2)
(iv) (0, 0), (6, 0) and (4, 3)

Solution:
(i) Given (3, 8), (-4, 2) and (5, -1) are the vertices of the triangle.
We know that, if vertices of a triangle are (x_1, y_1), (x_2, y_2) and (x_3, y_3), then the area of the triangle is given by:

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Now, substituting given value in above formula

$$\Delta = \frac{1}{2}\begin{vmatrix} 3 & 8 & 1 \\ -4 & 2 & 1 \\ 5 & -1 & 1 \end{vmatrix}$$

Expanding along R_1

$$= \frac{1}{2}\left[3\begin{vmatrix} 2 & 1 \\ -1 & 1 \end{vmatrix} - 8\begin{vmatrix} -4 & 1 \\ 5 & 1 \end{vmatrix} + 1\begin{vmatrix} -4 & 2 \\ 5 & -1 \end{vmatrix}\right]$$

$$= \frac{1}{2}\left[3(3) - 8(-9) + 1(-6)\right]$$

$$= \frac{1}{2}[9 + 72 - 6]$$

$= \frac{75}{2}$ Square units

Thus area of triangle is $\frac{75}{2}$ square units

(ii) Given (2, 7), (1, 1) and (10, 8) are the vertices of the triangle.
We know that if vertices of a triangle are (x_1, y_1), (x_2, y_2) and (x_3, y_3), then the area of the

triangle is given by:

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Now, substituting given value in above formula

$$\Delta = \frac{1}{2}\begin{vmatrix} 2 & 7 & 1 \\ 1 & 1 & 1 \\ 10 & 8 & 1 \end{vmatrix}$$

Expanding along R_1

$$= \frac{1}{2}\left[2\begin{vmatrix} 1 & 1 \\ 8 & 1 \end{vmatrix} - 7\begin{vmatrix} 1 & 1 \\ 10 & 1 \end{vmatrix} + 1\begin{vmatrix} 1 & 1 \\ 10 & 8 \end{vmatrix}\right]$$

$$= \frac{1}{2}[2(-7) - 7(-9) + 1(-2)]$$

$$= \frac{1}{2}[-14 + 63 - 2]$$

$$= \frac{47}{2} \text{ Square units}$$

Thus area of triangle is $\frac{47}{2}$ square units

(iii) Given (-1, -8), (-2, -3) and (3, 2) are the vertices of the triangle.
We know that if vertices of a triangle are (x_1, y_1), (x_2, y_2) and (x_3, y_3), then the area of the triangle is given by:

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Now, substituting given value in above formula

$$\Delta = \frac{1}{2}\begin{vmatrix} -1 & -8 & 1 \\ -2 & -3 & 1 \\ 3 & 2 & 1 \end{vmatrix}$$

Expanding along R_1

$= \frac{1}{2}\left[-1\begin{vmatrix}-3 & 1\\ 2 & 1\end{vmatrix} - 8\begin{vmatrix}-2 & 1\\ 3 & 1\end{vmatrix} + 1\begin{vmatrix}-2 & -3\\ 3 & 2\end{vmatrix}\right]$

$= \frac{1}{2}[-1(-5) - 8(-5) + 1(5)]$

$= \frac{1}{2}[5 - 40 + 5]$

$= \frac{-30}{2}$ Square units

As we know area cannot be negative. Therefore, 15 square unit is the area

Thus area of triangle is 15 square units

(iv) Given (-1, -8), (-2, -3) and (3, 2) are the vertices of the triangle.

We know that if vertices of a triangle are (x_1, y_1), (x_2, y_2) and (x_3, y_3), then the area of the triangle is given by:

$$\Delta = \frac{1}{2}\begin{vmatrix}x_1 & y_1 & 1\\ x_2 & y_2 & 1\\ x_3 & y_3 & 1\end{vmatrix}$$

Now, substituting given value in above formula

$$\Delta = \frac{1}{2}\begin{vmatrix}0 & 0 & 1\\ 6 & 0 & 1\\ 4 & 3 & 1\end{vmatrix}$$

Expanding along R_1

$= \frac{1}{2}\left[0\begin{vmatrix}0 & 1\\ 3 & 1\end{vmatrix} - 0\begin{vmatrix}6 & 1\\ 4 & 1\end{vmatrix} + 1\begin{vmatrix}6 & 0\\ 4 & 3\end{vmatrix}\right]$

$= \frac{1}{2}[0 - 0 + 1(18)]$

$= \frac{1}{2}[18]$

= 9 square units

Thus area of triangle is 9 square units

2. Using the determinants show that the following points are collinear:

(i) (5, 5), (-5, 1) and (10, 7)

(ii) (1, -1), (2, 1) and (10, 8)
(iii) (3, -2), (8, 8) and (5, 2)
(iv) (2, 3), (-1, -2) and (5, 8)

Solution:
(i) Given (5, 5), (-5, 1) and (10, 7)
We have the condition that three points to be collinear, the area of the triangle formed by these points will be zero. Now, we know that, vertices of a triangle are (x_1, y_1), (x_2, y_2) and (x_3, y_3), then the area of the triangle is given by

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

Now, substituting given value in above formula

$$\Delta = \frac{1}{2}\begin{vmatrix} 5 & 5 & 1 \\ -5 & 1 & 1 \\ 10 & 7 & 1 \end{vmatrix} = 0$$

$$\frac{1}{2}\begin{vmatrix} 5 & 5 & 1 \\ -5 & 1 & 1 \\ 10 & 7 & 1 \end{vmatrix}$$

Expanding along R_1

$$= \frac{1}{2}\left[5\begin{vmatrix} 1 & 1 \\ 7 & 1 \end{vmatrix} - 5\begin{vmatrix} -5 & 1 \\ 10 & 1 \end{vmatrix} + 1\begin{vmatrix} -5 & 1 \\ 10 & 7 \end{vmatrix}\right]$$

$$= \frac{1}{2}[5(-6) - 5(-15) + 1(-45)]$$

$$= \frac{1}{2}[-35 + 75 - 45]$$

= 0

Since, Area of triangle is zero

Hence, points are collinear

(ii) Given (1, -1), (2, 1) and (10, 8)
We have the condition that three points to be collinear, the area of the triangle formed

by these points will be zero. Now, we know that, vertices of a triangle are (x_1, y_1), (x_2, y_2) and (x_3, y_3), then the area of the triangle is given by,

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

Now, by substituting given value in above formula

$$\Delta = \frac{1}{2}\begin{vmatrix} 1 & -1 & 1 \\ 2 & 1 & 1 \\ 4 & 5 & 1 \end{vmatrix} = 0$$

$$\frac{1}{2}\begin{vmatrix} 1 & -1 & 1 \\ 2 & 1 & 1 \\ 4 & 5 & 1 \end{vmatrix}$$

Expanding along R_1

$$= \frac{1}{2}\left[1\begin{vmatrix} 1 & 1 \\ 5 & 1 \end{vmatrix} + 1\begin{vmatrix} 2 & 1 \\ 4 & 1 \end{vmatrix} + 1\begin{vmatrix} 2 & 1 \\ 4 & 5 \end{vmatrix}\right]$$

$$= \frac{1}{2}[1 - 5 + 2 - 4 + 10 - 4]$$

$$= \frac{1}{2}[0]$$

= 0

Since, Area of triangle is zero.

Hence, points are collinear.

(iii) Given (3, -2), (8, 8) and (5, 2)

We have the condition that three points to be collinear, the area of the triangle formed by these points will be zero. Now, we know that, vertices of a triangle are (x_1, y_1), (x_2, y_2) and (x_3, y_3), then the area of the triangle is given by,

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

Now, by substituting given value in above formula

$$\Delta = \frac{1}{2}\begin{vmatrix} 3 & -2 & 1 \\ 8 & 8 & 1 \\ 5 & 2 & 1 \end{vmatrix} = 0$$

$$\frac{1}{2}\begin{vmatrix} 3 & -2 & 1 \\ 8 & 8 & 1 \\ 5 & 2 & 1 \end{vmatrix}$$

Expanding along R_1

$$= \frac{1}{2}\left[3\begin{vmatrix} 8 & 1 \\ 2 & 1 \end{vmatrix} + 2\begin{vmatrix} 8 & 1 \\ 5 & 1 \end{vmatrix} + 1\begin{vmatrix} 8 & 8 \\ 5 & 2 \end{vmatrix}\right]$$

$$= \frac{1}{2}[3(6) + 2(3) + 1(-24)]$$

$$= \frac{1}{2}[0]$$

$= 0$

Since, Area of triangle is zero

Hence, points are collinear.

(iv) Given (2, 3), (-1, -2) and (5, 8)

We have the condition that three points to be collinear, the area of the triangle formed by these points will be zero. Now, we know that, vertices of a triangle are (x_1, y_1), (x_2, y_2) and (x_3, y_3), then the area of the triangle is given by,

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

Now, by substituting given value in above formula

$$\Delta = \frac{1}{2}\begin{vmatrix} 2 & 3 & 1 \\ -1 & -2 & 1 \\ 5 & 8 & 1 \end{vmatrix} = 0$$

$$\frac{1}{2}\begin{vmatrix} 2 & 3 & 1 \\ -1 & -2 & 1 \\ 5 & 8 & 1 \end{vmatrix}$$

Expanding along R_1

$$= \frac{1}{2}\left[2\begin{vmatrix}-2 & 1\\ 8 & 1\end{vmatrix} - 3\begin{vmatrix}-1 & 1\\ 5 & 1\end{vmatrix} + 1\begin{vmatrix}-1 & -2\\ 5 & 8\end{vmatrix}\right]$$

$$= \frac{1}{2}[2(-10) - 3(-1-5) + 1(-8+10)]$$

$$= \frac{1}{2}[-20 + 18 + 2]$$

= 0

Since, Area of triangle is zero

Hence, points are collinear.

3. If the points (a, 0), (0, b) and (1, 1) are collinear, prove that a + b = ab

Solution:

Given (a, 0), (0, b) and (1, 1) are collinear

We have the condition that three points to be collinear, the area of the triangle formed by these points will be zero. Now, we know that, vertices of a triangle are (x_1, y_1), (x_2, y_2) and (x_3, y_3), then the area of the triangle is given by,

$$\Delta = \frac{1}{2}\begin{vmatrix}x_1 & y_1 & 1\\ x_2 & y_2 & 1\\ x_3 & y_3 & 1\end{vmatrix} = 0$$

Thus

$$\frac{1}{2}\begin{vmatrix}a & 0 & 1\\ 0 & b & 1\\ 1 & 1 & 1\end{vmatrix} = 0$$

Expanding along R_1

$$\Rightarrow 0 = \frac{1}{2}\left[a\begin{vmatrix}b & 1\\ 1 & 1\end{vmatrix} - 0\begin{vmatrix}0 & 1\\ 1 & 1\end{vmatrix} + 1\begin{vmatrix}0 & b\\ 1 & 1\end{vmatrix}\right]$$

$$\Rightarrow \frac{1}{2}[a(b-1) - 0(-1) + 1(-b)] = 0$$

$$\Rightarrow \frac{1}{2}[ab - a - b] = 0$$

⇒ a + b = ab

Hence Proved

4. Using the determinants prove that the points (a, b), (a', b') and (a - a', b - b) are

collinear if a b' = a' b.

Solution:

Given (a, b), (a', b') and (a - a', b - b) are collinear

We have the condition that three points to be collinear, the area of the triangle formed by these points will be zero. Now, we know that, vertices of a triangle are (x_1, y_1), (x_2, y_2) and (x_3, y_3), then the area of the triangle is given by,

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

Thus

$$\frac{1}{2}\begin{vmatrix} a & b & 1 \\ a' & b' & 1 \\ a-a' & b-b' & 1 \end{vmatrix} = 0$$

Expanding along R_1

$$\Rightarrow 0 = \frac{1}{2}\left[a\begin{vmatrix} b' & 1 \\ b-b' & 1 \end{vmatrix} - b\begin{vmatrix} a' & 1 \\ a-a' & 1 \end{vmatrix} + 1\begin{vmatrix} a' & b' \\ a-a' & b-b' \end{vmatrix}\right]$$

$$\Rightarrow \frac{1}{2}\left[a(b'-b+b') - b(a'-a+a') + 1(a'b - a'b' - ab' + a'b')\right] = 0$$

$$\Rightarrow \frac{1}{2}\left[a'b - ab + ab' - a'b + ab + a'b + a'b - a'b' - ab' + a'b'\right] = 0$$

$\Rightarrow ab' - a'b = 0$

$\Rightarrow$ a b' = a' b

Hence, the proof.

5. Find the value of λ so that the points (1, -5), (-4, 5) and (λ, 7) are collinear.

Solution:

Given (1, -5), (-4, 5) and (λ, 7) are collinear

We have the condition that three points to be collinear, the area of the triangle formed by these points will be zero. Now, we know that, vertices of a triangle are (x_1, y_1), (x_2, y_2) and (x_3, y_3), then the area of the triangle is given by,

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

Now, by substituting given value in above formula

$$\Delta = \frac{1}{2}\begin{vmatrix} 1 & -5 & 1 \\ -4 & 5 & 1 \\ \lambda & 7 & 1 \end{vmatrix} = 0$$

Expanding along R_1

$$\Rightarrow \frac{1}{2}\left[1\begin{vmatrix} 5 & 1 \\ 7 & 1 \end{vmatrix} + 5\begin{vmatrix} -4 & 1 \\ \lambda & 1 \end{vmatrix} + 1\begin{vmatrix} -4 & 5 \\ \lambda & 7 \end{vmatrix}\right] = 0$$

$$\Rightarrow \frac{1}{2}[1(-2) + 5(-4-\lambda) + 1(-28-5\lambda)] = 0$$

$$\Rightarrow \frac{1}{2}[-2-20-5\lambda-28-5\lambda] = 0$$

$\Rightarrow - 50 – 10\lambda = 0$

$\Rightarrow \lambda = – 5$

6. Find the value of x if the area of Δ is 35 square cms with vertices (x, 4), (2, -6) and (5, 4).

Solution:

Given (x, 4), (2, -6) and (5, 4) are the vertices of a triangle.

We have the condition that three points to be collinear, the area of the triangle formed by these points will be zero. Now, we know that, vertices of a triangle are (x_1, y_1), (x_2, y_2) and (x_3, y_3), then the area of the triangle is given by,

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Now, by substituting given value in above formula

$$\Rightarrow 35 = \left|\frac{1}{2}\begin{vmatrix} x & 4 & 1 \\ 2 & -6 & 1 \\ 5 & 4 & 1 \end{vmatrix}\right|$$

Removing modulus

$\Rightarrow \pm 2 \times 35 = \begin{vmatrix} x & 4 & 1 \\ 2 & -6 & 1 \\ 5 & 4 & 1 \end{vmatrix}$

Expanding along R_1

$\Rightarrow \left[x\begin{vmatrix} -6 & 1 \\ 4 & 1 \end{vmatrix} - 4\begin{vmatrix} 2 & 1 \\ 5 & 1 \end{vmatrix} + 1\begin{vmatrix} 2 & -6 \\ 5 & 4 \end{vmatrix}\right] = \pm 70$

⇒ [x (– 10) – 4(– 3) + 1(8 – 30)] = ± 70

⇒ [– 10x + 12 + 38] = ± 70

⇒ ±70 = – 10x + 50

Taking positive sign, we get

⇒ + 70 = – 10x + 50

⇒ 10x = – 20

⇒ x = – 2

Taking –negative sign, we get

⇒ – 70 = – 10x + 50

⇒ 10x = 120

⇒ x = 12

Thus x = – 2, 12

EXERCISE 6.4 PAGE NO: 6.84

Solve the following system of linear equations by Cramer's rule:
1. x – 2y = 4
-3x + 5y = -7

Solution:
Given x – 2y = 4
-3x + 5y = -7
Let there be a system of n simultaneous linear equations and with n unknown given by

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2$$

$$\vdots \; \vdots \; \vdots$$

$$a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n = b_n$$

$$\text{Let } D = \begin{vmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n1} & \ldots & a_{nn} \end{vmatrix}$$

Let D_j be the determinant obtained from D after replacing the j^{th} column by

$$\begin{vmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{vmatrix}$$

Then,

$$x_1 = \frac{D_1}{D}, x_2 = \frac{D_2}{D}, \ldots, x_n = \frac{D_n}{D} \text{ Provided that } D \neq 0$$

Now, here we have

x – 2y = 4

– 3x + 5y = – 7

So by comparing with the theorem, let's find D, D_1 and D_2

$\Rightarrow \ D = \begin{vmatrix} 1 & -2 \\ -3 & 5 \end{vmatrix}$

Solving determinant, expanding along 1st row

$\Rightarrow$ D = 5(1) – (– 3) (– 2)

$\Rightarrow$ D = 5 – 6

$\Rightarrow$ D = – 1

Again,

$\Rightarrow \ D_1 = \begin{vmatrix} 4 & -2 \\ -7 & 5 \end{vmatrix}$

Solving determinant, expanding along 1st row

$\Rightarrow D_1 = 5(4) - (-7)(-2)$

$\Rightarrow D_1 = 20 - 14$

$\Rightarrow D_1 = 6$

And

$\Rightarrow \ D_2 = \begin{vmatrix} 1 & 4 \\ -3 & -7 \end{vmatrix}$

Solving determinant, expanding along 1st row

$\Rightarrow D_2 = 1(-7) - (-3)(4)$

$\Rightarrow D_2 = -7 + 12$

$\Rightarrow D_2 = 5$

Thus by Cramer's Rule, we have

$\Rightarrow x = \frac{D_1}{D}$

$\Rightarrow x = \frac{6}{-1}$

$\Rightarrow$ x = – 6

And

$\Rightarrow y = \frac{D_2}{D}$

$\Rightarrow y = \frac{5}{-1}$

$\Rightarrow$ y = – 5

2. 2x – y = 1
7x – 2y = -7

Solution:

Given 2x – y = 1 and

$7x - 2y = -7$

Let there be a system of n simultaneous linear equations and with n unknown given by

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2$$

$$\vdots\ \vdots\ \vdots$$

$$a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n = b_n$$

$$\text{Let } D = \begin{vmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n1} & \ldots & a_{nn} \end{vmatrix}$$

Let D_j be the determinant obtained from D after replacing the j^{th} column by

$$\begin{vmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{vmatrix}$$

Then,

$x_1 = \frac{D_1}{D}, x_2 = \frac{D_2}{D}, \ldots, x_n = \frac{D_n}{D}$ Provided that $D \neq 0$

Now, here we have

$2x - y = 1$

$7x - 2y = -7$

So by comparing with the theorem, let's find D, D_1 and D_2

$$\Rightarrow D = \begin{vmatrix} 2 & -1 \\ 7 & -2 \end{vmatrix}$$

Solving determinant, expanding along 1^{st} row

$\Rightarrow D_1 = 1(-2) - (-7)(-1)$

$\Rightarrow D_1 = -2 - 7$

$\Rightarrow D_1 = -9$

And

$$\Rightarrow D_2 = \begin{vmatrix} 2 & 1 \\ 7 & -7 \end{vmatrix}$$

Solving determinant, expanding along 1^{st} row

⇒ $D_2 = 2(-7) - (7)(1)$

⇒ $D_2 = -14 - 7$

⇒ $D_2 = -21$

Thus by Cramer's Rule, we have

⇒ $x = \frac{D_1}{D}$

⇒ $x = \frac{-9}{3}$

⇒ $x = -3$

And ⇒ $y = \frac{D_2}{D}$

⇒ $y = \frac{-21}{3}$

⇒ $y = -7$

3. 2x – y = 17
3x + 5y = 6

Solution:

Given 2x – y = 17 and

3x + 5y = 6

Let there be a system of n simultaneous linear equations and with n unknown given by

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$

$$\vdots \; \vdots \; \vdots$$

$$a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = b_n$$

$$\text{Let } D = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n1} & \dots & a_{nn} \end{vmatrix}$$

Let D_j be the determinant obtained from D after replacing the j^{th} column by

$$\begin{vmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{vmatrix}$$

Then,

$x_1 = \frac{D_1}{D}, x_2 = \frac{D_2}{D}, \ldots, x_n = \frac{D_n}{D}$ Provided that $D \neq 0$

Now, here we have

$2x - y = 17$

$3x + 5y = 6$

So by comparing with the theorem, let's find D, D_1 and D_2

$\Rightarrow D = \begin{vmatrix} 2 & -1 \\ 3 & 5 \end{vmatrix}$

Solving determinant, expanding along 1st row

$\Rightarrow D_1 = 17(5) - (6)(-1)$

$\Rightarrow D_1 = 85 + 6$

$\Rightarrow D_1 = 91$

$\Rightarrow D_2 = \begin{vmatrix} 2 & 17 \\ 3 & 6 \end{vmatrix}$

Solving determinant, expanding along 1st row

$\Rightarrow D_2 = 2(6) - (17)(3)$

$\Rightarrow D_2 = 12 - 51$

$\Rightarrow D_2 = -39$

Thus by Cramer's Rule, we have

$\Rightarrow x = \frac{D_1}{D}$

$\Rightarrow x = \frac{91}{13}$

$\Rightarrow x = 7$

And $\Rightarrow y = \frac{D_2}{D}$

$\Rightarrow y = \frac{-39}{13}$

$\Rightarrow y = -3$

4. 3x + y = 19
3x – y = 23

Solution:
Let there be a system of n simultaneous linear equations and with n unknown given by

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2$$

$$\vdots \; \vdots \; \vdots$$

$$a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n = b_n$$

$$\text{Let } D = \begin{vmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n1} & \ldots & a_{nn} \end{vmatrix}$$

Let D_j be the determinant obtained from D after replacing the j^{th} column by

$$\begin{vmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{vmatrix}$$

Then,

$$x_1 = \frac{D_1}{D}, x_2 = \frac{D_2}{D}, \ldots, x_n = \frac{D_n}{D} \text{ Provided that } D \neq 0$$

Now, here we have

3x + y = 19

3x – y = 23

So by comparing with the theorem, let's find D, D_1 and D_2

$$\Rightarrow D = \begin{vmatrix} 3 & 1 \\ 3 & -1 \end{vmatrix}$$

Solving determinant, expanding along 1^{st} row

⇒ D = 3(– 1) – (3) (1)

⇒ D = – 3 – 3

$\Rightarrow$ D = – 6

Again,

$\Rightarrow \ D_1 = \begin{vmatrix} 19 & 1 \\ 23 & -1 \end{vmatrix}$

Solving determinant, expanding along 1st row

$\Rightarrow D_1 = 19(-1) - (23)(1)$

$\Rightarrow D_1 = -19 - 23$

$\Rightarrow D_1 = -42$

$\Rightarrow \ D_2 = \begin{vmatrix} 3 & 19 \\ 3 & 23 \end{vmatrix}$

Solving determinant, expanding along 1st row

$\Rightarrow D_2 = 3(23) - (19)(3)$

$\Rightarrow D_2 = 69 - 57$

$\Rightarrow D_2 = 12$

Thus by Cramer's Rule, we have

$\Rightarrow x = \frac{D_1}{D}$

$\Rightarrow x = \frac{-42}{-6}$

$\Rightarrow x = 7$

And $\Rightarrow y = \frac{D_2}{D}$

$\Rightarrow y = \frac{12}{-6}$

$\Rightarrow y = -2$

5. 2x – y = -2
3x + 4y = 3

Solution:

Given 2x – y = -2 and

3x + 4y = 3

Let there be a system of n simultaneous linear equations and with n unknown given by

$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$

$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$

$\vdots \ \vdots \ \vdots$

$a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = b_n$

Let $D = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n1} & \dots & a_{nn} \end{vmatrix}$

Let D_j be the determinant obtained from D after replacing the j^{th} column by

$\begin{vmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{vmatrix}$

Then,

$x_1 = \frac{D_1}{D}, x_2 = \frac{D_2}{D}, \dots, x_n = \frac{D_n}{D}$ Provided that $D \neq 0$

Now, here we have

$2x - y = -2$

$3x + 4y = 3$

So by comparing with the theorem, let's find D, D_1 and D_2

$\Rightarrow \quad D = \begin{vmatrix} 2 & -1 \\ 3 & 4 \end{vmatrix}$

Solving determinant, expanding along 1^{st} row

$\Rightarrow D = 2(4) - (3)(-1)$

$\Rightarrow D = 8 + 3$

$\Rightarrow D = 11$

Again,

$\Rightarrow \quad D_1 = \begin{vmatrix} -2 & -1 \\ 3 & 4 \end{vmatrix}$

Solving determinant, expanding along 1^{st} row

$\Rightarrow D_1 = -2(4) - (3)(-1)$

$\Rightarrow D_1 = -8 + 3$

$\Rightarrow D_1 = -5$

$\Rightarrow \quad D_2 = \begin{vmatrix} 2 & -2 \\ 3 & 3 \end{vmatrix}$

Solving determinant, expanding along 1st row

$\Rightarrow D_2 = 3(2) - (-2)(3)$

$\Rightarrow D_2 = 6 + 6$

$\Rightarrow D_2 = 12$

Thus by Cramer's Rule, we have

$\Rightarrow x = \frac{D_1}{D}$

$\Rightarrow x = \frac{-5}{11}$

And $\Rightarrow y = \frac{D_2}{D}$

$\Rightarrow y = \frac{12}{11}$

6. 3x + ay = 4
2x + ay = 2, a ≠ 0

Solution:

Given 3x + ay = 4 and

2x + ay = 2, a ≠ 0

Let there be a system of n simultaneous linear equations and with n unknown given by

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$

$$\vdots \quad \vdots \quad \vdots$$

$$a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = b_n$$

$$\text{Let } D = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n1} & \dots & a_{nn} \end{vmatrix}$$

Let D_j be the determinant obtained from D after replacing the j^{th} column by

$$\begin{vmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{vmatrix}$$

Then,

$x_1 = \frac{D_1}{D}, x_2 = \frac{D_2}{D}, \ldots, x_n = \frac{D_n}{D}$ Provided that $D \neq 0$

3x + ay = 4

2x + ay = 2, a≠0

So by comparing with the theorem, let's find D, D_1 and D_2

$\Rightarrow D = \begin{vmatrix} 3 & a \\ 2 & a \end{vmatrix}$

Solving determinant, expanding along 1st row

⇒ D = 3(a) – (2) (a)

⇒ D = 3a – 2a

⇒ D = a

Again,

$\Rightarrow D_1 = \begin{vmatrix} 4 & a \\ 2 & a \end{vmatrix}$

Solving determinant, expanding along 1st row

⇒ D_1 = 4(a) – (2) (a)

⇒ D = 4a – 2a

⇒ D = 2a

$\Rightarrow D_2 = \begin{vmatrix} 3 & 4 \\ 2 & 2 \end{vmatrix}$

Solving determinant, expanding along 1st row

⇒ D_2 = 3(2) – (2) (4)

⇒ D = 6 – 8

⇒ D = – 2

Thus by Cramer's Rule, we have

$\Rightarrow x = \frac{D_1}{D}$

$\Rightarrow x = \frac{2a}{a}$

⇒ x = 2

$\Rightarrow y = \frac{D_2}{D}$

$\Rightarrow y = \frac{-2}{a}$

7. 2x + 3y = 10

x + 6y = 4

Solution:

Let there be a system of n simultaneous linear equations and with n unknown given by

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$

$$\vdots \; \vdots \; \vdots$$

$$a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = b_n$$

$$\text{Let } D = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n1} & \dots & a_{nn} \end{vmatrix}$$

Let D_j be the determinant obtained from D after replacing the j^{th} column by

$$\begin{vmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{vmatrix}$$

Then,

$x_1 = \frac{D_1}{D}, x_2 = \frac{D_2}{D}, \dots, x_n = \frac{D_n}{D}$ Provided that $D \neq 0$

Now, here we have

$2x + 3y = 10$

$x + 6y = 4$

So by comparing with the theorem, let's find D, D_1 and D_2

$$\Rightarrow \; D = \begin{vmatrix} 2 & 3 \\ 1 & 6 \end{vmatrix}$$

Solving determinant, expanding along 1^{st} row

$\Rightarrow D = 2 (6) – (3) (1)$

$\Rightarrow D = 12 – 3$

$\Rightarrow D = 9$

Again,

$\Rightarrow \ D_1 = \begin{vmatrix} 10 & 3 \\ 4 & 6 \end{vmatrix}$

Solving determinant, expanding along 1st row

⇒ D_1 = 10 (6) – (3) (4)

⇒ D = 60 – 12

⇒ D = 48

$\Rightarrow \ D_2 = \begin{vmatrix} 2 & 10 \\ 1 & 4 \end{vmatrix}$

Solving determinant, expanding along 1st row

⇒ D_2 = 2 (4) – (10) (1)

⇒ D_2 = 8 – 10

⇒ D_2 = – 2

Thus by Cramer's Rule, we have

$\Rightarrow x = \frac{D_1}{D}$

$\Rightarrow x = \frac{48}{9}$

$\Rightarrow x = \frac{16}{3}$

$\Rightarrow y = \frac{D_2}{D}$

$\Rightarrow y = \frac{-2}{9}$

$\Rightarrow y = \frac{-2}{9}$

8. 5x + 7y = -2
4x + 6y = -3

Solution:

Let there be a system of n simultaneous linear equations and with n unknown given by

$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1$

$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2$

$\vdots \vdots \vdots$

$a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n = b_n$

$\text{Let } D = \begin{vmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n1} & \ldots & a_{nn} \end{vmatrix}$

Let D_j be the determinant obtained from D after replacing the j^{th} column by

$$\begin{vmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{vmatrix}$$

Then,

$x_1 = \frac{D_1}{D}, x_2 = \frac{D_2}{D}, \ldots, x_n = \frac{D_n}{D}$ Provided that $D \neq 0$

Now, here we have

$5x + 7y = -2$

$4x + 6y = -3$

So by comparing with the theorem, let's find D, D_1 and D_2

$\Rightarrow D = \begin{vmatrix} 5 & 7 \\ 4 & 6 \end{vmatrix}$

Solving determinant, expanding along 1^{st} row

$\Rightarrow D = 5(6) - (7)(4)$

$\Rightarrow D = 30 - 28$

$\Rightarrow D = 2$

Again,

$\Rightarrow D_1 = \begin{vmatrix} -2 & 7 \\ -3 & 6 \end{vmatrix}$

Solving determinant, expanding along 1^{st} row

$\Rightarrow D_1 = -2(6) - (7)(-3)$

$\Rightarrow D_1 = -12 + 21$

$\Rightarrow D_1 = 9$

$\Rightarrow D_2 = \begin{vmatrix} 5 & -2 \\ 4 & -3 \end{vmatrix}$

Solving determinant, expanding along 1^{st} row

$\Rightarrow D_2 = -3(5) - (-2)(4)$

$\Rightarrow D_2 = -15 + 8$

$\Rightarrow D_2 = -7$

Thus by Cramer's Rule, we have

$\Rightarrow x = \frac{D_1}{D}$

$\Rightarrow x = \frac{9}{2}$

$\Rightarrow x = \frac{9}{2}$

$\Rightarrow y = \frac{D_2}{D}$

$\Rightarrow y = \frac{-7}{2}$

$\Rightarrow y = \frac{-7}{2}$

9. 9x + 5y = 10
3y – 2x = 8

Solution:
Let there be a system of n simultaneous linear equations and with n unknown given by

$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1$

$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2$

$\vdots \vdots \vdots$

$a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n = b_n$

Let $D = \begin{vmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n1} & \ldots & a_{nn} \end{vmatrix}$

Let D_j be the determinant obtained from D after replacing the j^{th} column by

$\begin{vmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{vmatrix}$

Then,

$x_1 = \frac{D_1}{D}, x_2 = \frac{D_2}{D}, \ldots, x_n = \frac{D_n}{D}$ Provided that $D \neq 0$

Now, here we have

9x + 5y = 10

3y – 2x = 8

So by comparing with the theorem, let's find D, D_1 and D_2

$\Rightarrow\ D = \begin{vmatrix} 9 & 5 \\ -2 & 3 \end{vmatrix}$

Solving determinant, expanding along 1st row

$\Rightarrow$ D = 3(9) – (5) (– 2)

$\Rightarrow$ D = 27 + 10

$\Rightarrow$ D = 37

Again,

$\Rightarrow\ D_1 = \begin{vmatrix} 10 & 5 \\ 8 & 3 \end{vmatrix}$

Solving determinant, expanding along 1st row

$\Rightarrow D_1 = 10(3) – (8)(5)$

$\Rightarrow D_1 = 30 – 40$

$\Rightarrow D_1 = – 10$

$\Rightarrow\ D_2 = \begin{vmatrix} 9 & 10 \\ -2 & 8 \end{vmatrix}$

Solving determinant, expanding along 1st row

$\Rightarrow D_2 = 9(8) – (10)(– 2)$

$\Rightarrow D_2 = 72 + 20$

$\Rightarrow D_2 = 92$

Thus by Cramer's Rule, we have

$\Rightarrow x = \frac{D_1}{D}$

$\Rightarrow x = \frac{-10}{37}$

$\Rightarrow x = \frac{-10}{37}$

$\Rightarrow y = \frac{D_2}{D}$

$\Rightarrow y = \frac{92}{37}$

$\Rightarrow y = \frac{92}{37}$

10. x + 2y = 1
3x + y = 4

Solution:

Let there be a system of n simultaneous linear equations and with n unknown given by

$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$

$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$

$\vdots \ \vdots \ \vdots$

$a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = b_n$

Let $D = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n1} & \dots & a_{nn} \end{vmatrix}$

Let D_j be the determinant obtained from D after replacing the j^{th} column by

$\begin{vmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{vmatrix}$

Then,

$x_1 = \frac{D_1}{D}, x_2 = \frac{D_2}{D}, \dots, x_n = \frac{D_n}{D}$ Provided that $D \neq 0$

Now, here we have

$x + 2y = 1$

$3x + y = 4$

So by comparing with theorem, now we have to find D, D_1 and D_2

$\Rightarrow \ D = \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix}$

Solving determinant, expanding along 1^{st} row

$\Rightarrow D = 1(1) – (3)(2)$

$\Rightarrow D = 1 – 6$

$\Rightarrow D = – 5$

Again,

$\Rightarrow \ D_1 = \begin{vmatrix} 1 & 2 \\ 4 & 1 \end{vmatrix}$

Solving determinant, expanding along 1^{st} row

$\Rightarrow D_1 = 1(1) – (2)(4)$

$\Rightarrow D_1 = 1 - 8$
$\Rightarrow D_1 = -7$
$\Rightarrow \ D_2 = \begin{vmatrix} 1 & 1 \\ 3 & 4 \end{vmatrix}$
Solving determinant, expanding along 1st row
$\Rightarrow D_2 = 1(4) - (1)(3)$
$\Rightarrow D_2 = 4 - 3$
$\Rightarrow D_2 = 1$
Thus by Cramer's Rule, we have
$\Rightarrow x = \frac{D_1}{D}$
$\Rightarrow x = \frac{-7}{-5}$
$\Rightarrow x = \frac{7}{5}$
$\Rightarrow y = \frac{D_2}{D}$
$\Rightarrow y = \frac{1}{-5}$
$\Rightarrow y = -\frac{1}{5}$

Solve the following system of linear equations by Cramer's rule:
11. 3x + y + z = 2
2x – 4y + 3z = -1
4x + y – 3z = -11

Solution:
Let there be a system of n simultaneous linear equations and with n unknown given by
$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1$
$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2$
$\vdots \vdots \vdots$
$a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n = b_n$
$\text{Let } D = \begin{vmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n1} & \ldots & a_{nn} \end{vmatrix}$
Let D_j be the determinant obtained from D after replacing the j^{th} column by

$$\begin{vmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{vmatrix}$$

Then,

$x_1 = \frac{D_1}{D}, x_2 = \frac{D_2}{D}, \ldots, x_n = \frac{D_n}{D}$ Provided that $D \neq 0$

Now, here we have

$3x + y + z = 2$

$2x - 4y + 3z = -1$

$4x + y - 3z = -11$

So by comparing with the theorem, let's find D, D_1, D_2 and D_3

$$\Rightarrow D = \begin{vmatrix} 3 & 1 & 1 \\ 2 & -4 & 3 \\ 4 & 1 & -3 \end{vmatrix}$$

Solving determinant, expanding along 1st row

$\Rightarrow D = 3[(-4)(-3) - (3)(1)] - 1[(2)(-3) - 12] + 1[2 - 4(-4)]$

$\Rightarrow D = 3[12 - 3] - [-6 - 12] + [2 + 16]$

$\Rightarrow D = 27 + 18 + 18$

$\Rightarrow D = 63$

Again,

$$\Rightarrow D_1 = \begin{vmatrix} 2 & 1 & 1 \\ -1 & -4 & 3 \\ -11 & 1 & -3 \end{vmatrix}$$

Solving determinant, expanding along 1st row

$\Rightarrow D_1 = 2[(-4)(-3) - (3)(1)] - 1[(-1)(-3) - (-11)(3)] + 1[(-1) - (-4)(-11)]$

$\Rightarrow D_1 = 2[12 - 3] - 1[3 + 33] + 1[-1 - 44]$

$\Rightarrow D_1 = 2[9] - 36 - 45$

$\Rightarrow D_1 = 18 - 36 - 45$

$\Rightarrow D_1 = -63$

Again

$$\Rightarrow D_2 = \begin{vmatrix} 3 & 2 & 1 \\ 2 & -1 & 3 \\ 4 & -11 & -3 \end{vmatrix}$$

Solving determinant, expanding along 1st row

$\Rightarrow D_2 = 3[3 + 33] - 2[-6 - 12] + 1[-22 + 4]$

$\Rightarrow D_2 = 3[36] - 2(-18) - 18$

$\Rightarrow D_2 = 126$

$$\Rightarrow D_3 = \begin{vmatrix} 3 & 1 & 2 \\ 2 & -4 & -1 \\ 4 & 1 & -11 \end{vmatrix}$$

Solving determinant, expanding along 1st row

$\Rightarrow D_3 = 3[44 + 1] – 1[– 22 + 4] + 2[2 + 16]$

$\Rightarrow D_3 = 3[45] – 1(– 18) + 2(18)$

$\Rightarrow D_3 = 135 + 18 + 36$

$\Rightarrow D_3 = 189$

Thus by Cramer's Rule, we have

$\Rightarrow x = \frac{D_1}{D}$

$\Rightarrow x = \frac{-63}{63}$

$\Rightarrow x = -1$

$\Rightarrow y = \frac{D_2}{D}$

$\Rightarrow y = \frac{126}{63}$

$\Rightarrow y = 2$

$\Rightarrow z = \frac{D_3}{D}$

$\Rightarrow z = \frac{189}{63}$

$\Rightarrow z = 3$

12. x – 4y – z = 11
2x – 5y + 2z = 39
-3x + 2y + z = 1

Solution:

Given,

x – 4y – z = 11

2x – 5y + 2z = 39

-3x + 2y + z = 1

Let there be a system of n simultaneous linear equations and with n unknown given by

$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1$

$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2$

$\vdots \; \vdots \; \vdots$

$a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n = b_n$

$$\text{Let } D = \begin{vmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n1} & \ldots & a_{nn} \end{vmatrix}$$

Let D_j be the determinant obtained from D after replacing the j^{th} column by

$$\begin{vmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{vmatrix}$$

Then,

$x_1 = \frac{D_1}{D}, x_2 = \frac{D_2}{D}, \ldots, x_n = \frac{D_n}{D}$ Provided that $D \neq 0$

Now, here we have

$x - 4y - z = 11$

$2x - 5y + 2z = 39$

$-3x + 2y + z = 1$

So by comparing with theorem, now we have to find D, D_1 and D_2

$$\Rightarrow \; D = \begin{vmatrix} 1 & -4 & -1 \\ 2 & -5 & 2 \\ -3 & 2 & 1 \end{vmatrix}$$

Solving determinant, expanding along 1^{st} row

$\Rightarrow D = 1[(-5)(1) - (2)(2)] + 4[(2)(1) + 6] - 1[4 + 5(-3)]$

$\Rightarrow D = 1[-5 - 4] + 4[8] - [-11]$

$\Rightarrow D = -9 + 32 + 11$

$\Rightarrow D = 34$

Again,

$$\Rightarrow \; D_1 = \begin{vmatrix} 11 & -4 & -1 \\ 39 & -5 & 2 \\ 1 & 2 & 1 \end{vmatrix}$$

Solving determinant, expanding along 1^{st} row

$\Rightarrow D_1 = 11[(-5)(1) - (2)(2)] + 4[(39)(1) - (2)(1)] - 1[2(39) - (-5)(1)]$
$\Rightarrow D_1 = 11[-5-4] + 4[39-2] - 1[78+5]$
$\Rightarrow D_1 = 11[-9] + 4(37) - 83$
$\Rightarrow D_1 = -99 - 148 - 45$
$\Rightarrow D_1 = -34$
Again
$$\Rightarrow \quad D_2 = \begin{vmatrix} 1 & 11 & -1 \\ 2 & 39 & 2 \\ -3 & 1 & 1 \end{vmatrix}$$
Solving determinant, expanding along 1st row
$\Rightarrow D_2 = 1[39-2] - 11[2+6] - 1[2+117]$
$\Rightarrow D_2 = 1[37] - 11(8) - 119$
$\Rightarrow D_2 = -170$
And,
$$\Rightarrow \quad D_3 = \begin{vmatrix} 1 & -4 & 11 \\ 2 & -5 & 39 \\ -3 & 2 & 1 \end{vmatrix}$$
Solving determinant, expanding along 1st row
$\Rightarrow D_3 = 1[-5-(39)(2)] - (-4)[2-(39)(-3)] + 11[4-(-5)(-3)]$
$\Rightarrow D_3 = 1[-5-78] + 4(2+117) + 11(4-15)$
$\Rightarrow D_3 = -83 + 4(119) + 11(-11)$
$\Rightarrow D_3 = 272$
Thus by Cramer's Rule, we have
$\Rightarrow x = \frac{D_1}{D}$
$\Rightarrow x = \frac{-34}{34}$
$\Rightarrow x = -1$
Again,
$\Rightarrow y = \frac{D_2}{D}$
$\Rightarrow y = \frac{-170}{34}$
$\Rightarrow y = -5$
$\Rightarrow z = \frac{D_3}{D} = (272/34) = 8$

13. 6x + y – 3z = 5
x + 3y – 2z = 5

2x + y + 4z = 8

Solution:

Given

6x + y – 3z = 5

x + 3y – 2z = 5

2x + y + 4z = 8

Let there be a system of n simultaneous linear equations and with n unknown given by

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$

$$\vdots \vdots \vdots$$

$$a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = b_n$$

$$\text{Let } D = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n1} & \dots & a_{nn} \end{vmatrix}$$

Let D_j be the determinant obtained from D after replacing the j^th^ column by

$$\begin{vmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{vmatrix}$$

Then,

$x_1 = \frac{D_1}{D}, x_2 = \frac{D_2}{D}, \dots, x_n = \frac{D_n}{D}$ Provided that $D \neq 0$

Now, here we have

6x + y – 3z = 5

x + 3y – 2z = 5

2x + y + 4z = 8

So by comparing with theorem, now we have to find D , D_1 and D_2

$$\Rightarrow D = \begin{vmatrix} 6 & 1 & -3 \\ 1 & 3 & -2 \\ 2 & 1 & 4 \end{vmatrix}$$

Solving determinant, expanding along 1st Row

$\Rightarrow D = 6[(4)(3) - (1)(-2)] - 1[(4)(1) + 4] - 3[1 - 3(2)]$
$\Rightarrow D = 6[12 + 2] - [8] - 3[-5]$
$\Rightarrow D = 84 - 8 + 15$
$\Rightarrow D = 91$
Again, Solve D_1 formed by replacing 1st column by B matrices
Here

$$B = \begin{vmatrix} 5 \\ 5 \\ 8 \end{vmatrix}$$

$$\Rightarrow D_1 = \begin{vmatrix} 5 & 1 & -3 \\ 5 & 3 & -2 \\ 8 & 1 & 4 \end{vmatrix}$$

Solving determinant, expanding along 1st Row
$\Rightarrow D_1 = 5[(4)(3) - (-2)(1)] - 1[(5)(4) - (-2)(8)] - 3[(5) - (3)(8)]$
$\Rightarrow D_1 = 5[12 + 2] - 1[20 + 16] - 3[5 - 24]$
$\Rightarrow D_1 = 5[14] - 36 - 3(-19)$
$\Rightarrow D_1 = 70 - 36 + 57$
$\Rightarrow D_1 = 91$
Again, Solve D_2 formed by replacing 1st column by B matrices
Here

$$B = \begin{vmatrix} 5 \\ 5 \\ 8 \end{vmatrix}$$

$$\Rightarrow D_2 = \begin{vmatrix} 6 & 5 & -3 \\ 1 & 5 & -2 \\ 2 & 8 & 4 \end{vmatrix}$$

Solving determinant
$\Rightarrow D_2 = 6[20 + 16] - 5[4 - 2(-2)] + (-3)[8 - 10]$
$\Rightarrow D_2 = 6[36] - 5(8) + (-3)(-2)$
$\Rightarrow D_2 = 182$
And, Solve D_3 formed by replacing 1st column by B matrices
Here

$$B = \begin{vmatrix} 5 \\ 5 \\ 8 \end{vmatrix}$$

$$\Rightarrow D_3 = \begin{vmatrix} 6 & 1 & 5 \\ 1 & 3 & 5 \\ 2 & 1 & 8 \end{vmatrix}$$

Solving determinant, expanding along 1^{st} Row

$\Rightarrow D_3 = 6[24 - 5] - 1[8 - 10] + 5[1 - 6]$

$\Rightarrow D_3 = 6[19] - 1(-2) + 5(-5)$

$\Rightarrow D_3 = 114 + 2 - 25$

$\Rightarrow D_3 = 91$

Thus by Cramer's Rule, we have

$\Rightarrow x = \frac{D_1}{D}$

$\Rightarrow x = \frac{91}{91}$

$\Rightarrow x = 1$

$\Rightarrow y = \frac{D_2}{D}$

$\Rightarrow y = \frac{182}{91}$

$\Rightarrow y = 2$

$\Rightarrow z = \frac{D_3}{D}$

$\Rightarrow z = \frac{91}{91}$

$\Rightarrow z = 1$

14. x + y = 5
y + z = 3
x + z = 4

Solution:

Given x + y = 5

y + z = 3

x + z = 4

Let there be a system of n simultaneous linear equations and with n unknown given by

$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$

$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$

$\vdots \; \vdots \; \vdots$

$a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = b_n$

$$\text{Let } D = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n1} & \dots & a_{nn} \end{vmatrix}$$

Let D_j be the determinant obtained from D after replacing the j^{th} column by

$$\begin{vmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{vmatrix}$$

Then,

$x_1 = \frac{D_1}{D}, x_2 = \frac{D_2}{D}, \ldots, x_n = \frac{D_n}{D}$ Provided that $D \neq 0$

Now, here we have

$x + y = 5$

$y + z = 3$

$x + z = 4$

So by comparing with theorem, now we have to find D, D_1 and D_2

$$\Rightarrow \quad D = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix}$$

Solving determinant, expanding along 1^{st} Row

$\Rightarrow D = 1[1] - 1[-1] + 0[-1]$

$\Rightarrow D = 1 + 1 + 0$

$\Rightarrow D = 2$

Again, Solve D_1 formed by replacing 1^{st} column by B matrices

Here

$$B = \begin{vmatrix} 5 \\ 3 \\ 4 \end{vmatrix}$$

$$\Rightarrow \quad D_1 = \begin{vmatrix} 5 & 1 & 0 \\ 3 & 1 & 1 \\ 4 & 0 & 1 \end{vmatrix}$$

Solving determinant, expanding along 1^{st} Row

$\Rightarrow D_1 = 5[1] - 1[(3)(1) - (4)(1)] + 0[0 - (4)(1)]$

$\Rightarrow D_1 = 5 - 1[3 - 4] + 0[-4]$

$\Rightarrow D_1 = 5 - 1[-1] + 0$

$\Rightarrow D_1 = 5 + 1 + 0$

$\Rightarrow D_1 = 6$

Again, Solve D_2 formed by replacing 1^{st} column by B matrices

Here

$$B = \begin{vmatrix} 5 \\ 3 \\ 4 \end{vmatrix}$$

$$\Rightarrow \quad D_2 = \begin{vmatrix} 1 & 5 & 0 \\ 0 & 3 & 1 \\ 1 & 4 & 1 \end{vmatrix}$$

Solving determinant

$\Rightarrow D_2 = 1[3 – 4] – 5[– 1] + 0[0 – 3]$

$\Rightarrow D_2 = 1[– 1] + 5 + 0$

$\Rightarrow D_2 = 4$

And, Solve D_3 formed by replacing 1^{st} column by B matrices

Here

$$B = \begin{vmatrix} 5 \\ 3 \\ 4 \end{vmatrix}$$

$$\Rightarrow \quad D_3 = \begin{vmatrix} 1 & 1 & 5 \\ 0 & 1 & 3 \\ 1 & 0 & 4 \end{vmatrix}$$

Solving determinant, expanding along 1^{st} Row

$\Rightarrow D_3 = 1[4 – 0] – 1[0 – 3] + 5[0 – 1]$

$\Rightarrow D_3 = 1[4] – 1(– 3) + 5(– 1)$

$\Rightarrow D_3 = 4 + 3 – 5$

$\Rightarrow D_3 = 2$

Thus by Cramer's Rule, we have

$\Rightarrow x = \frac{D_1}{D}$

$\Rightarrow x = \frac{6}{2}$

$\Rightarrow x = 3$

$\Rightarrow y = \frac{D_2}{D}$

$\Rightarrow y = \frac{4}{2}$

$\Rightarrow y = 2$

$\Rightarrow z = \frac{D_3}{D}$

$\Rightarrow z = \frac{2}{2}$

$\Rightarrow z = 1$

15. 2y – 3z = 0
x + 3y = -4
3x + 4y = 3

Solution:
Given
2y – 3z = 0
x + 3y = -4
3x + 4y = 3
Let there be a system of n simultaneous linear equations and with n unknown given by

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$

$$\vdots \; \vdots \; \vdots$$

$$a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = b_n$$

$$\text{Let } D = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n1} & \dots & a_{nn} \end{vmatrix}$$

Let D_j be the determinant obtained from D after replacing the j^{th} column by

$$\begin{vmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{vmatrix}$$

Then,

$x_1 = \frac{D_1}{D}, x_2 = \frac{D_2}{D}, \dots, x_n = \frac{D_n}{D}$ Provided that $D \neq 0$

Now, here we have
2y – 3z = 0
x + 3y = – 4
3x + 4y = 3
So by comparing with theorem, now we have to find D, D_1 and D_2

$$\Rightarrow \; D = \begin{vmatrix} 0 & 2 & -3 \\ 1 & 3 & 0 \\ 3 & 4 & 0 \end{vmatrix}$$

Solving determinant, expanding along 1st Row
⇒ D = 0[0] – 2[(0) (1) – 0] – 3[1 (4) – 3 (3)]
⇒ D = 0 – 0 – 3[4 – 9]
⇒ D = 0 – 0 + 15
⇒ D = 15
Again, Solve D_1 formed by replacing 1st column by B matrices
Here

$$B = \begin{vmatrix} 0 \\ -4 \\ 3 \end{vmatrix}$$

$$\Rightarrow \ D_1 = \begin{vmatrix} 0 & 2 & -3 \\ -4 & 3 & 0 \\ 3 & 4 & 0 \end{vmatrix}$$

Solving determinant, expanding along 1st Row
⇒ D_1 = 0[0] – 2[(0) (– 4) – 0] – 3[4 (– 4) – 3(3)]
⇒ D_1 = 0 – 0 – 3[– 16 – 9]
⇒ D_1 = 0 – 0 – 3(– 25)
⇒ D_1 = 0 – 0 + 75
⇒ D_1 = 75
Again, Solve D_2 formed by replacing 2nd column by B matrices
Here

$$B = \begin{vmatrix} 0 \\ -4 \\ 3 \end{vmatrix}$$

$$\Rightarrow \ D_2 = \begin{vmatrix} 0 & 0 & -3 \\ 1 & -4 & 0 \\ 3 & 3 & 0 \end{vmatrix}$$

Solving determinant
⇒ D_2 = 0[0] – 0[(0) (1) – 0] – 3[1 (3) – 3(– 4)]
⇒ D_2 = 0 – 0 + (– 3) (3 + 12)
⇒ D_2 = – 45
And, Solve D_3 formed by replacing 3rd column by B matrices
Here

$$B = \begin{vmatrix} 0 \\ -4 \\ 3 \end{vmatrix}$$

$$\Rightarrow D_3 = \begin{vmatrix} 0 & 2 & 0 \\ 1 & 3 & -4 \\ 3 & 4 & 3 \end{vmatrix}$$

Solving determinant, expanding along 1st Row

$\Rightarrow D_3 = 0[9 - (-4) 4] - 2[(3)(1) - (-4)(3)] + 0[1 (4) - 3 (3)]$

$\Rightarrow D_3 = 0[25] - 2(3 + 12) + 0(4 - 9)$

$\Rightarrow D_3 = 0 - 30 + 0$

$\Rightarrow D_3 = -30$

Thus by Cramer's Rule, we have

$\Rightarrow x = \frac{D_1}{D}$

$\Rightarrow x = \frac{75}{15}$

$\Rightarrow x = 5$

$\Rightarrow y = \frac{D_2}{D}$

$\Rightarrow y = \frac{-45}{15}$

$\Rightarrow y = -3$

$\Rightarrow z = \frac{D_3}{D}$

$\Rightarrow z = \frac{-30}{15}$

$\Rightarrow z = -2$

16. 5x – 7y + z = 11
6x – 8y – z = 15
3x + 2y – 6z = 7

Solution:

Given

$5x - 7y + z = 11$

$6x - 8y - z = 15$

$3x + 2y - 6z = 7$

Let there be a system of n simultaneous linear equations and with n unknown given by

$a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = b_1$

$a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = b_2$

$\vdots \vdots \vdots$

$a_{n1}x_1 + a_{n2}x_2 + ... + a_{nn}x_n = b_n$

Let $D = \begin{vmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n1} & ... & a_{nn} \end{vmatrix}$

Let D_j be the determinant obtained from D after replacing the j^{th} column by

$\begin{vmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{vmatrix}$

Then,

$x_1 = \frac{D_1}{D}, x_2 = \frac{D_2}{D}, ..., x_n = \frac{D_n}{D}$ Provided that $D \neq 0$

Now, here we have

5x – 7y + z = 11

6x – 8y – z = 15

3x + 2y – 6z = 7

So by comparing with theorem, now we have to find D, D_1 and D_2

$\Rightarrow \quad D = \begin{vmatrix} 5 & -7 & 1 \\ 6 & -8 & -1 \\ 3 & 2 & -6 \end{vmatrix}$

Solving determinant, expanding along 1^{st} Row

⇒ D = 5[(– 8) (– 6) – (– 1) (2)] – 7[(– 6) (6) – 3(– 1)] + 1[2(6) – 3(– 8)]

⇒ D = 5[48 + 2] – 7[– 36 + 3] + 1[12 + 24]

⇒ D = 250 – 231 + 36

⇒ D = 55

Again, Solve D_1 formed by replacing 1^{st} column by B matrices

Here

$B = \begin{vmatrix} 11 \\ 15 \\ 7 \end{vmatrix}$

$$\Rightarrow \quad D_1 = \begin{vmatrix} 11 & -7 & 1 \\ 15 & -8 & -1 \\ 7 & 2 & -6 \end{vmatrix}$$

Solving determinant, expanding along 1st Row

$\Rightarrow D_1 = 11[(-8)(-6) - (2)(-1)] - (-7)[(15)(-6) - (-1)(7)] + 1[(15)2 - (7)(-8)]$

$\Rightarrow D_1 = 11[48 + 2] + 7[-90 + 7] + 1[30 + 56]$

$\Rightarrow D_1 = 11[50] + 7[-83] + 86$

$\Rightarrow D_1 = 550 - 581 + 86$

$\Rightarrow D_1 = 55$

Again, Solve D_2 formed by replacing 2nd column by B matrices

Here

$$B = \begin{vmatrix} 11 \\ 15 \\ 7 \end{vmatrix}$$

$$\Rightarrow \quad D_2 = \begin{vmatrix} 5 & 11 & 1 \\ 6 & 15 & -1 \\ 3 & 7 & -6 \end{vmatrix}$$

Solving determinant, expanding along 1st Row

$\Rightarrow D_2 = 5[(15)(-6) - (7)(-1)] - 11[(6)(-6) - (-1)(3)] + 1[(6)7 - (15)(3)]$

$\Rightarrow D_2 = 5[-90 + 7] - 11[-36 + 3] + 1[42 - 45]$

$\Rightarrow D_2 = 5[-83] - 11(-33) - 3$

$\Rightarrow D_2 = -415 + 363 - 3$

$\Rightarrow D_2 = -55$

And, Solve D_3 formed by replacing 3rd column by B matrices

Here

$$B = \begin{vmatrix} 11 \\ 15 \\ 7 \end{vmatrix}$$

$$\Rightarrow \quad D_3 = \begin{vmatrix} 5 & -7 & 11 \\ 6 & -8 & 15 \\ 3 & 2 & 7 \end{vmatrix}$$

Solving determinant, expanding along 1st Row

$\Rightarrow D_3 = 5[(-8)(7) - (15)(2)] - (-7)[(6)(7) - (15)(3)] + 11[(6)2 - (-8)(3)]$

$\Rightarrow D_3 = 5[-56 - 30] - (-7)[42 - 45] + 11[12 + 24]$

$\Rightarrow D_3 = 5[-86] + 7[-3] + 11[36]$

$\Rightarrow D_3 = -430 - 21 + 396$

$\Rightarrow D_3 = -55$

Thus by Cramer's Rule, we have

$\Rightarrow x = \frac{D_1}{D}$

$\Rightarrow x = \frac{55}{55}$

$\Rightarrow x = 1$

$\Rightarrow y = \frac{D_2}{D}$

$\Rightarrow y = \frac{-55}{55}$

$\Rightarrow y = -1$

$\Rightarrow z = \frac{D_3}{D}$

$\Rightarrow z = \frac{-55}{55}$

$\Rightarrow z = -1$

EXERCISE 6.5 PAGE NO: 6.89

Solve each of the following system of homogeneous linear equations:

1. x + y – 2z = 0
2x + y – 3z =0
5x + 4y – 9z = 0

Solution:

Given x + y – 2z = 0
2x + y – 3z =0
5x + 4y – 9z = 0
Any system of equation can be written in matrix form as AX = B

Now finding the Determinant of these set of equations,

$$D = \begin{vmatrix} 1 & 1 & -2 \\ 2 & 1 & -3 \\ 5 & 4 & -9 \end{vmatrix}$$

$$|A| = 1\begin{vmatrix} 1 & -3 \\ 4 & -9 \end{vmatrix} - 1\begin{vmatrix} 2 & -3 \\ 5 & -9 \end{vmatrix} - 2\begin{vmatrix} 2 & 1 \\ 5 & 4 \end{vmatrix}$$

= 1(1 × (– 9) – 4 × (– 3)) – 1(2 × (– 9) – 5 × (– 3)) – 2(4 × 2 – 5 × 1)

= 1(– 9 + 12) – 1(– 18 + 15) – 2(8 – 5)

= 1 × 3 –1 × (– 3) – 2 × 3

= 3 + 3 – 6

= 0

Since D = 0, so the system of equation has infinite solution.

Now let z = k

⇒ x + y = 2k

And 2x + y = 3k

Now using the Cramer's rule

$$x = \frac{D_1}{D}$$

$$x = \frac{\begin{vmatrix} 2k & 1 \\ 3k & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix}}$$

$$x = \frac{-k}{-1}$$

x = k

Similarly,

$$y = \frac{D_2}{D}$$

$$y = \frac{\begin{vmatrix} 1 & 2k \\ 2 & 3k \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix}}$$

$$y = \frac{-k}{-1}$$

y = k

Hence, x = y = z = k.

2. 2x + 3y + 4z = 0
x + y + z = 0
2x + 5y – 2z = 0

Solution:
Given
2x + 3y + 4z = 0
x + y + z = 0
2x + 5y – 2z = 0
Any system of equation can be written in matrix form as AX = B
Now finding the Determinant of these set of equations,

$$D = \begin{vmatrix} 2 & 3 & 4 \\ 1 & 1 & 1 \\ 2 & 5 & -2 \end{vmatrix}$$

$|A| = 2\begin{vmatrix}1 & 1\\5 & -2\end{vmatrix} - 3\begin{vmatrix}1 & 1\\2 & -2\end{vmatrix} + 4\begin{vmatrix}1 & 1\\2 & 5\end{vmatrix}$

= 2(1 × (– 2) – 1 × 5) – 3(1 × (– 2) – 2 × 1) + 4(1 × 5 – 2 × 1)

= 2(– 2 – 5) – 3(– 2 – 2) + 4(5 – 2)

= 1 × (– 7) – 3 × (– 4) + 4 × 3

= – 7 + 12 + 12

= 17

Since D ≠ 0, so the system of equation has infinite solution.

Therefore the system of equation has only solution as x = y = z = 0.