EXERCISE 19.23

Evaluate the following integrals:

1. $\int \frac{1}{5+4\cos x}\,dx$

Solution:

Given $I = \int \frac{1}{5+4\cos x}dx$

We know that $\cos x = \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}$

$$\Rightarrow \int \frac{1}{5+4\cos x}dx = \int \frac{1}{5+4\left(\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right)}dx$$

$$= \int \frac{1+\tan^2\frac{x}{2}}{5\left(1+\tan^2\frac{x}{2}\right)+4(1-\tan^2\frac{x}{2})}dx$$

Replacing $1 + \tan^2 x/2$ in numerator by $\sec^2 x/2$,

$$\Rightarrow \int \frac{1+\tan^2\frac{x}{2}}{5\left(1+\tan^2\frac{x}{2}\right)+4(1-\tan^2\frac{x}{2})}dx = \int \frac{\sec^2\frac{x}{2}}{\tan^2\frac{x}{2}+9}dx$$

Putting $\tan x/2 = t$ and $\sec^2(x/2)\,dx = 2dt$,

$$\Rightarrow \int \frac{\sec^2\frac{x}{2}}{\tan^2\frac{x}{2}+9}dx = \int \frac{2dt}{t^2+9}$$

$$= 2\int \frac{1}{t^2+9}dt$$

We know that $\int \frac{1}{a^2+x^2}dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$

$$\Rightarrow 2\int \frac{1}{t^2+9}dt = 2\left(\frac{1}{3}\right)\tan^{-1}\left(\frac{t}{3}\right) + c$$

$$= \frac{2}{3}\tan^{-1}\left(\frac{\tan x/2}{3}\right) + c$$

2. $\int \frac{1}{5 - 4\sin x}\, dx$

Solution:

Given $I = \int \frac{1}{5-4\sin x} dx$

We know that $\sin x = \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}$

$$\Rightarrow \int \frac{1}{5 - 4\sin x} dx = \int \frac{1}{5 - 4\left(\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right)} dx$$

$$= \int \frac{1 + \tan^2\frac{x}{2}}{5\left(1 + \tan^2\frac{x}{2}\right) - 4\left(2\tan\frac{x}{2}\right)} dx$$

Replacing $1 + \tan^2 x/2$ in numerator by $\sec^2 x/2$,

$$\Rightarrow \int \frac{1 + \tan^2\frac{x}{2}}{5\left(1 + \tan^2\frac{x}{2}\right) - 4\left(2\tan\frac{x}{2}\right)} dx = \int \frac{\sec^2\frac{x}{2}}{5 + 5\tan^2\frac{x}{2} - 8\tan\frac{x}{2}} dx$$

Putting $\tan x/2 = t$ and $\sec^2(x/2)\, dx = 2dt$,

$$\Rightarrow \int \frac{\sec^2\frac{x}{2}}{5 + 5\tan^2\frac{x}{2} - 8\tan\frac{x}{2}} dx = \int \frac{2dt}{5 + 5t^2 - 8t}$$

$$= \frac{2}{5}\int \frac{1}{t^2 - \frac{8}{5}t + 1} dt$$

$$= \frac{2}{5}\int \frac{1}{\left(t - \frac{4}{5}\right)^2 + \left(\frac{3}{5}\right)^2} dt$$

We know that $\int \frac{1}{a^2+x^2} dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$

$$\Rightarrow \frac{2}{5}\int \frac{1}{\left(t-\frac{4}{5}\right)^2 + \left(\frac{3}{5}\right)^2} dt = \frac{2}{5}\left(\frac{1}{\frac{3}{5}}\right)\tan^{-1}\left(\frac{t-\frac{4}{5}}{\frac{3}{5}}\right) + c$$

$$= \frac{2}{3}\tan^{-1}\left(\frac{5\tan x/2 - 4}{3}\right) + c$$

3. $\int \frac{1}{1-2\sin x}\, dx$

Solution:

Given $I = \int \frac{1}{1-2\sin x} dx$

We know that $\sin x = \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}$

$$\Rightarrow \int \frac{1}{1-2\sin x} dx = \int \frac{1}{1-2\left(\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right)} dx$$

$$= \int \frac{1+\tan^2\frac{x}{2}}{1\left(1+\tan^2\frac{x}{2}\right) - 2\left(2\tan\frac{x}{2}\right)} dx$$

Replacing $1 + \tan^2 x/2$ in numerator by $\sec^2 x/2$,

$$\Rightarrow \int \frac{1+\tan^2\frac{x}{2}}{1\left(1+\tan^2\frac{x}{2}\right) - 2\left(2\tan\frac{x}{2}\right)} dx = \int \frac{\sec^2\frac{x}{2}}{1+\tan^2\frac{x}{2} - 4\tan\frac{x}{2}} dx$$

Putting $\tan x/2 = t$ and $\sec^2(x/2)\, dx = 2dt$,

$$\Rightarrow \int \frac{\sec^2\frac{x}{2}}{1+\tan^2\frac{x}{2} - 4\tan\frac{x}{2}} dx = \int \frac{2dt}{1+t^2-4t}$$

$$= 2\int \frac{1}{t^2-4t+1} dt$$

$$= 2\int \frac{1}{(t-2)^2 - (\sqrt{3})^2} dt$$

We know that $\int \frac{1}{x^2-a^2} dx = \frac{1}{2a} \log\left|\frac{x-a}{x+a}\right| + c$

$$\Rightarrow 2\int \frac{1}{(t-2)^2 - (\sqrt{3})^2} dt = 2\left(\frac{1}{2\sqrt{3}}\right) \log\left|\left(\frac{t-2-\sqrt{3}}{t-2+\sqrt{3}}\right)\right| + c$$

$$= \frac{1}{\sqrt{3}} \log\left|\left(\frac{\tan x - (2+\sqrt{3})}{\tan x + (2+\sqrt{3})}\right)\right| + c$$

$$= \frac{1}{\sqrt{3}} \log\left|\frac{\tan\frac{x}{2} - 2 - \sqrt{3}}{\tan\frac{x}{2} - 2 + \sqrt{3}}\right| + c$$

4. $\int \frac{1}{4\cos x - 1} dx$

Solution:

Given $I = \int \frac{1}{4\cos x - 1} dx$

We know that $\cos x = \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}$

$$\Rightarrow \int \frac{1}{-1+4\cos x} dx = \int \frac{1}{-1+4\left(\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right)} dx$$

$$= \int \frac{1+\tan^2\frac{x}{2}}{-1\left(1+\tan^2\frac{x}{2}\right) + 4(1-\tan^2\frac{x}{2})} dx$$

Replacing $1 + \tan^2 x/2$ in numerator by $\sec^2 x/2$,

$$\Rightarrow \int \frac{1+\tan^2\frac{x}{2}}{-1\left(1+\tan^2\frac{x}{2}\right) + 4(1-\tan^2\frac{x}{2})} dx = \int \frac{\sec^2\frac{x}{2}}{-5\tan^2\frac{x}{2} + 3} dx$$

Putting $\tan\frac{x}{2} = t$ and $\sec^2(\frac{x}{2})dx = dt$,

$$\Rightarrow \int \frac{\sec^2\frac{x}{2}}{-5\tan^2\frac{x}{2}+3}dx = \int \frac{2\,dt}{3-5t^2}$$

$$= \frac{2}{5}\int \frac{1}{\frac{3}{5}-t^2}dt$$

We know that $\int \frac{1}{a^2-x^2}dx = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + c$

$$\Rightarrow \frac{2}{5}\int \frac{1}{\frac{3}{5}-t^2}dt = \frac{2}{5}\left(\frac{1}{2\sqrt{\frac{3}{5}}}\right)\log\left|\frac{\sqrt{\frac{3}{5}}+t}{\sqrt{\frac{3}{5}}-t}\right| + c$$

$$= \frac{1}{\sqrt{15}}\log\left|\frac{\sqrt{3}+\sqrt{5}\tan\frac{x}{2}}{\sqrt{3}-\sqrt{5}\tan\frac{x}{2}}\right| + c$$

$$\therefore I = \int \frac{1}{4\cos x - 1}dx = \frac{1}{\sqrt{15}}\log\left|\frac{\sqrt{3}+\sqrt{5}\tan\frac{x}{2}}{\sqrt{3}-\sqrt{5}\tan\frac{x}{2}}\right| + c$$

5. $\int \frac{1}{1-\sin x+\cos x}dx$

Solution:

Given $I = \int \frac{1}{1-\sin x+\cos x}dx$

We know that $\sin x = \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}$ and $\cos x = \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}$

$$\Rightarrow \int \frac{1}{1-\sin x+\cos x}dx = \int \frac{1}{1-\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}+\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}}dx$$

$$= \int \frac{1+\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}-2\tan\frac{x}{2}+1-\tan^2\frac{x}{2}}dx$$

Replacing $1 + \tan^2 x/2$ in numerator by $\sec^2 x/2$ and putting $\tan x/2 = t$ and $\sec^2 x/2\, dx = 2dt$,

$$\Rightarrow \int \frac{1+\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}-2\tan\frac{x}{2}+1-\tan^2\frac{x}{2}}dx = \int \frac{\sec^2\frac{x}{2}}{2-2\tan\frac{x}{2}}dx$$

$$= \int \frac{2dt}{2-2t}$$

$$= \int \frac{1}{1-t}dt$$

We know that $\int \frac{1}{x}dx = \log|x| + c$

$$\Rightarrow \int \frac{1}{1-t}dt = -\log|1-t| + c$$

$$= -\log\left|1-\tan\frac{x}{2}\right| + c$$

$$\therefore I = \int \frac{1}{1-\sin x+\cos x}dx = -\log\left|1-\tan\frac{x}{2}\right| + c$$