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Evaluate the following integrals:

1. $\int \frac{x^2+x+1}{x^2-x}\,dx$

Solution:

Given $I = \int \frac{x^2+x+1}{x^2-x}\,dx$

Expressing the integral $\int \frac{P(x)}{ax^2+bx+c}\,dx = \int Q(x)\,dx + \int \frac{R(x)}{ax^2+bx+c}\,dx$

$\Rightarrow \int \frac{x^2+x+1}{(x-1)x}\,dx$

$\Rightarrow \int (\frac{2x+1}{(x-1)x} + 1)\,dx$

$\Rightarrow \int \frac{2x+1}{(x-1)x}\,dx + \int 1\ dx$

Consider $\int \frac{2x+1}{(x-1)x}\,dx$

By partial fraction decomposition,

$\Rightarrow \frac{2x+1}{(x-1)x} = \frac{A}{x-1} + \frac{B}{x}$

$\Rightarrow 2x + 1 = Ax + B(x - 1)$

$\Rightarrow 2x + 1 = Ax + Bx - B$

$\Rightarrow 2x + 1 = (A + B)x - B$

$\therefore B = -1$ and $A + B = 2$

$\therefore A = 2 + 1 = 3$

Thus, $\Rightarrow \frac{2x+1}{(x-1)x} = \frac{3}{x-1} - \frac{1}{x}$

$$\Rightarrow \int (\frac{3}{x-1} - \frac{1}{x})dx$$

$$\Rightarrow 3\int \frac{1}{x-1}dx - \int \frac{1}{x}\ dx$$

Consider $\int \frac{1}{x-1}dx$

Substitute u = x – 1 → dx = du.

$$\Rightarrow \int \frac{1}{x-1}dx\ = \int \frac{1}{u}du$$

We know that $\int \frac{1}{x}dx = \log|x| + c$

$$\therefore \int \frac{1}{u}du\ = \log|u| = \log|x-1|$$

Then,

$$\Rightarrow 3\int \frac{1}{x-1}dx - \int \frac{1}{x}\ dx = 3(\log|x-1|) - \int \frac{1}{x}\ dx$$

$$= 3(\log|x-1|) - \log|x|$$

$$\therefore \int \frac{2x+1}{(x-1)x}dx = 3(\log|x-1|) - \log|x|$$

Then,

$$\Rightarrow \int \frac{2x+1}{(x-1)x}dx + \int 1\ dx = 3(\log|x-1|) - \log|x| + \int 1\ dx$$

We know that $\int 1\ dx = x + c$

$$\Rightarrow \int \frac{2x+1}{(x-1)x}dx + \int 1\ dx = 3(\log|x-1|) - \log|x| + x + c$$

$$\therefore I = \int \frac{x^2+x+1}{x^2-x}dx = -\log|x| + x + 3(\log|x-1|) + c$$

2. $\int \frac{x^2+x-1}{x^2+x-6}\,dx$

Solution:

Consider $I = \int \frac{x^2+x-1}{x^2+x-6} dx$

Expressing the integral $\int \frac{P(x)}{ax^2+bx+c} dx = \int Q(x)\, dx + \int \frac{R(x)}{ax^2+bx+c} dx$

Let $x^2 + x - 1 = x^2 + x - 6 + 5$

$$\Rightarrow \int \frac{x^2+x-1}{x^2+x-6} dx = \int \left(\frac{x^2+x-6}{x^2+x-6} + \frac{5}{x^2+x-6}\right) dx$$

$$= \int \left(\frac{5}{x^2+x-6} + 1\right) dx$$

$$= 5 \int \left(\frac{1}{x^2+x-6}\right) dx + \int 1\, dx$$

Consider $\int \frac{1}{x^2+x-6} dx$

Factorizing the denominator,

$$\Rightarrow \int \frac{1}{x^2+x-6} dx = \int \frac{1}{(x-2)(x+3)} dx$$

By partial fraction decomposition,

$$\Rightarrow \frac{1}{(x-2)(x+3)} = \frac{A}{x-2} + \frac{B}{x+3}$$

$\Rightarrow 1 = A(x + 3) + B(x - 2)$

$\Rightarrow 1 = Ax + 3A + Bx - 2B$

$\Rightarrow 1 = (A + B) x + (3A - 2B)$

$\Rightarrow$ Then $A + B = 0$ … (1)

And $3A - 2B = 1$ … (2)

Solving (1) and (2),

$2 \times (1) \rightarrow 2A + 2B = 0$

$1 \times (2) \rightarrow 3A - 2B = 1$

$5A = 1$

$\therefore A = 1/5$

Substituting A value in (1),

$\Rightarrow A + B = 0$

$\Rightarrow 1/5 + B = 0$

$\therefore B = -1/5$

Thus, $\frac{1}{(x-2)(x+3)} = \frac{1}{5(x-2)} - \frac{1}{5(x+3)}$

$$= \frac{1}{5}\int \frac{1}{x-2}dx - \frac{1}{5}\int \frac{1}{x+3}dx$$

Let $x - 2 = u \rightarrow dx = du$

And $x + 3 = v \rightarrow dx = dv$.

$$\Rightarrow \frac{1}{5}\int \frac{1}{u}du - \frac{1}{5}\int \frac{1}{v}dv$$

We know that $\int \frac{1}{x}dx = \log|x| + c$

$$\Rightarrow \frac{1}{5}\log|u| - \frac{1}{5}\log|v|$$

$$\Rightarrow \frac{1}{5}\log|x-2| - \frac{1}{5}\log|x+3|$$

$$\Rightarrow \frac{1}{5}(\log|x-2| - \log|x+3|)$$

Then,

$$\Rightarrow 5\int \left(\frac{1}{x^2+x-6}\right)dx + \int 1\,dx = 5\left(\frac{1}{5}(\log|x-2| - \log|x+3|)\right) + \int 1\,dx$$

We know that $\int 1\ dx = x + c$

$$\Rightarrow (\log|x-2| - \log|x+3|) + x + c$$

$\therefore I = \int \frac{x^2+x-1}{x^2+x-6}dx = -\log|x+3| + x + \log|x-2| + c$

Or I = log|(x - 2)/(x + 3)| + x + c

3. $\int \frac{(1-x^2)}{x(1-2x)} dx$

Solution:

Given $I = \int \frac{1-x^2}{(1-2x)x} dx$

Rewriting, we get $\int \frac{x^2-1}{x(2x-1)} dx$

Expressing the integral $\int \frac{P(x)}{ax^2+bx+c} dx = \int Q(x)\, dx + \int \frac{R(x)}{ax^2+bx+c} dx$

$$\Rightarrow \int \frac{x^2-1}{x(2x-1)} dx = \int \left(\frac{x-2}{2x(2x-1)} + \frac{1}{2}\right) dx$$

$$= \frac{1}{2}\int \frac{x-2}{x(2x-1)} dx + \frac{1}{2}\int 1dx$$

Consider $\int \frac{x-2}{x(2x-1)} dx$

By partial fraction decomposition,

$$\Rightarrow \frac{x-2}{x(2x-1)} = \frac{A}{x} + \frac{B}{2x-1}$$

$\Rightarrow x - 2 = A\,(2x - 1) + Bx$

$\Rightarrow x - 2 = 2Ax - A + Bx$

$\Rightarrow x - 2 = (2A + B)\,x - A$

$\therefore A = 2$ and $2A + B = 1$

$\therefore B = 1 - 4 = -3$

Thus, $\Rightarrow \frac{x-2}{x(2x-1)} = \frac{2}{x} - \frac{3}{2x-1}$

$$\Rightarrow \int \left(\frac{2}{x} - \frac{3}{2x-1}\right) dx$$

$$\Rightarrow 2\int \frac{1}{x} dx - 3\int \frac{1}{2x-1}\, dx$$

Consider $\int \frac{1}{x} dx$

We know that $\int \frac{1}{x} dx = \log|x| + c$

$$\Rightarrow \int \frac{1}{x} dx = \log|x|$$

And consider $\int \frac{1}{2x-1} dx$

Let u = 2x – 1 → dx = 1/2 du

$$\Rightarrow \int \frac{1}{2x-1} dx = \frac{1}{2} \int \frac{1}{u} du$$

We know that $\int \frac{1}{x} dx = \log|x| + c$

$$\Rightarrow \frac{1}{2} \int \frac{1}{u} du = \frac{\log|u|}{2} = \frac{\log|2x-1|}{2}$$

Then,

$$\Rightarrow \int \frac{x-2}{x(2x-1)} dx = 2 \int \frac{1}{x} dx - 3 \int \frac{1}{2x-1} dx$$

$$= 2(\log|x|) - 3\left(\frac{\log|2x-1|}{2}\right)$$

Then,

$$\Rightarrow \int \frac{x^2-1}{x(2x-1)} dx = \frac{1}{2} \int \frac{x-2}{x(2x-1)} dx + \frac{1}{2} \int 1 dx$$
$$= \frac{1}{2}\left(2(\log|x|) - 3\left(\frac{\log|2x-1|}{2}\right)\right) + \frac{1}{2} \int 1 \, dx$$

We know that $\int 1 \; dx = x + c$

$$\Rightarrow \log|x| - \frac{3\log|2x-1|}{4} + \frac{x}{2} + c$$

$$\therefore I = \int \frac{1-x^2}{(1-2x)x} dx = -\frac{3\log|2x-1|}{4} + \log|x| + \frac{x}{2} + c$$

4. $\int \frac{x^2+1}{x^2-5x+6}\,dx$

Solution:

Let $u = x^2 - 5x + 6 \rightarrow dx = \frac{1}{2x-5}du$

$\Rightarrow \int \frac{2x-5}{(x^2-5x+6)}dx = \int \frac{2x-5}{u}\frac{1}{2x-5}du$

$= \int \frac{1}{u}du$

We know that $\int \frac{1}{x}dx = \log|x| + c$

$\Rightarrow \int \frac{1}{u}du = \log|u| = \log|x^2 - 5x + 6|$

Now consider $\int \frac{1}{x^2-5x+6}dx$

$\Rightarrow \int \frac{1}{x^2-5x+6}dx = \int \frac{1}{(x-3)(x-2)}dx$

By partial fraction decomposition,

$\Rightarrow \frac{1}{(x-3)(x-2)} = \frac{A}{x-3} + \frac{B}{x-2}$

$\Rightarrow 1 = A(x-2) + B(x-3)$

$\Rightarrow 1 = Ax - 2A + Bx - 3B$

$\Rightarrow 1 = (A + B)x - (2A + 3B)$

$\Rightarrow A + B = 0$ and $2A + 3B = -1$

Solving the two equations,

$\Rightarrow \quad A + B = 0$

2A + 3B = -1

-B = 1

∴ B = -1 and A = 1

$$\Rightarrow \int \frac{1}{(x-3)(x-2)} dx = \int \left(\frac{1}{x-3} - \frac{1}{x-2}\right) dx$$

$$= \int \frac{1}{x-3} dx - \int \frac{1}{x-2} dx$$

Consider $\int \frac{1}{x-3} dx$

Let u = x – 3 → dx = du

$$\Rightarrow \int \frac{1}{x-3} dx = \int \frac{1}{u} du$$

We know that $\int \frac{1}{x} dx = \log|x| + c$

$$\Rightarrow \int \frac{1}{u} du = \log|u| = \log|x-3|$$

Similarly $\int \frac{1}{x-2} dx$

Let u = x – 2 → dx = du

$$\Rightarrow \int \frac{1}{x-2} dx = \int \frac{1}{u} du$$

We know that $\int \frac{1}{x} dx = \log|x| + c$

$$\Rightarrow \int \frac{1}{u} du = \log|u| = \log|x-2|$$

Then,

$$\Rightarrow \int \frac{1}{x^2-5x+6} dx = \int \frac{1}{(x-3)(x-2)} dx = \int \frac{1}{x-3} dx - \int \frac{1}{x-2} dx$$

$$= \log|x-3| - \log|x-2|$$

$$\Rightarrow \int \frac{x-1}{x^2-5x+6}dx = \frac{1}{2}\int \frac{2x-5}{(x^2-5x+6)}dx + \frac{3}{2}\int \frac{1}{x^2-5x+6}dx$$

$$= \frac{1}{2}(\log|x^2-5x+6|) + \frac{3}{2}(\log|x-3| - \log|x-2|)$$

$$= \frac{\log|x^2-5x+6|}{2} + \frac{3\log|x-3|}{2} - \frac{3\log|x-2|}{2}$$

Then,

$$\Rightarrow \int \frac{x^2+1}{x^2-5x+6}dx = 5\int \frac{x-1}{x^2-5x+6}dx + \int 1\,dx$$

We know that $\int 1\ dx = x + c$

$$\Rightarrow 5\int \frac{x-1}{x^2-5x+6}dx + \int 1\,dx$$

$$= \frac{5\log|x^2-5x+6|}{2} + \frac{15\log|x-3|}{2} - \frac{15\log|x-2|}{2} + x + c$$

$$= \frac{5\log|x-2|\log|x-3|}{2} + \frac{15\log|x-3|}{2} - \frac{15\log|x-2|}{2} + x + c$$

$$= x - 5\log|x-2| + 10\log|x-3| + c$$

$$\therefore I = \int \frac{x^2+1}{x^2-5x+6}dx = x - 5\log|x-2| + 10\log|x-3| + c$$

5. $\displaystyle\int \frac{x^2}{x^2+7x+10}\,dx$

Solution:

Given $I = \int \frac{x^2}{x^2+7x+10}dx$

Expressing the integral $\int \frac{P(x)}{ax^2+bx+c}dx = \int Q(x)\,dx + \int \frac{R(x)}{ax^2+bx+c}dx$

$$\Rightarrow \int \frac{x^2}{x^2+7x+10}dx = \int (\frac{-7x-10}{x^2+7x+10} + 1)dx$$

$$= -\int \frac{7x+10}{x^2+7x+10}dx + \int 1\,dx$$

Consider $\int \frac{7x+10}{x^2+7x+10}dx$

Let $7x+10 = \frac{7}{2}(2x+7) - \frac{29}{2}$ and split,

$$\Rightarrow \int \frac{7x+10}{x^2+7x+10}dx = \int \left(\frac{7(2x+7)}{2(x^2+7x+10)} - \frac{29}{2(x^2+7x+10)}\right)dx$$

$$= \frac{7}{2}\int \frac{2x+7}{x^2+7x+10}dx - \frac{29}{2}\int \frac{1}{x^2+7x+10}dx$$

Consider $\int \frac{2x+7}{x^2+7x+10}dx$

Let $u = x^2+7x+10 \to dx = \frac{1}{2x+7}du$

$$\Rightarrow \int \frac{2x+7}{(x^2+7x+10)}dx = \int \frac{2x+7}{u}\frac{1}{2x+7}du$$

$$= \int \frac{1}{u}du$$

We know that $\int \frac{1}{x}dx = \log|x| + c$

$$\Rightarrow \int \frac{1}{u}du = \log|u| = \log|x^2+7x+10|$$

Now consider $\int \frac{1}{x^2+7x+10}dx$

$$\Rightarrow \int \frac{1}{x^2+7x+10}dx = \int \frac{1}{(x+2)(x+5)}dx$$

By partial fraction decomposition,

$$\Rightarrow \frac{1}{(x+2)(x+5)} = \frac{A}{x+2} + \frac{B}{x+5}$$

$\Rightarrow 1 = A(x+2) + B(x+5)$

$\Rightarrow 1 = Ax + 2A + Bx + 5B$

$\Rightarrow$ A + B = 0 and 2A + 5B = 1

Solving the two equations,

$\Rightarrow$ 2A + 2B = 0

2A + 5B = 1

-3B = -1

$\therefore$ B = 1/3 and A = -1/3

$$\Rightarrow \int \frac{1}{(x+2)(x+5)} dx = \int \left(\frac{-1}{3(x+2)} + \frac{1}{3(x+5)}\right) dx$$

$$= -\frac{1}{3}\int \frac{1}{x+2} dx + \frac{1}{3}\int \frac{1}{x+5} dx$$

Consider $\int \frac{1}{x+2} dx$

Let u = x + 2 → dx = du

$$\Rightarrow \int \frac{1}{x+2} dx = \int \frac{1}{u} du$$

We know that $\int \frac{1}{x} dx = \log|x| + c$

$$\Rightarrow \int \frac{1}{u} du = \log|u| = \log|x+2|$$

Similarly $\int \frac{1}{x+5} dx$

Let u = x + 5 → dx = du

$$\Rightarrow \int \frac{1}{x+5} dx = \int \frac{1}{u} du$$

We know that $\int \frac{1}{x} dx = \log|x| + c$

$$\Rightarrow \int \frac{1}{u} du = \log|u| = \log|x+5|$$

Then,

$$\Rightarrow \int \frac{1}{x^2+7x+10}dx = \int \frac{1}{(x+2)(x+5)}dx = -\frac{1}{3}\int \frac{1}{x+2}dx + \frac{1}{3}\int \frac{1}{x+5}dx$$

$$= \frac{-\log|x+2|}{3} + \frac{\log|x+5|}{3}$$

Then,

$$\Rightarrow \int \frac{7x+10}{x^2+7x+10}dx = \frac{7}{2}\int \frac{2x+7}{x^2+7x+10}dx - \frac{29}{2}\int \frac{1}{x^2+7x+10}dx$$

$$= \frac{7}{2}(\log|x^2+7x+10|) - \frac{29}{2}(\frac{-\log|x+2|}{3} + \frac{\log|x+5|}{3})$$

$$= \frac{7\log|x^2+7x+10|}{2} + \frac{29\log|x+2|}{6} - \frac{29\log|x+5|}{6}$$

Then,

$$\Rightarrow \int \frac{x^2}{x^2+7x+10}dx = -\int \frac{7x+10}{x^2+7x+10}dx + \int 1\,dx$$

We know that $\int 1 \ dx = x + c$

$$\Rightarrow -\int \frac{7x+10}{x^2+7x+10}dx + \int 1\,dx$$

$$= \frac{-7\log|x^2+7x+10|}{2} - \frac{29\log|x+2|}{6} + \frac{29\log|x+5|}{6} + x + c$$

Hence,

$$I = x - \frac{7}{2}\log\left|x^2+7x+10\right| + \frac{29}{6}\log\left|\frac{x+2}{x+5}\right| + c$$