EXERCISE 19.24 PAGE NO: 19.122

Evaluate the following integrals:

1. $\int \frac{1}{1-\cot x}\,dx$

Solution:

Let, I = $\int \frac{1}{1-\cot x} dx$

To solve such integrals involving trigonometric terms in numerator and denominators. If I has the form $\int \frac{a\sin x+b\cos x+c}{d\sin x+e\cos x+f} dx$

Then substitute numerator as

$$a\sin x + b\cos x + c = A\frac{d}{dx}(d\sin x + e\cos x + f) + B(d\sin x + e\cos x + f) + C$$

Where A, B and C are constants

We have, I = $\int \frac{1}{1-\cot x} dx = \int \frac{1}{1-\frac{\cos x}{\sin x}} dx = \int \frac{\sin x}{\sin x-\cos x} dx$

As I matches with the form described above, so we will take the steps as described.

$\therefore \sin x = A\frac{d}{dx}(\sin x - \cos x) + B(\sin x - \cos x) + C$

$\Rightarrow \sin x = A(\cos x + \sin x) + B(\sin x - \cos x) + C \; \{\because \frac{d}{dx}\cos x = -\sin x\}$

$\Rightarrow \sin x = \sin x\,(B + A) + \cos x\,(A - B) + C$

Comparing both sides we have:

C = 0

A – B = 0 ⇒ A = B

B + A = 1 ⇒ 2A = 1 ⇒ A = ½

∴ A = B = ½

Thus I can be expressed as:

I = $\int \frac{\frac{1}{2}(\cos x+\sin x)+\frac{1}{2}(\sin x-\cos x)}{\sin x-\cos x}dx$

I = $\int \frac{\frac{1}{2}(\cos x+\sin x)}{\sin x-\cos x}dx + \int \frac{\frac{1}{2}(\sin x-\cos x)}{\sin x-\cos x}dx$

∴ Let $I_1 = \frac{1}{2}\int \frac{(\cos x+\sin x)}{\sin x-\cos x}dx$ and $I_2 = \frac{1}{2}\int \frac{(\sin x-\cos x)}{\sin x-\cos x}dx$

⇒ $I = I_1 + I_2$ ....equation 1

$I_1 = \frac{1}{2}\int \frac{(\cos x+\sin x)}{\sin x-\cos x}dx$

Let, u = sin x – cos x ⇒ du = (cos x + sin x) dx

So, I_1 reduces to:

$I_1 = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\log|u| + C_1$

∴ $I_1 = \frac{1}{2}\log|\sin x - \cos x| + C_1$ ..... Equation 2

As, $I_2 = \frac{1}{2}\int \frac{(\sin x-\cos x)}{\sin x-\cos x}dx = \frac{1}{2}\int dx$

∴ $I_2 = \frac{x}{2} + C_2$ ..... Equation 3

From equation 1, 2 and 3 we have:

$I = \frac{1}{2}\log|\sin x - \cos x| + C_1 + \frac{x}{2} + C_2$

∴ $I = \frac{1}{2}\log|\sin x - \cos x| + \frac{x}{2} + C$

2. $\int \frac{1}{1-\tan x}dx$

Solution:

Let, I = $\int \frac{1}{1-\tan x}dx$

To solve such integrals involving trigonometric terms in numerator and denominators. If I has the form $\int \frac{a\sin x+b\cos x+c}{d\sin x+e\cos x+f}dx$

Then substitute numerator as

$a\sin x+b\cos x+c = A\frac{d}{dx}(d\sin x+e\cos x+f)+B(d\sin x+e\cos x+f)+C$

Where A, B and C are constants

We have, I = $\int \frac{1}{1-\tan x}dx = \int \frac{1}{1-\frac{\sin x}{\cos x}}dx = \int \frac{\cos x}{\cos x-\sin x}dx$

As I matches with the form described above, so we will take the steps as described.

$\therefore \cos x = A\frac{d}{dx}(\cos x-\sin x)+B(\cos x-\sin x)+C$

$\Rightarrow \cos x = A(-\sin x-\cos x)+B(\cos x-\sin x)+C \left\{\because \frac{d}{dx}\cos x = -\sin x\right\}$

$\Rightarrow \cos x = -\sin x\,(B+A)+\cos x\,(B-A)+C$

Comparing both sides we have:

C = 0

B – A = 1 ⇒ A = B - 1

B + A = 0 ⇒ 2B - 1 = 0 ⇒ B = ½
∴ A = B - 1 = - ½

Thus I can be expressed as:

$$I = \int \frac{-\frac{1}{2}(-\sin x-\cos x)+\frac{1}{2}(\cos x-\sin x)}{(\cos x-\sin x)}dx$$

$$I = \int \frac{\frac{1}{2}(\cos x+\sin x)+\frac{1}{2}(\cos x-\sin x)}{(\cos x-\sin x)}dx$$

$$I = \int \frac{\frac{1}{2}(\cos x+\sin x)}{(\cos x-\sin x)}dx + \int \frac{\frac{1}{2}(\cos x-\sin x)}{(\cos x-\sin x)}dx$$

$\therefore$ Let $I_1 = \frac{1}{2}\int \frac{(\cos x+\sin x)}{(\cos x-\sin x)} dx$ and $I_2 = \frac{1}{2}\int \frac{(\cos x-\sin x)}{(\cos x-\sin x)} dx$

$\Rightarrow I = I_1 + I_2$ ....equation 1

$I_1 = \frac{1}{2}\int \frac{(\cos x+\sin x)}{(\cos x-\sin x)} dx$

Let, $u = \cos x - \sin x \Rightarrow du = -(\cos x + \sin x)\,dx$

So, I_1 reduces to:

$I_1 = -\frac{1}{2}\int \frac{du}{u} = -\frac{1}{2}\log|u| + C_1$

$\therefore I_1 = -\frac{1}{2}\log|\cos x - \sin x| + C_1$ ..... Equation 2

As, $I_2 = \frac{1}{2}\int \frac{(\cos x-\sin x)}{(\cos x-\sin x)} dx = \frac{1}{2}\int dx$

$\therefore I_2 = \frac{x}{2} + C_2$ ..... Equation 3

From equation 1, 2 and 3 we have:

$I = -\frac{1}{2}\log|\cos x - \sin x| + C_1 + \frac{x}{2} + C_2$

$\therefore I = -\frac{1}{2}\log|\cos x - \sin x| + \frac{x}{2} + C$

3. $\int \frac{3 + 2\cos x + 4\sin x}{2\sin x + \cos x + 3} dx$

Solution:

Let, $I = \int \frac{3+2\cos x+4\sin x}{2\sin x+\cos x+3} dx$

To solve such integrals involving trigonometric terms in numerator and denominators. If I has the form $\int \frac{a\sin x+b\cos x+c}{d\sin x+e\cos x+f} dx$

Then substitute numerator as

$$a\sin x + b\cos x + c = A\frac{d}{dx}(d\sin x + e\cos x + f) + B(d\sin x + e\cos x + f) + C$$

Where A, B and C are constants

We have, I = $\int \frac{3+2\cos x+4\sin x}{2\sin x+\cos x+3}dx$

As I matches with the form described above, so we will take the steps as described.

$\therefore 3 + 2\cos x + 4\sin x = A\frac{d}{dx}(2\sin x + \cos x + 3) + B(2\sin x + \cos x + 3) + C$

$\Rightarrow 3 + 2\cos x + 4\sin x = A(2\cos x - \sin x) + B(2\sin x + \cos x + 3) + C$

$\{\because \frac{d}{dx}\cos x = -\sin x\}$

$\Rightarrow 3 + 2\cos x + 4\sin x = \sin x\,(2B - A) + \cos x\,(B + 2A) + 3B + C$

Comparing both sides we have:

3B+ C = 3

B + 2A = 2

2B - A = 4

On solving for A, B and C we have:

A = 0, B = 2 and C = -3

Thus I can be expressed as:

I = $\int \frac{2(2\sin x+\cos x+3)-3}{2\sin x+\cos x+3}dx$

I = $\int \frac{2(2\sin x+\cos x+3)}{2\sin x+\cos x+3}dx + \int \frac{-3}{2\sin x+\cos x+3}dx$

$\therefore$ Let $I_1 = 2\int \frac{(2\sin x+\cos x+3)}{2\sin x+\cos x+3}dx$ and $I_2 = -3\int \frac{1}{2\sin x+\cos x+3}dx$

$\Rightarrow I = I_1 + I_2$ ....equation 1

$I_1 = 2\int \frac{(2\sin x+\cos x+3)}{2\sin x+\cos x+3}dx$

So, I_1 reduces to:

$I_1 = 2\int dx = 2x + C_1$ ..... Equation 2

As, $I_2 = -3\int \frac{1}{2\sin x+\cos x+3}dx$

To solve the integrals of the form $\int \frac{1}{a\sin x+b\cos x+c}dx$

To apply substitution method we take following procedure.

We substitute:

$$\sin x = \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}} \text{ and } \cos x = \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}$$

$\therefore I_2 = -3\int \frac{1}{2\sin x+\cos x+3}dx$

$\Rightarrow I_2 = -3\int \frac{1}{2\left(\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right)+3\left(\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right)+3}dx$

$\Rightarrow I_2 = -3\int \frac{1+\tan^2\frac{x}{2}}{4\tan\frac{x}{2}+1-\tan^2\frac{x}{2}+3(1+\tan^2\frac{x}{2})}dx$

$\Rightarrow I_2 = -3\int \frac{\sec^2\frac{x}{2}}{2(2\tan\frac{x}{2}+2+1\tan^2\frac{x}{2})}dx$

Let, $t = \tan\frac{x}{2} \Rightarrow dt = \frac{1}{2}\sec^2\frac{x}{2}dx$

$\therefore I_2 = -3\int \frac{1}{(2t+2+t^2)}dt$

As, the denominator is polynomial without any square root term. So one of the special integral will be used to solve I_2.

$I_2 = -3\int \frac{1}{(2t+2+t^2)}dt$

$\Rightarrow I_2 = -3\int \frac{1}{(t^2+2(1)t+1)+1}\,dt$

$\therefore I_2 = -3\int \frac{1}{(t+1)^2+1}\,dt$ $\{\because a^2 + 2ab + b^2 = (a + b)^2\}$

As, I_2 matches with the special integral form

$$\int \frac{1}{x^2+a^2}dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$$

$I_2 = -3\tan^{-1}(t+1)$

Putting value of t we have:

$\therefore I_2 = -3\tan^{-1}\left(\tan\frac{x}{2}+1\right) + C_2$ ......equation 3

From equation 1, 2 and 3:

$I = 2x + C_1 - 3\tan^{-1}\left(\tan\frac{x}{2}+1\right) + C_2$

$\therefore I = 2x - 3\tan^{-1}\left(\tan\frac{x}{2}+1\right) + C$

4. $\int \frac{1}{p+q\tan x}\,dx$

Solution:

Let, $I = \int \frac{1}{p+q\tan x}dx$

To solve such integrals involving trigonometric terms in numerator and denominators. If I has the form $\int \frac{a\sin x+b\cos x+c}{d\sin x+e\cos x+f}\,dx$

Then substitute numerator as

$$a\sin x + b\cos x + c = A\frac{d}{dx}(d\sin x + e\cos x + f) + B(d\sin x + e\cos x + f) + C$$

Where A, B and C are constants

We have, I = $\int \frac{1}{p+q\tan x}dx = \int \frac{1}{p+q\frac{\sin x}{\cos x}}dx = \int \frac{\cos x}{p\cos x+q\sin x}dx$

As I matches with the form described above, so we will take the steps as described.

$\therefore \cos x = A\frac{d}{dx}(p\cos x + q\sin x) + B(p\cos x + q\sin x) + C$

$\Rightarrow \cos x = A(-p\sin x + q\cos x) + B(p\cos x + q\sin x) + C \{\because \frac{d}{dx}\cos x = -\sin x\}$

$\Rightarrow \cos x = \sin x\,(Bq - Ap) + \cos x\,(Bp + Aq) + C$

Comparing both sides we have:

C = 0

B p + A q = 1

B q − A p = 0

On solving above equations, we have:

A = $\frac{q}{p^2+q^2}$ B = $\frac{p}{p^2+q^2}$ and C = 0

Thus I can be expressed as:

I = $\int \frac{\frac{q}{p^2+q^2}(-p\sin x+q\cos x)+\frac{p}{p^2+q^2}(p\cos x+q\sin x)}{(p\cos x+q\sin x)}dx$

I = $\int \frac{\frac{q}{p^2+q^2}(-p\sin x+q\cos x)}{(p\cos x+q\sin x)}dx + \int \frac{\frac{p}{p^2+q^2}(p\cos x+q\sin x)}{(p\cos x+q\sin x)}dx$

∴ Let $I_1 = \frac{q}{p^2+q^2}\int \frac{(-p\sin x+q\cos x)}{(p\cos x+q\sin x)}dx$ and $I_2 = \frac{p}{p^2+q^2}\int \frac{(p\cos x+q\sin x)}{(p\cos x+q\sin x)}dx$

⇒ $I = I_1 + I_2$ ....equation 1

$I_1 = \frac{q}{p^2+q^2}\int \frac{(-p\sin x+q\cos x)}{(p\cos x+q\sin x)}dx$

Let, u = p cos x + q sin x ⇒ du = (-p sin x + q cos x) dx

So, I_1 reduces to:

$I_1 = \frac{q}{p^2+q^2}\int \frac{du}{u} = \frac{q}{p^2+q^2}\log|u| + C_1$

$\therefore I_1 = \frac{q}{p^2+q^2}\log|(p\cos x + q\sin x)| + C_1$ ..... Equation 2

As, $I_2 = \frac{p}{p^2+q^2}\int \frac{(p\cos x+q\sin x)}{(p\cos x+q\sin x)}dx = \frac{p}{p^2+q^2}\int dx$

$\therefore I_2 = \frac{px}{p^2+q^2} + C_2$ ..... Equation 3

From equation 1, 2 and 3 we have:

$I = \frac{q}{p^2+q^2}\log|(p\cos x + q\sin x)| + C_1 + \frac{px}{p^2+q^2} + C_2$

$\therefore I = \frac{q}{p^2+q^2}\log|(p\cos x + q\sin x)| + \frac{px}{p^2+q^2} + C$

5. $\int \frac{5\cos x + 6}{2\cos x + \sin x + 3}dx$

Solution:

Let, $I = \int \frac{5\cos x+6}{2\cos x+\sin x+3}dx$

To solve such integrals involving trigonometric terms in numerator and denominators. If I has the form $\int \frac{a\sin x+b\cos x+c}{d\sin x+e\cos x+f}dx$

Then substitute numerator as

$$a\sin x + b\cos x + c = A\frac{d}{dx}(d\sin x + e\cos x + f) + B(d\sin x + e\cos x + f) + C$$

Where A, B and C are constants

We have, $I = \int \frac{5\cos x+6}{2\cos x+\sin x+3}dx$

As I matches with the form described above, so we will take the steps as described.

$\therefore 5\cos x + 6 = A\frac{d}{dx}(2\cos x + \sin x + 3) + B(2\cos x + \sin x + 3) + C$

$\Rightarrow 5\cos x + 6 = A(-2\sin x + \cos x) + B(2\cos x + \sin x + 3) + C$

$\because \frac{d}{dx}\cos x = -\sin x \}$

$\Rightarrow 5\cos x + 6 = \sin x\,(B - 2A) + \cos x\,(2B + A) + 3B + C$

Comparing both sides we have:

3B+ C = 6

2B + A = 5

B - 2A = 0

On solving for A, B and C we have:

A = 1, B = 2 and C = 0

Thus I can be expressed as:

$I = \int \frac{(-2\sin x + \cos x) + 2(2\cos x + \sin x + 3)}{2\cos x + \sin x + 3} dx$

$I = \int \frac{(-2\sin x + \cos x)}{2\cos x + \sin x + 3} dx + \int \frac{2(2\cos x + \sin x + 3)}{2\cos x + \sin x + 3} dx$

$\therefore$ Let $I_1 = \int \frac{(-2\sin x + \cos x)}{2\cos x + \sin x + 3} dx$ and $I_2 = \int \frac{2(2\cos x + \sin x + 3)}{2\cos x + \sin x + 3} dx$

$\Rightarrow I = I_1 + I_2$ ....equation 1

$I_1 = \int \frac{(-2\sin x + \cos x)}{2\cos x + \sin x + 3} dx$

Let, 2 cos x + sin x + 3 = u

$\Rightarrow$ (-2sin x + cos x) dx = du

So, I_1 reduces to:

$I_1 = \int \frac{du}{u} = \log|u| + C_1$

$\therefore I_1 = \log|2\cos x + \sin x + 3| + C_1$ ..... Equation 2

As, $I_2 = \int \frac{2(2\cos x + \sin x + 3)}{2\cos x + \sin x + 3} dx$

$\Rightarrow I_2 = 2\int dx = 2x + C_2$ ..... Equation 3

From equation 1, 2 and 3 we have:

$I = \log|2\cos x + \sin x + 3| + C_1 + 2x + C_2$

$\therefore I = \log|2\cos x + \sin x + 3| + 2x + C$