EXERCISE 19.31 PAGE NO: 19.190

Evaluate the following integrals:

1. $\int \frac{x^2+1}{x^4+x^2+1}\,dx$

Solution:

The given equation can be written as,

$$\int \frac{1+\frac{1}{x^2}}{x^2+1+\frac{1}{x^2}}\,dx$$

$$= \int \frac{1+\frac{1}{x^2}}{\left(x-\frac{1}{x}\right)^2+3}\,dx$$

Let $x-\frac{1}{x} = t$

Then, $\left(1+\frac{1}{x^2}\right)dx = dt$

$$= \int \frac{1}{t^2+3}\,dt$$

Using standard identity we get

$$= \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{t}{\sqrt{3}}\right)+c$$

Substituting t as $x-\frac{1}{x}$, we get

$$= \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{\left(x-\frac{1}{x}\right)}{\sqrt{3}}\right)+c$$

$$= \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x^2-1}{\sqrt{3}x}\right)+c$$

2. $\int \sqrt{\cot \theta}\, d\theta$

Solution:

Let cot θ = x^2

$$-\text{cosec}^2\theta d\theta = 2x dx$$

$$d\theta = -\frac{2x}{1+\cot^2\theta}dx$$

$$d\theta = -\frac{2x}{1+x^4}dx$$

$$\int -\frac{2x^2}{1+x^4}dx$$

Re-writing the given equation as

$$-\int \frac{1+\frac{1}{x^2}+1-\frac{1}{x^2}}{\frac{1}{x^2}+x^2}dx$$

$$-\int \frac{1+\frac{1}{x^2}}{\left(x-\frac{1}{x}\right)^2+2}dx - \int \frac{1-\frac{1}{x^2}}{\left(x+\frac{1}{x}\right)^2-2}dx$$

Let $x-\frac{1}{x}=t$ and $x+\frac{1}{x}=z$

So $\left(1+\frac{1}{x^2}\right)dx = dt$ and $\left(1-\frac{1}{x^2}\right)dx = dz$

$$-\int \frac{dt}{(t)^2+2} - \int \frac{dz}{(z)^2-2}$$

Using identity $\int \frac{1}{x^2+a^2}dx = 1/a \tan^{-1}(x)$ and $\int \frac{dz}{(z)^2-1} = \frac{1}{2}\log\left|\frac{z-1}{z+1}\right| + c$

$$-\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{t}{\sqrt{2}}\right) - \frac{1}{2\sqrt{2}}\log\left|\frac{z-\sqrt{2}}{z+\sqrt{2}}\right| + c$$

Now, substituting t as x – 1/x and z as x + 1/x we have

$$= -\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x^2-1}{\sqrt{2}x}\right) - \frac{1}{2\sqrt{2}}\log\left|\frac{x^2+1-\sqrt{2}x}{x^2+1+\sqrt{2}x}\right| + c$$

Lastly, substituting x^2 as $\cot\theta$ we get

$$I = -\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\cot\theta-1}{\sqrt{2\cot\theta}}\right) - \frac{1}{2\sqrt{2}}\log\left|\frac{\cot\theta+1-\sqrt{2\cot\theta}}{\cot\theta+1-\sqrt{2\cot\theta}}\right| + c$$

3. $\int \frac{x^2+9}{x^4+81}\,dx$

Solution:

$$\int \frac{1+\frac{9}{x^2}}{x^2+\frac{81}{x^2}}\,dx$$

$$= \int \frac{1+\frac{9}{x^2}}{\left(x-\frac{9}{x}\right)^2+18}\,dx \quad \text{(By completing the square)}$$

Let $x-\frac{9}{x} = t$

$$\left(1+\frac{9}{x^2}\right)dx = dt$$

$$= \int \frac{dt}{t^2+18}$$

Using identity $\int \frac{1}{x^2+1}\,dx = \tan^{-1}(x)$

$$= \frac{1}{3\sqrt{2}}\tan^{-1}\left(\frac{t}{3\sqrt{2}}\right) + c$$

Substituting t as $x-\frac{1}{x}$

$$= \frac{1}{3\sqrt{2}}\tan^{-1}\left(\frac{x-\frac{1}{x}}{3\sqrt{2}}\right) + c$$

4. $\int \frac{1}{x^4 + x^2 + 1} dx$

Solution:

$$\int \frac{\frac{1}{x^2}}{x^2 + 1 + \frac{1}{x^2}} dx$$

$$= \frac{1}{2} \int \frac{1 + \frac{1}{x^2} + \frac{1}{x^2} - 1}{x^2 + 1 + \frac{1}{x^2}} dx$$ (Manipulating the numerator by multiplying and diving by 2)

$$= \frac{1}{2}\left[\int \frac{1 + \frac{1}{x^2}}{x^2 + 1 + \frac{1}{x^2}} dx + \int \frac{-1 + \frac{1}{x^2}}{x^2 + 1 + \frac{1}{x^2}} dx\right]$$

$$= \frac{1}{2}\left[\int \frac{1 + \frac{1}{x^2}}{\left(x - \frac{1}{x}\right)^2 + 3} dx + \int \frac{-1 + \frac{1}{x^2}}{\left(x + \frac{1}{x}\right)^2 - 1} dx\right]$$

Let $x - \frac{1}{x} = t$ and $x + \frac{1}{x} = z$

Then, $\left(1 + \frac{1}{x^2}\right) dx = dt$ and $\left(1 - \frac{1}{x^2}\right) dx = dz$

$$= \frac{1}{2}\left[\int \frac{dt}{(t)^2 + 3} - \int \frac{dz}{(z)^2 - 1}\right]$$

Using identity $\int \frac{1}{x^2+1} dx = \tan^{-1}(x)$ and $\int \frac{dz}{(z)^2-1} = \frac{1}{2}\log\left|\frac{z-1}{z+1}\right| + c$

$$= \frac{1}{2}\left[\frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{t}{\sqrt{3}}\right) - \frac{1}{2}\log\left|\frac{z-1}{z+1}\right|\right]$$

Substituting t as $x - \frac{1}{x}$ and z as $x + \frac{1}{x}$

We get,

$$= \frac{1}{2}\left[\frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{x - \frac{1}{x}}{\sqrt{3}}\right) - \frac{1}{2}\log\left|\frac{x + \frac{1}{x} - 1}{x + \frac{1}{x} + 1}\right|\right] + c$$

$$I = \frac{1}{2\sqrt{3}} \tan^{-1}\left(\frac{x^2 - 1}{\sqrt{3}x}\right) - \frac{1}{4}\log\left|\frac{x^2 + 1 - x}{x^2 + 1 + x}\right| + c$$

5. $\int \frac{x^2 - 3x + 1}{x^4 + x^2 + 1} dx$

Solution:

The given equation can be written as

$$\int \frac{1 - \frac{3}{x} + \frac{1}{x^2}}{x^2 + 1 + \frac{1}{x^2}} dx$$

$$\int \frac{1 + \frac{1}{x^2}}{\left(x - \frac{1}{x}\right)^2 + 3} dx - \int \frac{3x}{x^4 + x^2 + 1} dx$$

Substituting t as $x - \frac{1}{x}$ and z as x^2

$\left(1 + \frac{1}{x^2}\right) dx = dt$ And $2x dx = dz$

$$\int \frac{dt}{(t)^2 + 3} - \frac{3}{2}\int \frac{dz}{z^2 + z + 1}$$

$$\int \frac{dt}{(t)^2 + 3} - \frac{3}{2}\int \frac{dz}{\left(z + \frac{1}{2}\right)^2 + \frac{3}{4}}$$

Using identity $\int \frac{1}{x^2+1} dx = \tan^{-1}(x)$

$$\frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{t}{\sqrt{3}}\right) - \sqrt{3} \tan^{-1}\left(\frac{2z + 1}{\sqrt{3}}\right) + c$$

Substituting t as $x - \frac{1}{x}$ and z as x^2

$$= \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{x - \frac{1}{x}}{\sqrt{3}}\right) - \sqrt{3} \tan^{-1}\left(\frac{2x^2 + 1}{\sqrt{3}}\right) + c$$

Hence,

$$I = \frac{1}{\sqrt{3}} tan^{-1}\left(\frac{x^2 - 1}{\sqrt{3}x}\right) - \sqrt{3}\ tan^{-1}\left(\frac{2x^2 + 1}{\sqrt{3}}\right) + c$$