

EXERCISE 19.18

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Evaluate the following integrals:

$$1. \int \frac{x}{\sqrt{x^4 + a^4}} dx$$

Solution:

The given equation can be written as

$$\int \frac{x}{\sqrt{x^4 + a^4}} dx = \int \frac{x}{\sqrt{(x^2)^2 + (a^2)^2}} dx$$

Let $x^2 = t$, so $2x dx = dt$

Or, $x dx = dt/2$

$$\text{Hence, } \int \frac{x}{\sqrt{(x^2)^2 + (a^2)^2}} dx = \int \frac{1}{\sqrt{t^2 + (a^2)^2}} \frac{dt}{2} = \frac{1}{2} \int \frac{1}{\sqrt{t^2 + (a^2)^2}} dt$$

$$\text{Since, } \int \frac{1}{\sqrt{(x^2 + a^2)}} dx = \log|x + \sqrt{(x^2 + a^2)}| + c$$

$$\text{Hence, } \frac{1}{2} \int \frac{1}{\sqrt{t^2 + (a^2)^2}} dt = \frac{1}{2} \log |t + \sqrt{t^2 + (a^2)^2}| + c$$

Put $t = x^2$

$$= \frac{1}{2} \log |x^2 + \sqrt{(x^2)^2 + (a^2)^2}| + c$$

$$= \frac{1}{2} \log |x^2 + \sqrt{x^4 + a^4}| + c$$

$$2. \int \frac{\sec^2 x}{\sqrt{4 + \tan^2 x}} dx$$

Solution:

Let $\tan x = t$

Then $dt = \sec^2 x \, dx$

Therefore, $\int \frac{\sec^2 x}{\sqrt{4 + \tan^2 x}} dx = \int \frac{dt}{\sqrt{2^2 + t^2}}$

Since, $\int \frac{1}{\sqrt{(x^2 + a^2)}} dx = \log|x + \sqrt{(x^2 + a^2)}| + c$

Hence, $\int \frac{dt}{\sqrt{2^2 + t^2}} = \log|t + \sqrt{t^2 + 2^2}| + c$
 $= \log|\tan x + \sqrt{\tan^2 x + 4}| + c$

3. $\int \frac{e^x}{\sqrt{16 - e^{2x}}} dx$

Solution:

Let $e^x = t$

Then we have, $e^x dx = dt$

Substituting these values,

Therefore, $\int \frac{e^x}{\sqrt{16 - e^{2x}}} dx = \int \frac{dt}{\sqrt{4^2 - t^2}}$

Since we have, $\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\left(\frac{x}{a}\right) + c$

Hence, $\int \frac{dt}{\sqrt{4^2 - t^2}} = \sin^{-1}\left(\frac{e^x}{4}\right) + c$

4. $\int \frac{\cos x}{\sqrt{4 + \sin^2 x}} dx$

Solution:

Let $\sin x = t$

Then $dt = \cos x \, dx$

Now substituting these values we get

$$\text{Hence, } \int \frac{\cos x}{\sqrt{4 + \sin^2 x}} dx = \int \frac{dt}{\sqrt{2^2 + t^2}}$$

$$\text{Since we have, } \int \frac{1}{\sqrt{(x^2 + a^2)}} dx = \log|x + \sqrt{(x^2 + a^2)}| + c$$

$$\begin{aligned} \text{Therefore, } \int \frac{dt}{\sqrt{2^2 + t^2}} &= \log|t + \sqrt{t^2 + 2^2}| + c \\ &= \log|t + \sqrt{t^2 + 2^2}| + c = \log|\sin x + \sqrt{\sin^2 x + 4}| + c \end{aligned}$$

$$5. \int \frac{\sin x}{\sqrt{4 \cos^2 x - 1}} dx$$

Solution:

Let

$$2 \cos x = t$$

$$\text{Then } dt = -2 \sin x \, dx$$

$$\text{Or, } \sin x \, dx = -\frac{dt}{2}$$

Then substituting these values we get,

$$\text{Therefore, } \int \frac{\sin x}{\sqrt{4 \cos^2 x - 1}} dx = \int -\frac{dt}{2\sqrt{(t^2 - 1^2)}}$$

$$\text{Since, } \int \frac{1}{\sqrt{(x^2 - a^2)}} dx = \log|x + \sqrt{(x^2 - a^2)}| + c$$

$$\text{Therefore, } \int -\frac{dt}{2\sqrt{(t^2 - 1^2)}} = -\frac{1}{2} \log|t + \sqrt{t^2 - 1}| + c$$

On integrating we get

$$= -\frac{1}{2} \log|2 \cos x + \sqrt{4 \cos^2 x - 1}| + c$$

$$6. \int \frac{x}{\sqrt{4-x^4}} dx$$

Solution:

$$\text{Let } x^2 = t$$

$$2x dx = dt \text{ or } x dx = dt/2$$

Now substituting these values in the given equation we get

$$\text{Hence, } \int \frac{x}{\sqrt{4-x^4}} dx = \int \frac{dt}{2(\sqrt{2^2-t^2})}$$

$$\text{Since we have, } \int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1}\left(\frac{x}{a}\right) + c$$

$$\text{So, } \int \frac{dt}{2(\sqrt{2^2-t^2})} = \frac{1}{2} \sin^{-1}\left(\frac{t}{2}\right) + c$$

$$\text{Put } t = x^2$$

$$= \frac{1}{2} \sin^{-1}\left(\frac{t}{2}\right) + c = \frac{1}{2} \sin^{-1}\left(\frac{x^2}{2}\right) + c$$

$$7. \int \frac{1}{x\sqrt{4-9(\log x)^2}} dx$$

Solution:

$$\text{Let } 3 \log x = t$$

$$\text{We have } d(\log x) = 1/x$$

$$\text{Hence, } d(3 \log x) = dt = 3/x dx$$

$$\text{Or } 1/x dx = dt/3$$

$$\text{Hence, } \int \frac{1}{x\sqrt{4-9(\log x)^2}} dx = \int \frac{1}{3\sqrt{2^2-t^2}} dt$$

$$\text{Since we have, } \int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1}\left(\frac{x}{a}\right) + c$$

Hence, $\int \frac{1}{3\sqrt{2^2-t^2}} dt = \frac{1}{3} \sin^{-1}\left(\frac{t}{2}\right) + c$

Put $t = 3 \log x$

$$= \frac{1}{3} \sin^{-1}\left(\frac{t}{2}\right) + c = \frac{1}{3} \sin^{-1}\left(\frac{3 \log x}{2}\right) + c$$

8. $\int \frac{\sin 8x}{\sqrt{9 + \sin^4 4x}} dx$

Solution:

$$dt = 2 \sin 4x \cos 4x \times 4 dx$$

We know $\sin 2x = 2 \sin x \cos x$

Therefore, $dt = 4 \sin 8x dx$

$$\text{Or, } \sin 8x dx = dt/4$$

$$\int \frac{\sin 8x}{\sqrt{9 + \sin^4 4x}} dx = \frac{1}{4} \int \frac{dt}{\sqrt{3^2 + t^2}}$$

Since we have, $\int \frac{1}{\sqrt{(x^2 + a^2)}} dx = \log|x + \sqrt{(x^2 + a^2)}| + c$

$$= \frac{1}{4} \int \frac{dt}{\sqrt{3^2 + t^2}} = \frac{1}{4} \log|t + \sqrt{t^2 + 3^2}| + c$$

$$= \frac{1}{4} \log[\sin^2 4x + \sqrt{9 + \sin^4 4x}] + c$$

9. $\int \frac{\cos 2x}{\sqrt{\sin^2 2x + 8}} dx$

Solution:

$$\text{Let } t = \sin 2x$$

$$dt = 2 \cos 2x dx$$

$$\cos 2x dx = dt/2$$

$$\int \frac{\cos 2x}{\sqrt{\sin^2 2x + 8}} dx = \frac{1}{2} \int dt / \sqrt{t^2 + (2\sqrt{2})^2}$$

Since we have, $\int \frac{1}{\sqrt{x^2 + a^2}} dx = \log|x + \sqrt{x^2 + a^2}| + c$

$$= \frac{1}{2} \int dt / \sqrt{t^2 + (2\sqrt{2})^2} = \frac{1}{2} \log|t + \sqrt{t^2 + 8}| + c$$

$$= \frac{1}{2} \log|t + \sqrt{t^2 + 8}| + c = \frac{1}{2} \log|\sin 2x + \sqrt{\sin^2 2x + 8}| + c$$

