EXERCISE 19.1

1. Evaluate the following integrals:

(i) $\int x^4 dx$

Solution:

Given

$$\int x^4 dx$$

Now we have integrate the given function

$$= \frac{x^{4+1}}{4+1} + C$$

$$= \frac{x^5}{5} + C$$

(ii) $\int x^{5/4} dx$

Solution:

Given

$$\int x^{5/4} dx$$

Now we have to integrate the given function

$$= \frac{x^{\frac{5}{4}+1}}{\frac{5}{4}+1} + C$$

On simplifying, we get

$$= \frac{4}{9} x^{\frac{9}{4}} + C$$

(iii) $\int \frac{1}{x^5} dx$

Solution:

Given

$$\int \frac{1}{x^5} dx$$

We can write given question as

$$\int x^{-5} dx$$

Now by integrating, we get

$$= \frac{x^{-5+1}}{-5+1} + C$$

$$= -\frac{1}{4} x^{-4} + C$$

On simplifying we get

$$= -\frac{1}{4x^4} + C$$

(iv) $\int \frac{1}{x^{\frac{3}{2}}} dx$

Solution:

Given

$$\int \frac{1}{x^{\frac{3}{2}}} dx$$

Given equation can be written as

$$\int x^{\frac{-3}{2}} dx$$

Now by integrating the above equation we get

$$= \left[\frac{x^{-\frac{3}{2}+1}}{\frac{-3}{2}+1}\right] + C$$

$$= \left[\frac{x^{-\frac{1}{2}}}{-\frac{1}{2}}\right] + C$$

On simplifying we get

$$= -\frac{2}{\sqrt{x}} + C$$

(v) $\int 3^x dx$

Solution:
Given

$$\int 3^x dx$$

We know that

$$\int a^x dx = \frac{a^x}{\log_e a} + c$$

Now by integrating the given equation by using above integration formulae we get

$$\int 3^x dx = \frac{3^x}{\log 3} + c$$

(vi) $\int \frac{1}{\sqrt[3]{x^2}} dx$

Solution:
Given

$$\int \frac{1}{\sqrt[3]{x^2}} dx$$

now above equation can be written as

$$= \int \frac{dx}{x^{2/3}}$$

$$= \int x^{-2/3} dx$$

Now by integrating the above equation we get

$$= \frac{x^{-\frac{2}{3}+1}}{-\frac{2}{3}+1} + C$$

On simplifying

$$= 3x^{\frac{1}{3}} + C$$

(vii) $\int 3^{2\log_3 x} dx$

Solution:
Given

$$\int 3^{2\log_3 x} dx$$

Given equation can be written as

$$= \int 3\log_3 x^2 dx$$

On simplifying we get

$$= \int x^2 dx$$

Now by integrating the above equation we get

$$= \frac{x^3}{3} + C$$

(viii) $\int \log_x x \, dx$

Solution:

Given

$$\int \log_x x \, dx$$

Given equation can be written as

$$= \int 1 \cdot dx$$

By integrating we get

$$= x + C$$

2. Evaluate:

(i) $\int \sqrt{\frac{1+\cos 2x}{2}} \, dx$

Solution:

Given

$$\int \sqrt{\frac{1+\cos 2x}{2}} \, dx$$

Given equation can be written as

$$\int \sqrt{\frac{2\cos^2 x}{2}} dx \; [\because 1 + \cos 2A = 2\cos^2 A]$$

On simplifying, we get

$$= \int \cos\ x\, dx$$

On integrating

$$= \sin x + C$$

(ii) $\int \sqrt{\frac{1-\cos 2x}{2}}\, dx$

Solution:

Given

$$\int \sqrt{\frac{1-\cos 2x}{2}}\, dx$$

Given equation can be written as

$$= \int \sqrt{\frac{2\sin^2 x}{2}} dx \quad [\therefore 1-\cos 2x = 2\sin^2 x]$$

On simplifying we get

$$= \int \sin x\, dx$$

On integrating

$$= -\cos x + C$$

3. Evaluate:

$$\int \frac{e^{6\log_e x} - e^{5\log_e x}}{e^{4\log_e x} - e^{3\log_e x}} dx$$

Solution:

Given

$$\int \frac{e^{6\log_e x} - e^{5\log_e x}}{e^{4\log_e x} - e^{3\log_e x}} dx$$

$$= \int\left(\frac{e^{\log x^6} - e^{\log x^5}}{e^{\log x^4} - e^{\log x^3}}\right) dx$$

Above equation can be written as

$$= \int \left(\frac{x^6 - x^5}{x^4 - x^3}\right) dx$$

$$= \int \frac{x^5}{x^3} dx$$

$$= \int x^2 dx$$

Now by integrating we get

$$= \frac{x^3}{3} + C$$

EXERCISE 19.2 PAGE NO: 19.14

Evaluate the following integrals (1 - 44):

1. $\int (3x\sqrt{x} + 4\sqrt{x} + 5)\, dx$

Solution:

Given

$$\int (3x\sqrt{x} + 4\sqrt{x} + 5)\, dx$$

By Splitting, we get,

$$\Rightarrow \int ((3x\sqrt{x})dx + (4\sqrt{x})dx + 5dx)$$

$$\Rightarrow \int 3x\sqrt{x}dx + \int 4\sqrt{x}dx + \int 5dx$$

$$\Rightarrow \int 3x^{\frac{3}{2}}\, dx + \int 4x^{(\frac{1}{2})}\, dx + \int 5dx$$

By using the formula, $\int x^n\, dx = \frac{x^{n+1}}{n+1}$

$$\Rightarrow \frac{3x^{\frac{3}{2}+1}}{\frac{3}{2}+1} + \frac{4x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + \int 5dx$$

We know that

$$\int kdx = kx + c$$

$$\Rightarrow \frac{3x^{\frac{5}{2}}}{5/2} + \frac{4x^{\frac{3}{2}}}{3/2} + 5x + c$$

$$\Rightarrow \frac{6}{5}x^{\frac{5}{2}} + \frac{8}{3}x^{3/2} + 5x + c$$

2. $\int (2^x + \frac{5}{x} - \frac{1}{x^{\frac{1}{3}}})\, dx$

Solution:

Given

$$\int \left(2^x + \frac{5}{x} - \frac{1}{x^{1/3}}\right) dx$$

By splitting given equation we get,

$$\Rightarrow \int 2^x dx + \int \left(\frac{5}{x}\right) dx - \int \frac{1}{x^{1/3}} dx$$

By using the formula,

$$\int a^x\, dx = \frac{a^x}{\log a}$$

$$\Rightarrow \frac{2^x}{\log 2} + 5\int \left(\frac{1}{x}\right) dx - \int x^{-1/3} dx$$

Again by using the formula,

$$\int \left(\frac{1}{x}\right) dx = \log x$$

$$\Rightarrow \frac{2^x}{\log 2} + 5\log x - \int x^{-1/3}\, dx$$

By using the below formula, we get

$$\int x^n\, dx = \frac{x^{n+1}}{n+1}$$

$$\Rightarrow \frac{2^x}{\log 2} + 5\log x - \frac{x^{-\frac{1}{3}+1}}{-\frac{1}{3}+1}$$

$$\Rightarrow \frac{2^x}{\log 2} + 5\log x - \frac{x^{\frac{2}{3}}}{2/3}$$

On simplifying we get

$$\Rightarrow \frac{2^x}{\log 2} + 5\log x - \frac{3}{2} x^{2/3} + c$$

3. $\int \{\sqrt{x}(ax^2 + bx + c)\}\ dx$

Solution:

Given

$$\int \{\sqrt{x}(ax^2 + bx + c)\}dx$$

Now by multiplying we get

$$\Rightarrow \int (\sqrt{x}ax^2 + \sqrt{x}bx + \sqrt{x}c)\, dx$$

By splitting the given equation, we get,

$$\Rightarrow a\int x^2 \times x^{\frac{1}{2}}dx + b\int x^1 \times x^{\frac{1}{2}}dx + c\int x^{1/2}dx$$

$$\Rightarrow a\int x^{\frac{5}{2}}dx + b\int x^{\frac{3}{2}}dx + c\int x^{\frac{1}{2}}dx$$

By using the formula

$$\int x^n\, dx = \frac{x^{n+1}}{n+1}$$

We get

$$\Rightarrow \frac{ax^{\frac{5}{2}+1}}{\frac{5}{2}+1} + \frac{bx^{\frac{3}{2}+1}}{\frac{3}{2}+1} + \frac{cx^{\frac{1}{2}+1}}{\frac{1}{2}+1} + c$$

On simplifying

$$\Rightarrow \frac{ax^{\frac{7}{2}}}{7/2} + \frac{bx^{\frac{5}{2}}}{5/2} + \frac{cx^{\frac{3}{2}}}{3/2} + c$$

4. $\int (2 - 3x)(3 + 2x)(1 - 2x)\ dx$

Solution:

Given,

$\int(2 - 3x)(3 + 2x)(1 - 2x)\, dx$

$= \int(6 + 4x - 9x - 6x^2)(1 - 2x)\, dx$

$= \int(6 - 5x - 6x^2)(1 - 2x)\, dx$

$= \int(6 - 5x - 6x^2 - 12x + 10x^2 + 12x^3)\, dx$

$= \int(6 - 17x + 4x^2 + 12x^3)\, dx$

Upon splitting the above, we have

$= \int 6\, dx - \int 17x\, dx + \int 4x^2\, dx + \int 12x^3\, dx$

On integrating using formula,

$\int x^n\, dx = x^{n+1}/n+1$

we get

$= 6x - 17/(1+1)\, x^{1+1} + 4/(2+1)\, x^{2+1} + 12/(3+1)\, x^{3+1} + c$

$= 6x - 17x^2/2 + 4x^3/3 + 3x^4 + c$

5. $\int\left(\frac{m}{x}+\frac{x}{m}+m^x+x^m+mx\right)dx$

Solution:

Given

$$\int\left(\frac{m}{x}+\frac{x}{m}+m^x+x^m+mx\right)dx$$

By Splitting, we get,

$$\Rightarrow \int \frac{m}{x}dx + \int \frac{x}{m}dx + \int x^m dx + \int m^x dx + \int mxdx$$

We have

$$\int \frac{1}{x}dx = \log x + c$$

By applying the above formula, we get

$$\Rightarrow m\log x + \frac{1}{m}\int xdx + \int x^m dx + \int m^x dx + \int mxdx$$

By using this, we have

$$\int x^n\,dx = \frac{x^{n+1}}{n+1}$$

$$\Rightarrow m\log x + \frac{\frac{1}{m}x^{1+1}}{1+1} + \frac{x^{m+1}}{m+1} + \int m^x dx + \frac{mx^{1+1}}{1+1}$$

By using the formula,

$$\int a^x\,dx = \frac{a^x}{\log a}$$

$$\Rightarrow m\log x + \frac{\frac{1}{m}x^2}{2} + \frac{x^{m+1}}{m+1} + \frac{m^x}{\log m} + \frac{mx^2}{2} + c$$

6. $\int \left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^2 dx$

Solution:

$$\left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^2 dx$$

By applying $(a - b)^2 = a^2 - 2ab + b^2$ we get

$$\Rightarrow \int \left((\sqrt{x})^2 + \left(\frac{1}{\sqrt{x}}\right)^2 - 2(\sqrt{x})\left(\frac{1}{\sqrt{x}}\right)\right)dx$$

After computing or simplifying, we get

$$\Rightarrow \int \left(x + \frac{1}{x} - 2\right) dx$$

By splitting the above equation, we get,

$$\Rightarrow \int x dx + \int \frac{1}{x} dx - 2 \int 1\, dx$$

Now integrate by using standard integration formulae, we get

$$\Rightarrow \frac{x^{1+1}}{1+1} + \log x - 2x + c$$

$= ½ x^2 + \log |x| - 2x + c$

7. $\int \frac{(1+x)^3}{\sqrt{x}} dx$

Solution:

Given

$$\int \frac{(1+x)^3}{\sqrt{x}} dx$$

Now by applying this formula $(a + b)^3 = a^3 + b^3 + 3ab^2 + 3a^2b$ we get

$$\Rightarrow \int \frac{1+x^3+3x^2\times1+3\times1^2\times x}{\sqrt{x}} dx$$

$$\Rightarrow \int \frac{1+x^3+3x^2+3x}{\sqrt{x}} dx$$

By splitting the above equation, we get,

$$\Rightarrow \int \frac{1}{\sqrt{x}} dx + \int \frac{x^3}{\sqrt{x}} dx + \int \frac{3x^2}{\sqrt{x}} dx + \int \frac{3x}{\sqrt{x}} dx$$

$$\Rightarrow \int x^{-\frac{1}{2}} dx + \int x^3 \times x^{-\frac{1}{2}} dx + \int 3x^2 \times x^{-\frac{1}{2}} dx + \int 3x \times x^{-\frac{1}{2}} dx$$

$$\Rightarrow \int x^{-\frac{1}{2}} dx + \int x^{\frac{5}{2}} dx + 3 \int x^{\frac{3}{2}} dx + 3 \int x^{\frac{1}{2}} dx$$

Again we have formula,

$$\int x^n\,dx = \frac{x^{n+1}}{n+1}$$

By applying the above formula we get

$$\Rightarrow \frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + \frac{x^{\frac{5}{2}+1}}{\frac{5}{2}+1} + 3\frac{x^{\frac{3}{2}+1}}{\frac{3}{2}+1} + \frac{3x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + c$$

$$\Rightarrow \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + \frac{x^{\frac{7}{2}}}{\frac{7}{2}} + \frac{3x^{\frac{5}{2}}}{\frac{5}{2}} + \frac{3x^{\frac{3}{2}}}{\frac{3}{2}} + c$$

$$= 2\sqrt{x} + \frac{2}{7}x^{\frac{7}{2}} + 2x^{\frac{3}{2}} + \frac{6}{5}x^{\frac{5}{2}} + C$$

8. $\int \left\{x^2 + e^{\log x} + \left(\frac{e}{2}\right)^x\right\} dx$

Solution:

Given

$$\int \left\{x^2 + e^{\log x} + \left(\frac{e}{2}\right)^x\right\} dx$$

By splitting the above equation, we get,

$$\Rightarrow \int x^2\,dx + \int e^{\log x}dx + \int \left(\frac{e}{2}\right)^x dx$$

By applying formula,

$$\int x^n\,dx = \frac{x^{n+1}}{n+1}$$

We get

$$\Rightarrow \frac{x^{2+1}}{2+1} + \int e^{\log_e x}dx + \int \left(\frac{e}{2}\right)^x dx$$

$$\Rightarrow \frac{x^3}{3} + \int x\ dx + \frac{1}{\log(\frac{e}{2})}\left(\frac{e}{2}\right)^x$$

$$\Rightarrow \frac{x^3}{3} + \int xdx + \frac{1}{\log(\frac{e}{2})}\left(\frac{e}{2}\right)^x$$

Integrating and simplifying we get

$$\Rightarrow \frac{x^3}{3} + \frac{x^2}{2} + \frac{\left(\frac{e}{2}\right)^x}{\log\left(\frac{e}{2}\right)} + c$$

9. $\int (x^e + e^x + e^e)\, dx$

Solution:

$$\int (x^e + e^x + e^e)dx$$

By splitting the above equation, we get,

$$\Rightarrow \int x^e dx + \int e^x dx + \int e^e dx$$

By using the below formula,

$$\int x^n\, dx = \frac{x^{n+1}}{n+1}$$

We can write as

$$\Rightarrow \frac{x^{e+1}}{e+1} + \int e^x dx + \int e^e dx$$

Again by applying the formula,

$$\int a^x\, dx = \frac{a^x}{\log a}$$

We get

$$\Rightarrow \frac{x^{e+1}}{e+1} + \frac{e^x}{\log_e e} + \int e^e dx$$

We know that,

$$\int k dx = kx + c$$

So substituting this we have

$$\Rightarrow \frac{x^{e+1}}{e+1} + \frac{e^x}{\log_e e} + e^e x + c$$

10. $\int \sqrt{x}\left(x^3 - \frac{2}{x}\right) dx$

Solution:

Given

$$\int \sqrt{x}\left(x^3 - \frac{2}{x}\right) dx$$

Multiplying throughout the bracket, we get,

$$\Rightarrow \int (x^{\frac{1}{2}} \times x^3 - x^{\frac{1}{2}} \times \frac{2}{x}) dx$$

$$\Rightarrow \int (x^{\frac{1}{2}+3} - x^{\frac{1}{2}-1} \times 2) dx$$

Again by simplifying

$$\Rightarrow \int (x^{\frac{7}{2}} - 2x^{-\frac{1}{2}}) dx$$

By multiplying,

$$\Rightarrow \int x^{\frac{7}{2}} dx - 2\int x^{-\frac{1}{2}} dx$$

We have

$$\int x^n dx = \frac{x^{n+1}}{n+1}$$

Applying the above formula, we get

$$\Rightarrow \frac{x^{\frac{7}{2}+1}}{\frac{7}{2}+1} - 2\frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + c$$

$$\Rightarrow \frac{x^{\frac{9}{2}}}{\frac{9}{2}} - 2\frac{x^{\frac{1}{2}}}{\frac{1}{2}} + c$$

$$\Rightarrow \frac{2x^{\frac{9}{2}}}{9} - 4x^{\frac{1}{2}} + c$$

11. $\int \frac{1}{\sqrt{x}}\left(1+\frac{1}{x}\right) dx$

Solution:

Given

$$\int \frac{1}{\sqrt{x}}\left\{1+\frac{1}{x}\right\} dx$$

By multiplying $\frac{1}{\sqrt{x}}$ throughout the brackets,

$$\Rightarrow \int \left\{\frac{1}{\sqrt{x}} + \frac{1}{\sqrt{x}} \times \frac{1}{x}\right\} dx$$

The above equation can be written as

$$\Rightarrow \int \left\{\frac{1}{x^{\frac{1}{2}}} + \frac{1}{x^{\frac{1}{2}}} \times \frac{1}{x}\right\} dx$$

$$\Rightarrow \int \left\{\frac{1}{x^{\frac{1}{2}}} + \frac{1}{x^{\frac{1}{2}+1}}\right\} dx$$

$$\Rightarrow \int \left\{\frac{1}{x^{\frac{1}{2}}} + \frac{1}{x^{\frac{3}{2}}}\right\} dx$$

By splitting them, we get,

$$\Rightarrow \int x^{-\frac{1}{2}} dx + \int x^{-\frac{3}{2}} dx$$

We have

$$\int x^n \, dx = \frac{x^{n+1}}{n+1}$$

By applying the above formula and integrating, we get

$$\Rightarrow \frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + \frac{x^{-\frac{3}{2}+1}}{-\frac{3}{2}+1} + c$$

$$\Rightarrow \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + \frac{x^{-\frac{1}{2}}}{-\frac{1}{2}} + c$$

$$\Rightarrow 2x^{\frac{1}{2}} - 2x^{-\frac{1}{2}} + c$$

14. $\int \frac{(1+\sqrt{x})^2}{\sqrt{x}} dx$

Solution:

$$\int \frac{\left(1+\sqrt{x}\right)^2}{\sqrt{x}} dx$$

By applying $(a + b)^2 = a^2 + b^2 + 2ab$ we get

$$\Rightarrow \int \frac{(1)^2 + \left(\sqrt{x}\right)^2 + 2 \times 1 \times \sqrt{x}}{\sqrt{x}} dx$$

$$\Rightarrow \int \frac{1 + x + 2\sqrt{x}}{\sqrt{x}} dx$$

By splitting the above equation, we get,

$$\Rightarrow \int \left(\frac{1}{\sqrt{x}} + \frac{x}{\sqrt{x}} + \frac{2\sqrt{x}}{\sqrt{x}}\right) dx$$

$$\Rightarrow \int x^{-\frac{1}{2}}dx + \int x \times x^{-\frac{1}{2}}dx + 2\int dx$$

On simplifying and integrating

$$\Rightarrow \frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + \int x^{1-\frac{1}{2}}dx + 2x + c$$

$$\Rightarrow \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + \int x^{\frac{1}{2}}dx + 2x + c$$

Now by integrating, we get

$$\Rightarrow 2x^{\frac{1}{2}} + \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + 2x + c$$

$$\Rightarrow 2x^{\frac{1}{2}} + \frac{2x^{\frac{3}{2}}}{3} + 2x + c$$

15. $\int \sqrt{x}(3-5x)\,dx$

Solution:
Given

$$\int \sqrt{x}(3-5x)dx$$

By multiplying √x throughout the bracket we get,

$$\Rightarrow \int (3\sqrt{x} - 5x\sqrt{x})dx$$

$$\Rightarrow \int \left(3x^{\frac{1}{2}} - 5x^1 \times x^{\frac{1}{2}}\right)dx$$

$$\Rightarrow \int (3x^{\frac{1}{2}} - 5x^{1+\frac{1}{2}})dx$$

$$\Rightarrow \int (3x^{\frac{1}{2}} - 5x^{\frac{3}{2}})dx$$

By splitting the above equation, we get,

$$\Rightarrow 3\int x^{\frac{1}{2}} dx - 5\int x^{\frac{3}{2}} dx$$

By using the formula and integrating

$$\int x^n \, dx = \frac{x^{n+1}}{n+1}$$

$$\Rightarrow \frac{3x^{\frac{1}{2}+1}}{\frac{1}{2}+1} - \frac{5x^{\frac{3}{2}+1}}{\frac{3}{2}+1} + c$$

$$\Rightarrow \frac{3x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{5x^{\frac{5}{2}}}{\frac{5}{2}} + c$$

$$\Rightarrow 2x^{\frac{3}{2}} - 2x^{\frac{5}{2}} + c$$

16. $\int \frac{(x+1)(x-2)}{\sqrt{x}} dx$

Solution:

Given

$$\int \frac{(x+1)(x-2)}{\sqrt{x}} dx$$

Multiplying the above equation, we get

$$\Rightarrow \int \frac{x^2 - 2x + x - 2}{\sqrt{x}} dx$$

$$\Rightarrow \int \frac{x^2 - x - 2}{\sqrt{x}} dx$$

By splitting the above equation,

$$\Rightarrow \int \frac{x^2}{\sqrt{x}} dx - \int \frac{x}{\sqrt{x}} dx - \int \frac{2}{\sqrt{x}} dx$$

$$\Rightarrow \int x^2 \times x^{-\frac{1}{2}}dx - \int x \times x^{-\frac{1}{2}}dx - 2\int x^{-\frac{1}{2}}dx$$

$$\Rightarrow \int x^{2-\frac{1}{2}}dx - \int x^{1-\frac{1}{2}}dx - 2\int x^{-\frac{1}{2}}dx$$

$$\Rightarrow \int x^{\frac{3}{2}}dx - \int x^{\frac{1}{2}}dx - 2\int x^{-\frac{1}{2}}dx$$

We have the formula,

$$\int x^n\, dx = \frac{x^{n+1}}{n+1}$$

By applying the above formula we get

$$\Rightarrow \frac{x^{\frac{3}{2}+1}}{\frac{3}{2}+1} - \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} - \frac{2x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + c$$

$$\Rightarrow \frac{x^{\frac{5}{2}}}{\frac{5}{2}} - \frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{2x^{\frac{1}{2}}}{\frac{1}{2}} + c$$

$$\Rightarrow \frac{2}{5}x^{\frac{5}{2}} - \frac{2}{3}x^{\frac{3}{2}} - 4x^{\frac{1}{2}} + c$$

17. $\int \frac{x^5 + x^{-2} + 2}{x^2}\, dx$

Solution:

Given

$$\int \frac{x^5 + x^{-2} + 2}{x^2}dx$$

By splitting the above equation, we get,

$$\Rightarrow \int \left(\frac{x^5}{x^2} + \frac{x^{-2}}{x^2} + \frac{2}{x^2}\right)dx$$

The above equation can be written as

$$\Rightarrow \int (x^5 \times x^{-2} + x^{-2} \times x^{-2} + 2 \times x^{-2})dx$$

On simplifying,

$$\Rightarrow \int (x^{5-2} + x^{-2-2} + 2x^{-2})dx$$

$$\Rightarrow \int (x^3 + x^{-4} + 2x^{-2})dx$$

Again by splitting the above equation, we get,

$$\Rightarrow \int x^3 dx + \int x^{-4} dx + 2\int x^{-2} dx$$

By applying the formula,

$$\int x^n dx = \frac{x^{n+1}}{n+1}$$

Now by integrating by using the formula,

$$\Rightarrow \frac{x^{3+1}}{3+1} + \frac{x^{-4+1}}{-4+1} + \frac{2x^{-2+1}}{-2+1} + c$$

$$\Rightarrow \frac{x^4}{4} + \frac{x^{-3}}{-3} + \frac{2x^{-1}}{-1} + c$$

20. $\int \frac{5x^4 + 12x^3 + 7x^2}{x^2 + x} dx$

Solution:

Given

$$\int \frac{5x^4 + 12x^3 + 7x^2}{x^2 + x} dx$$

Now spilt $12x^3$ into $7x^3$ and $5x^3$

$$\Rightarrow \int \frac{5x^4 + 7x^3 + 5x^3 + 7x^2}{x^2 + x} dx$$

Now common $5x^3$ from two elements $7x^2$ from other two elements,

$$\Rightarrow \int \frac{5x^3(x+1) + 7x^2(x+1)}{x^2+x}dx$$

$$\Rightarrow \frac{\int (5x^3 + 7x^2)(x+1)}{x(x+1)}dx$$

$$\Rightarrow \int (5x^2 + 7x)dx$$

Now splitting the above equation, we get,

$$\Rightarrow \int 5x^2 dx + \int 7x\, dx$$

$$\Rightarrow \frac{5x^{2+1}}{2+1} + \frac{7x^{1+1}}{1+1} + c$$

$$\Rightarrow \frac{5x^3}{3} + \frac{7x^2}{2} + c$$

EXERCISE 19.3 PAGE NO: 19.23

1. $\int (2x-3)^5 + \sqrt{3x+2}\, dx$

Solution:

Let I = $\int (2x-3)^5 + \sqrt{3x+2}$

Then,

I = $\int (2x-3)^5 + (3x+2)^{\frac{1}{2}}$

Now by integrating the above equation, we get

= $\frac{(2x-3)^{5+1}}{2(5+1)} + \frac{(3x+2)^{\frac{1}{2}+1}}{3(\frac{1}{2}+1)}$

= $\frac{(2x-3)^6}{2(6)} + \frac{(3x+2)^{\frac{3}{2}}}{3(\frac{3}{2})}$

= $\frac{(2x-3)^6}{12} + \frac{2(3x+2)^{\frac{3}{2}}}{9}$

Hence, I = $\frac{(2x-3)^6}{12} + \frac{2(3x+2)^{\frac{3}{2}}}{9}$ +C

2. $\int \frac{1}{(7x-5)^3} + \frac{1}{\sqrt{5x-4}}\, dx$

Solution:

Let I = $\int \frac{1}{(7x-5)^3} + \frac{1}{\sqrt{5x-4}} dx$ then,

I= $\int (7x-5)^{-3} + (5x-4)^{-\frac{1}{2}}$

Integrating the above equation, we get

$= \frac{(7x-5)^{-3+1}}{7(-3+1)} + \frac{(5x-4)^{-\frac{1}{2}+1}}{5\left(-\frac{1}{2}+1\right)}$

$= \frac{(7x-5)^{-2}}{-14} + \frac{(5x-4)^{\frac{1}{2}}}{5\left(\frac{1}{2}\right)}$

Hence, I = $-\frac{1}{14}(7x-5)^{-2} + \frac{2}{5}\sqrt{5x-4}$ +C

3. $\int \frac{1}{2-3x} + \frac{1}{\sqrt{3x-2}}\, dx$

Solution:

Let I = $\int \frac{1}{2-3x} + \frac{1}{\sqrt{3x-2}} dx$

I= $\int \frac{1}{2-3x} + \frac{1}{\sqrt{3x-2}} dx$

We know $\int \frac{1}{x} dx = \log|x|$

By applying the above formula we get

$= \frac{\log|2-3x|}{-3} + \frac{2}{3}(3x-2)^{\frac{1}{2}}$

$= -\frac{1}{3}\log|2x-3| + \frac{2}{3}\sqrt{3x-2} + C$

4. $\int \frac{x+3}{(x+1)^4} dx$

Solution:

Let,

I = $\int \frac{x+3}{(x+1)^4} dx$

Splitting the above given equation

$= \int \frac{x+1}{(x+1)^4} dx + \int \frac{2}{(x+1)^4} dx$

$= \int \frac{1}{(x+1)^3} dx + \int \frac{2}{(x+1)^4} dx$

The above equation can be written as

$= \int (x+1)^{-3}\, dx + \int 2(x+1)^{-4}\, dx$

Integrating the above equation we get

$= \frac{[x+1]^{-3+1}}{-3+1} + \frac{2(x+1)^{-4+1}}{-4+1}$

$= \frac{[x+1]^{-2}}{-2} + \frac{2(x+1)^{-3}}{-3}$

Hence, I = $-\frac{1}{2(x+1)^2} - \frac{2}{3(x+1)^3} + C$

5. $\int \frac{1}{\sqrt{x+1} + \sqrt{x}} dx$

Solution:

Let I = $\int \frac{1}{\sqrt{x+1}+\sqrt{x}} dx$

Now multiply with the conjugate, we get

$= \int \frac{1}{\sqrt{x+1}+\sqrt{x}} . \frac{\sqrt{x+1}-\sqrt{x}}{\sqrt{x+1}-\sqrt{x}} dx$

$= \int \frac{\sqrt{x+1}-\sqrt{x}}{x+1-x} dx$

On simplification we get

$= \int \sqrt{x+1} - \sqrt{x}\, dx$

The above equation can be written as

$= \int (x+1)^{\frac{1}{2}} - x^{\frac{1}{2}}$

On integrating we get

$= \frac{(x+1)^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^{\frac{3}{2}}}{\frac{3}{2}}$

Hence I= $\frac{2}{3}(x+1)^{\frac{3}{2}} - \frac{2}{3}(x)^{\frac{3}{2}}$ +C

6. $\int \frac{1}{\sqrt{2x+3}+\sqrt{2x-3}} dx$

Solution:

Let I = $\int \frac{1}{\sqrt{2x+3}+\sqrt{2x-3}} dx$

Now, multiply with the conjugate, we get

$= \int \frac{1}{\sqrt{2x+3}+\sqrt{2x-3}} \times \frac{(\sqrt{2x+3}-\sqrt{2x-3})}{\sqrt{2x+3}-\sqrt{2x-3}} dx$

$= \int \frac{(\sqrt{2x+3}-\sqrt{2x-3})}{(\sqrt{2x+3})^2-(\sqrt{2x-3})^2} dx$

$= \int \frac{(\sqrt{2x+3}-\sqrt{2x-3})}{2x+3-2x+3} dx$

Om simplifying or computing we get

$= \int \frac{\sqrt{2x+3}}{6} dx - \int \frac{\sqrt{2x-3}}{6} dx$

Taking 1/6 as common

$= \frac{1}{6}\int (2x+3)^{\frac{1}{2}} dx - \frac{1}{6}\int (2x-3)^{\frac{1}{2}} dx$

On integrating we get

$= \frac{1}{6}\left(\frac{2x+3}{2}\right)^{\frac{1}{2}+1} - \frac{1}{6}\left[\frac{2x-3}{2}\right]^{\frac{1}{2}+1}$

$= \frac{1}{6}\left(\frac{2x+3}{2\times\frac{3}{2}}\right)^{\frac{3}{2}} - \frac{1}{6}\left(\frac{2x-3}{2\times\frac{3}{2}}\right)^{\frac{3}{2}}$

Hence, I= $\frac{1}{18}(2x+3)^{\frac{3}{2}} - \frac{1}{18}(2x-3)^{\frac{3}{2}} + C$

7. $\int \frac{2x}{(2x+1)^2}\,dx$

Solution:

Let I = $\int \frac{2x}{(2x+1)^2}\,dx$

Now by splitting the above equation we get

$= \int \frac{2x+1}{(2x+1)^2} - \frac{1}{(2x+1)^2}\,dx$

The above equation can be written as

$= \int \frac{1}{(2x+1)} - (2x+1)^{-2}\,dx$

On integrating we get

$= \frac{1}{2}\log|2x+1| - \frac{(2x+1)^{-2+1}}{-2+1(2)}$

$= \frac{1}{2}\log|2x+1| - \frac{(2x+1)^{-1}}{-2}$

Hence, I= $\frac{1}{2}\log|2x+1| + \frac{1}{2(2x+1)}$+C

8. $\int \frac{1}{\sqrt{x+a}+\sqrt{x+b}}\,dx$

Solution:

Let I = $\int \frac{1}{\sqrt{x+a}+\sqrt{x+b}}\,dx$

Now, multiply with conjugate, we get

$= \int \frac{1}{\sqrt{x+a}+\sqrt{x+b}} \times \frac{(\sqrt{x+a}-\sqrt{x+b})}{\sqrt{x+a}-\sqrt{(x+b)}} dx$

$= \int \frac{(\sqrt{x+a}-\sqrt{x+b})}{(\sqrt{x+a})^2-\sqrt{(x+b)}^2} dx$

On computing, we get

$= \int \frac{(\sqrt{x+a}-\sqrt{x+b})}{a-b} dx$

On integrating the above equation we get

$= \frac{1}{a-b}\left[\frac{2}{3}(x+a)^{\frac{3}{2}} - \frac{2}{3}(x+b)^{\frac{3}{2}}\right]$

Hence, I= $\frac{2}{3(a-b)}\left[(x+a)^{\frac{3}{2}} - (x+b)^{\frac{3}{2}}\right] + C$

9. $\int \sin x \sqrt{1+\cos 2x}\, dx$

Solution:

Let I = $\int \sin x \sqrt{(1+\cos 2x)} dx$

$= \int \sin x \sqrt{(1+\cos 2x)} dx$

By substituting the formula, we get

$= \int \sin x \sqrt{2\cos^2 x} dx$

$= \int \sin x \sqrt{2} \cos x\, dx$

$= \sqrt{2} \int \sin x \ \cos x\, dx$

Now, multiply and Divide by 2 we get,

$= \frac{\sqrt{2}}{2} \int 2 \sin x \ \cos x\, dx$

$= \frac{\sqrt{2}}{2}\int \sin 2x \, dx$

On integrating

$= \frac{\sqrt{2}}{2}\frac{-\cos 2x}{2}$

Hence, I= $-\frac{1}{2\sqrt{2}}\cos 2x$ +C

EXERCISE 19.4

1. $\int \frac{x^2+5x+2}{x+2}\,dx$

Solution:

Given

$$\int \frac{x^2+5x+2}{x+2}\,dx$$

By performing long division of the given equation we get

Quotient = x + 3

Remainder = –4

∴ We can write the above equation as

$\Rightarrow x+3-\frac{4}{x+2}$

∴ The above equation becomes

$\Rightarrow \int x+3-\frac{4}{x+2}\,dx$

By splitting

$\Rightarrow \int x\,dx+3\int dx-4\int \frac{1}{x+2}\,dx$

We know $\int x\,dx=\frac{x^n}{n+1}; \int \frac{1}{x}dx=\ln x$

$\Rightarrow \frac{x^2}{2}+3x-4\ln(x+2)$ +c. (Where c is some arbitrary constant)

$$=\frac{x^2}{2}+3x-4\log|x+2|+c$$

2. $\int \frac{x^3}{x-2}\,dx$

Solution:

Given

$$\int \frac{x^3}{x-2}\,dx$$

By performing long division of the given equation we get

Quotient = x^2+2x+4

Remainder = 8

∴ We can write the above equation as

$$\Rightarrow x^2 + 2x + 4 + \frac{8}{x-2}$$

∴ The above equation becomes

$$\Rightarrow \int x^2 + 2x + 4 + \frac{8}{x-2}\,dx$$

$$\Rightarrow \int x^2\,dx + 2\int x\,dx + 4\int dx + 8\int \frac{1}{x-2}\,dx$$

We know $\int x\,dx = \frac{x^n}{n+1}; \int \frac{1}{x}dx = \ln x$

$$\Rightarrow \frac{x^3}{3} + 2\frac{x^2}{2} + 4x + 8\ln(x-2) + c$$

$\Rightarrow \frac{x^3}{3} + x^2 + 4x + 8\ln(x-2) + c$. (Where c is some arbitrary constant)

$$= \frac{x^3}{3} + x^2 + 4x + 8\log|x-2| + c$$

3. $\int \frac{x^2+x+5}{3x+2}\,dx$

Solution:

Given

$$\int \frac{x^2+x+5}{3x+2}\,dx$$

By doing long division of the given equation we get

Quotient = $\frac{x}{3}+\frac{1}{9}$

Remainder = $\frac{43}{9}$

∴ We can write the above equation as

$\Rightarrow \frac{x}{3}+\frac{1}{9}+\frac{43}{9}\left(\frac{1}{3x+2}\right)$

∴ The above equation becomes

$\Rightarrow \int \frac{x}{3}+\frac{1}{9}+\frac{43}{9}\left(\frac{1}{3x+2}\right) dx$

$\Rightarrow \frac{1}{3}\int x dx+\frac{1}{9}\int dx+\frac{43}{9}\int \frac{1}{3x+2} dx$

We know $\int x^n dx=\frac{x^{n+1}}{n+1}; \int \frac{1}{x} dx=\ln x$

$\Rightarrow \frac{1}{3}\times\frac{x^2}{2}+\frac{1}{9}\times x+\frac{43}{9}\times\frac{1}{3}\ln(3x+2)+c$

$= \frac{x^2}{6}+\frac{1}{9}x+\frac{43}{27}\log|3x+2|+c$ (Where c is some arbitrary constant)

EXERCISE 19.5 PAGE NO: 19.33

1. $\int \frac{x+1}{\sqrt{2x+3}} dx$

Solution:

Given

$$\int \frac{x+1}{\sqrt{2x+3}} dx$$

In this type of questions, little manipulation makes the questions easier to solve

Here we have multiply and divide by 2 to given equation

$$\Rightarrow \frac{1}{2}\int \frac{2x+2}{\sqrt{2x+3}} dx$$

Add and subtract 1 from the numerator

$$\Rightarrow \frac{1}{2}\int \frac{2x+2+1-1}{\sqrt{2x+3}} dx$$

$$\Rightarrow \frac{1}{2}\int \frac{2x+3-1}{\sqrt{2x+3}} dx$$

Splitting the above equation we get

$$\Rightarrow \frac{1}{2}\int \frac{2x+3}{\sqrt{2x+3}} dx - \frac{1}{2}\int \frac{1}{\sqrt{2x+3}} dx$$

Taking ½ common from the above equation

$$\Rightarrow \frac{1}{2}\left(\int \sqrt{2x + 3}\, dx - \int (2x + 3)^{\frac{-1}{2}} dx\right)$$

Now by integrating the above equation we get

$$\Rightarrow \frac{1}{2} \times \frac{(2x+3)^{\frac{3}{2}}}{2 \times \frac{3}{2}} - \frac{1}{2} \times \frac{(2x+3)^{\frac{1}{2}}}{2 \times \frac{1}{2}} + c$$

$\Rightarrow \frac{(2x+3)^{\frac{3}{2}}}{6} - \frac{(2x+3)^{\frac{1}{2}}}{2} + c$

2. $\int x\sqrt{x+2}\,dx$

Solution:

Given

$\int x\sqrt{x+2}\,dx$

In this type of questions, little manipulation makes the questions easier to solve

Here add and subtract 2 from x in the given equation

We get

$\Rightarrow \int (x + 2 - 2)\sqrt{x + 2}dx$

$\Rightarrow \int (x + 2)^{\frac{3}{2}}dx - \int 2\sqrt{x + 2}dx$

On integrating we get

$\Rightarrow \frac{2(x+2)^{\frac{5}{2}}}{5} - \frac{4(x+2)^{\frac{3}{2}}}{3} + c$

3. $\int \frac{x-1}{\sqrt{x+4}}dx$

Solution:

Given

$$\int \frac{x-1}{\sqrt{x+4}}\,dx$$

In this type of questions, little manipulation makes the questions easier to solve

Add and subtract 5 from the numerator

$$\Rightarrow \int \frac{x+5-5-1}{\sqrt{x+4}}\,dx$$

$$\Rightarrow \int \frac{x+4-5}{\sqrt{x+4}}\,dx$$

By splitting the above equation

$$\Rightarrow \int \frac{x+4}{\sqrt{x+4}}\,dx - \int \frac{5}{\sqrt{x+4}}\,dx$$

$$\Rightarrow \left(\int \sqrt{x+4}\,dx - 5\int (x+4)^{\frac{-1}{2}}\,dx\right)$$

Now by integrating, we get

$$\Rightarrow \frac{(x+4)^{\frac{3}{2}}}{\frac{3}{2}} - 5 \times \frac{(x+4)^{\frac{1}{2}}}{\frac{1}{2}} + c$$

By computing

$$\Rightarrow \frac{2(x+4)^{\frac{3}{2}}}{3} - 10(x+4)^{\frac{1}{2}} + c$$

4. $\int (x+2)\sqrt{3x+5}\,dx$

Solution:

Let

$$I = \int (x+2)\sqrt{3x+5}\text{dx}$$

Substitute 3x + 5 = t

$$\Rightarrow x = \frac{t-5}{3}$$

$$\Rightarrow 3dx = dt$$

$$\Rightarrow dx = \frac{dt}{3}$$

$$\therefore I = \int\left(\frac{t-5}{3}+2\right)\sqrt{t}\ \frac{dt}{3}$$
$$= \frac{1}{3}\ \int\left(\frac{t-5+6}{3}\right)\sqrt{t}\ dt$$

By taking 3 as common and multiplying, we get

$$= \frac{1}{9}\int\left(t^{\frac{3}{2}}+t^{\frac{1}{2}}\right)dt$$

On integrating we get

$$= \frac{1}{9}\left[\frac{t^{\frac{3}{2}+1}}{\frac{3}{2}+1}+\frac{t^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right]+C$$

On simplifying

$$= \frac{1}{9}\left[\frac{2}{5}t^{\frac{5}{2}} + \frac{2}{3}t^{\frac{3}{2}}\right] + C$$

By substituting the value of t

$$= \frac{1}{9}\left[\frac{2}{5}(3x+5)^{\frac{5}{2}} + \frac{2}{3}(3x+5)^{\frac{3}{2}}\right] + C$$

$$= \frac{2}{9}\left[(3x+5)^{\frac{3}{2}}\left\{\frac{3x+5}{5} + \frac{1}{3}\right\}\right] + C$$

$$= \frac{2}{9}\left[(3x+5)^{\frac{3}{2}}\left\{\frac{9x+15+5}{15}\right\}\right] + C$$

$$= \frac{2}{9}\left[(3x+5)^{\frac{3}{2}}\left\{\frac{9x+20}{15}\right\}\right] + C$$

$$= \frac{2}{135}(3x+5)^{\frac{3}{2}}(9x+20) + C$$

5. $\int \frac{2x+1}{\sqrt{3x+2}}\,dx$

Solution:

Given

$$\int \left(\frac{2x+1}{\sqrt{3x+2}}\right) dx$$

Multiply and divide by 3 in the above equation we get

$$= \frac{1}{3}\int \left(\frac{6x+3}{\sqrt{3x+2}}\right) dx$$

The above equation can be written as

$$= \frac{1}{3} \int \left(\frac{6x + 4 - 1}{\sqrt{3x + 2}} \right) dx$$

Taking 2 as common and subtracting

$$= \frac{1}{3} \int \left(\frac{2(3x + 2)}{\sqrt{3x + 2}} - \frac{1}{\sqrt{3x + 2}} \right) dx$$

On simplifying

$$= \frac{1}{3} \int \left(2\sqrt{3x + 2} - \frac{1}{\sqrt{3x + 2}} \right) dx$$

By splitting the integral

$$= \frac{1}{3} \left[\int 2(3x + 2)^{\frac{1}{2}} dx - \int (3x + 2)^{-\frac{1}{2}} dx \right]$$

On integrating we get

$$= \frac{1}{3} \left[2 \left\{ \frac{(3x + 2)^{\frac{1}{2}} + 1}{3\left(\frac{1}{2} + 1\right)} \right\} - \frac{(3x + 2)^{-\frac{1}{2}+1}}{\left(-\frac{1}{2} + 1\right) \times 3} \right] + C$$

$$= \frac{1}{3} \left[\frac{4}{9} (3x + 2)^{\frac{3}{2}} - \frac{2}{3} (3x + 2)^{\frac{1}{2}} \right] + C$$

On simplifying we get

$$= \frac{4}{27} (3x + 2)^{\frac{3}{2}} - \frac{2}{9} (3x + 2)^{\frac{1}{2}} + C$$

$$= \sqrt{3x + 2} \left(\frac{4}{27} (3x + 2) - \frac{2}{9} \right) + C$$

$$= \sqrt{3x + 2} \left(\frac{4(3x + 2) - 6}{27} \right) + C$$

$$= \sqrt{3x + 2} \left(\frac{12x + 8 - 6}{27} \right) + C$$

$$= \frac{2}{27} (6x + 1) \sqrt{3x + 2} + C$$

EXERCISE 19.6 PAGE NO: 19.36

1. $\int \sin^2(2x+5)\,dx$

Solution:

We know that

$\sin^2 x = \frac{1-\cos 2x}{2}$

By substituting the above formula

∴ The given equation becomes,

$\Rightarrow \int \frac{1-\cos 2(2x+5)}{2}\,dx$

We know $\int \cos ax\,dx = \frac{1}{a}\sin ax + c$

$\Rightarrow \frac{1}{2}\int dx - \frac{1}{2}\int \cos(4x+10)\,dx$

On integrating we get

$\Rightarrow \frac{x}{2} - \frac{1}{8}\sin(4x+10) + c$

2. $\int \sin^3(2x+1)\,dx$

Solution:

We know that $\sin 3x = -4\sin^3 x + 3\sin x$

The above formula can be written as

$\Rightarrow 4\sin^3 x = 3\sin x - \sin 3x$

The above equation becomes

$\Rightarrow \sin^3 x = \frac{3\sin x - \sin 3x}{4}$

Now applying above formula to the given question we get

$\Rightarrow \int \sin^3(2x+1)dx = \int \frac{3\sin(2x+1) - \sin 3(2x+1)}{4} dx$

We know $\int \sin ax\, dx = \frac{-1}{a}\cos ax + c$

By substituting the above formula we get

$\Rightarrow \frac{3}{8}\int \sin(2x+1)dx - \frac{1}{4}\int \sin(6x+3)dx$

On integrating we get

$\Rightarrow \frac{-3}{8}\cos(2x+1) + \frac{1}{24}\cos(6x+3) + c$.

3. $\int \cos^4 2x\, dx$

Solution:

Consider,

$\text{Cos}^4 2x = (\cos^2 2x)^2$

We know that

$\Rightarrow \cos^2 x = \frac{1+\cos 2x}{2}$

The above equation

$\Rightarrow (\cos^2 2x)^2 = \left(\frac{1+\cos 4x}{2}\right)^2$

$\Rightarrow \left(\frac{1+\cos 4x}{2}\right)^2 = \left(\frac{1+2\cos 4x+\cos^2 4x}{4}\right)$

$\Rightarrow \cos^2 4x = \frac{1+\cos 8x}{2}$

$\Rightarrow \left(\frac{1+2\cos 4x+\cos^2 4x}{4} = \frac{1}{4} + \frac{\cos 4x}{2} + \frac{1+\cos 8x}{8}\right)$

Now the question becomes,

$\Rightarrow \frac{1}{4}\int dx + \frac{1}{2}\int \cos 4x\, dx + \frac{1}{8}\int dx + \frac{1}{8}\int \cos 8x\, dx$

We know $\int \cos ax\, dx = \frac{1}{a}\sin ax + c$

$\Rightarrow \frac{x}{4} + \frac{1}{8}\sin 4x + \frac{x}{8} + \frac{\sin 8x}{64} + c$

$\Rightarrow \frac{24x + 8\sin 4x + \sin 8x}{64} + c$

4. $\int \sin^2 bx\, dx$

Solution:

We know that

$\sin^2 x = \frac{1-\cos 2x}{2}$

By substituting this formula,

∴ The given equation becomes,

$\Rightarrow \int \frac{1-\cos 2bx}{2}\, dx$

We know $\int \cos ax\, dx = \frac{1}{a}\sin ax + c$

$\Rightarrow \frac{1}{2}\int dx - \frac{1}{2}\int \cos(2bx)\, dx$

On integration

$\Rightarrow \frac{x}{2} - \frac{1}{4b}\sin(2bx) + c$

EXERCISE 19.7 PAGE NO: 19.38

Integrate the following integrals:

1. $\int \sin 4x \cos 7x \, dx$

Solution:

Given

$$\int \sin 4x \cos 7x \, dx$$

We know that 2 Sin A cos B = sin (A + B) + sin (A – B)

Now by substituting this formula in given question we get

∴ Sin 4x cos 7x = $\frac{\sin 11x + \sin(-3x)}{2}$

We know sin (– θ) = – sin θ

Hence Sin (– 3x) = – sin3x

∴ the above equation becomes

$\Rightarrow \int \frac{1}{2}(\sin 11x - \sin 3x)dx$

$\Rightarrow \frac{1}{2}(\int \sin 11x \, dx - \int \sin 3x \, dx)$

We know $\int \sin ax \, dx = \frac{-1}{a}\cos ax + c$

$\Rightarrow \frac{1}{2}\left(\frac{-1}{11}\cos 11x + \frac{1}{3}\cos 3x\right)$

$= -\frac{1}{22}\cos 11x + \frac{1}{6}\cos 3x + c$

2. $\int \cos 3x \cos 4x \, dx$

Solution:

Given

$$\int \cos 4x \cos 3x \, dx$$

Multiply and divide the given equation by 2

$$= \frac{1}{2}\int 2\cos 4x \ \cos \ 3x \ dx$$

We know that 2 cos A cos B = cos (A + B) + cos (A – B)

$$= \frac{1}{2}\int [\cos(4x+3x)+\cos\ (4x-3x)]\,dx$$

Now by simplifying we get

$$= \frac{1}{2}\int (\cos(7x)+\cos \mathrm{x})\,dx$$

On integration we get

$$= \frac{1}{2}\left[\frac{\sin 7x}{7}+\sin x\right]+C$$

$$= \frac{1}{14}\sin 7x+\frac{1}{2}\sin x + C$$

3. $\int$ **cos mx cos nx dx, m** $\neq$ **n**

Solution:

Given

$$\int \cos mx \cos nx \, dx, \, m \neq n$$

We know 2cosAcosB = cos (A – B) + cos (A + B)

Now substituting the above formula we get,

∴ Cos mx cos n x = $\frac{\cos(m-n)x+\cos(m+n)x}{2}$

∴ The above equation becomes

$$\Rightarrow \int \frac{1}{2}(\cos(m-n)x + \cos(m+n)x)dx$$

We know $\int \cos ax\, dx = \frac{1}{a}\sin ax + c$

Applying the above

$$\Rightarrow \frac{1}{2}\left(\frac{1}{m-n}\sin(m-n)x + \frac{1}{m+n}\sin(m+n)x\right)$$

$$\Rightarrow \frac{1}{2}\left(\frac{(m+n)\sin(m-n)x + (m-n)\sin(m+n)x}{m^2-n^2}\right) + c$$

We know that $a^2 - b^2 = (a+b)(a-b)$

By substituting the above formula and simplifying we get

$$\frac{1}{2}\left\{\frac{\sin(m+n)x}{m+n} + \frac{\sin(m-n)x}{m-n}\right\} + c$$

EXERCISE 19.8 PAGE NO: 19.47

Evaluate the following integrals:

1. $\int \frac{1}{\sqrt{1-\cos 2x}}\,dx$

Solution:

Given

$$\int \frac{1}{\sqrt{1-\cos 2x}}\,dx$$

In the given equation $\cos 2x = \cos^2 x - \sin^2 x$

Also we know $\cos^2 x + \sin^2 x = 1$.

Substituting the values in the above equation we get

$$\Rightarrow \int \frac{1}{\sqrt{\sin^2 x + \cos^2 x - (-\sin^2 x + \cos^2 x)}}\,dx$$

$$\Rightarrow \int \frac{1}{\sqrt{\sin^2 x + \cos^2 x + \sin^2 x - \cos^2 x}}\,dx$$

$$\Rightarrow \int \frac{1}{\sqrt{2\sin^2 x}}\,dx$$

$$\Rightarrow \int \frac{1}{\sqrt{2}\sin x}\,dx$$

$$\frac{1}{\sqrt{2}}\int cosec\ x\,dx$$

$$\Rightarrow \frac{1}{\sqrt{2}}\int \csc x\,dx$$

$$\Rightarrow \frac{1}{\sqrt{2}}\log\left|\frac{\tan x}{2}\right| + c$$

3. $\int \frac{\sqrt{1+\cos 2x}}{\sqrt{1-\cos 2x}}\,dx$

Solution:

Given,

$$\int \frac{\sqrt{1+\cos 2x}}{\sqrt{1-\cos 2x}}\, dx$$

We know that

$1 + \cos 2x = 2\cos^2 x$

$1 - \cos 2x = 2\sin^2 x$

By substituting these formulae in the given equation we get

$$\Rightarrow \int \sqrt{\frac{2\cos^2 x}{2\sin^2 x}}\, dx$$

Again by applying standard formula, we get

$$\Rightarrow \int \sqrt{\cot^2 x}\, dx$$

By simplifying we get

$$\Rightarrow \int \cot x\, dx$$

$\Rightarrow \log|\sin x| + c$

4. $\displaystyle \int \frac{\sqrt{1-\cos x}}{\sqrt{1+\cos x}}\, dx$

Solution:

Given,

$$\int \frac{\sqrt{1-\cos x}}{\sqrt{1+\cos x}}\, dx$$

We know that

$1 - \cos x = 2\sin^2 \frac{x}{2}$

$1 + \cos x = 2\cos^2 \frac{x}{2}$

By substituting these formulae in the given equation we get

$\Rightarrow \int \sqrt{\tan^2 \frac{x}{2}} dx$

On simplification,

$\Rightarrow \int \tan \frac{x}{2} dx$

$\Rightarrow -2 \ln \left|\cos \frac{x}{2}\right| + c$

5. $\int \frac{\sec x}{\sec 2x} dx$

Solution:

Here first of all convert sec x in terms of cos x

We know

$\Rightarrow \sec x = \frac{1}{\cos x}, \sec 2x = \frac{1}{\cos 2x}$

Therefore the above equation becomes,

$\Rightarrow \frac{\frac{1}{\cos x}}{\frac{1}{\cos 2x}}$

$= \frac{\cos 2x}{\cos x}$

∴ The equation now becomes

$\Rightarrow \int \frac{\cos 2x}{\cos x} dx$

We know

Cos 2x = 2 $\cos^2 x$ - 1

∴ We can write the above equation as

$\Rightarrow \int \frac{2\cos^2 x - 1}{\cos x} dx$

$\Rightarrow \int 2 \cos x \, dx - \int \frac{1}{\cos x} dx$

$\Rightarrow 2 \sin x - \int \sec x \, dx$

$(\int \sec x \, dx = \ln|\sec x + \tan x| + c$

$\Rightarrow 2 \sin x - \log |\sec x + \tan x| + c$

6. $\int \frac{\cos 2x}{(\cos x + \sin x)^2} dx$

Solution:

Let

$$I = \int \frac{\cos 2x}{(\cos x + \sin x)^2} dx$$

By substituting the formula, we get

$$= \int \frac{\cos^2 x - \sin^2 x}{(\cos x + \sin x)^2} dx$$

On simplification, we get

$$= \int \frac{\cos x - sinx}{\cos x + \sin x} dx$$

Put sin x + cos x = t

$$\Rightarrow -\sin x + \cos x = \frac{dt}{dx}$$

On rearranging

$$\Rightarrow (\cos x - \sin x) dx = dt$$

$$\therefore I = \int \frac{1}{t} dt$$

$$= \ln |t| + C$$

Now substitute the value of t, we get

$$= In |\cos x + \sin x| + C$$

7. $\int \frac{\sin(x-a)}{\sin(x-b)}$

Solution:

To solve these types of questions, it is better to eliminate the denominator.

$\Rightarrow \int \frac{\sin(x-a)}{\sin(x-b)} dx$

Add and subtract b in (x - a)

$\Rightarrow \int \frac{\sin(x-a+b-b)}{\sin(x-b)} dx$

$\Rightarrow \int \frac{\sin(x-b+b-a)}{\sin(x-b)}$

Numerator is of the form sin (A + B) = sin A cos B + cos A sin B

Where A = x - b; B = b - a

$\Rightarrow \int \frac{\sin(x-b)\cos(b-a)+\cos(x-b)\sin(b-a)}{\sin(x-b)} dx$

$\Rightarrow \int \frac{\sin(x-b)\cos(b-a)}{\sin(x-b)} dx + \int \frac{\cos(x-b)\sin(b-a)}{\sin(x-b)} dx$

$\Rightarrow \int \cos(b-a)\, dx + \int \cot(x-b)\sin(b-a)\, dx$

$\Rightarrow \cos(b-a) \int dx + \sin(b-a) \int \cot(x-b)\, dx$

As $\int \cot(x)\, dx = \ln|\sin x|$

⇒ Cos (b - a) x + sin (b - a) log |sin (x - b)|

Therefore,

= cos (b - a)x + sin(b - a) log |sin(x - b)| + c, where c is an arbitrary constant.

EXERCISE 19.9

PAGE NO: 19.57

Evaluate the following integrals:

1. $\int \frac{\log x}{x}$ **dx**

Solution:

Assume log x = t

⇒ d (log x) = dt

$\Rightarrow \frac{1}{x}dx = dt$

Substituting t and dt in above equation we get

$\Rightarrow \int t.dt$

$\Rightarrow \frac{t^2}{2} + c$

But t = log(x)

$\Rightarrow \frac{\log^2 x}{2} + c$.

2. $\int \frac{\log\left(1+\frac{1}{x}\right)}{x(1+x)} dx$

Solution:

Assume $\log\left(1 + \frac{1}{x}\right) = t$

$\Rightarrow d(\log\left(1 + \frac{1}{x}\right)) = dt$

$\Rightarrow \frac{1}{1+\frac{1}{x}} \times \frac{-1}{x^2}dx = dt$

$\Rightarrow \frac{x}{x+1} \times \frac{-1}{x^2}dx = dt$

$\Rightarrow \frac{-1.dx}{x(x+1)} = dt$

$\Rightarrow \frac{dx}{x(x+1)} = -dt$

∴ Substituting t and dt in the given equation we get

$\Rightarrow \int -t.\,dt$

$\Rightarrow -\int t.\,dt$

$\Rightarrow \frac{-t^2}{2} + c$

But $\log\left(1 + \frac{1}{x}\right) = t$

$\Rightarrow -\frac{1}{2}\left\{\log\left(1+\frac{1}{x}\right)\right\}^2 + c$

3. $\int \frac{(1+\sqrt{x})^2}{\sqrt{x}}\,dx$

Solution:

Assume 1 + √x = t

⇒ d (1 + √x) = dt

$\Rightarrow \frac{1}{2\sqrt{x}}dx = dt$

$\Rightarrow \frac{1}{\sqrt{x}}dx = 2dt$

∴ Substituting t and dt in the given equation we get

$\Rightarrow \int 2t^2.\,dt$

$\Rightarrow 2\int t^2.\,dt$

$\Rightarrow \frac{2t^3}{3} + c$

But $1 + \sqrt{x} = t$

$\Rightarrow \frac{2(1+\sqrt{x})^3}{3} + c$.

4. $\int \sqrt{1+e^x}\, e^x \, dx$

Solution:

Assume $1 + e^x = t$

$\Rightarrow d(1 + e^x) = dt$

$\Rightarrow e^x dx = dt$

∴ Substituting t and dt in given equation we get

$\Rightarrow \int \sqrt{t}. dt$

$\Rightarrow \int t^{1/2} . dt$

$\Rightarrow \frac{2t^{\frac{3}{2}}}{3} + c$

But $1 + e^x = t$

$\Rightarrow \frac{2(1+e^x)^{3/2}}{3} + c$.

5. $\int \sqrt[3]{\cos^2 x} \sin x \, dx$

Solution:

Assume $\cos x = t$

$\Rightarrow d(\cos x) = dt$

$\Rightarrow -\sin x \, dx = dt$

$\Rightarrow dx = \frac{-dt}{\sin x}$

∴ Substituting t and dt in the given equation we get

$\Rightarrow \int \sqrt[3]{t^2} \sin x . \frac{-dt}{\sin x}$

$\Rightarrow \int -t^{2/3} . dt$

$\Rightarrow -\frac{3}{5} t^{\frac{5}{3}} + c$

But cos x = t

$\Rightarrow -\frac{3}{5} \cos^{\frac{5}{3}} x + c$.

6. $\int \frac{e^x}{(1+e^x)^2} dx$

Solution:

Assume $1 + e^x = t$

$\Rightarrow d(1 + e^x) = dt$

$\Rightarrow e^x dx = dt$

∴ Substituting t and dt in given equation we get

$\Rightarrow \int \frac{1}{t^2} dt$

$\Rightarrow \int t^{-2} . dt$

$\Rightarrow \frac{-1}{t} + c$

But $1 + e^x = t$

$\Rightarrow \frac{-1}{1 + e^x} + c$.

7. $\int \cot^3 x \ cosec^2 \ x \, dx$

Solution:

Assume cot x = t

⇒ d (cot x) = dt

⇒ - $\operatorname{cosec}^2 x.dx$ = dt

⇒ $dx = \frac{-dt}{\csc^2 x}$

∴ Substituting t and dt in the given equation we get

⇒ $\int t^3 \csc^2 x . \frac{-dt}{\csc^2 x}$

⇒ $\int -t^3 . dt$

⇒ $-\int t^3 . dt$

⇒ $\frac{-t^4}{4} + c$

But t = cot x

⇒ $\frac{-\cot^4 x}{4} + c$.

8. $\int \frac{\left\{e^{\sin^{-1} x}\right\}^2}{\sqrt{1-x^2}} dx$

Solution:

Assume $\sin^{-1} x$ = t

⇒ d ($\sin^{-1} x$) = dt

⇒ $\frac{dx}{\sqrt{1-x^2}} = dt$

∴ Substituting t and dt in the given equation we get

⇒ $\int e^{t^2} dt$

⇒ $\int e^{2t} . dt$

$\Rightarrow \frac{e^{2t}}{2} + c$

But t = $\sin^{-1}x$

$\Rightarrow \frac{1}{2}\left\{e^{\sin^{-1} x}\right\}^2 + c$

9. $\int \frac{1+\sin x}{\sqrt{x-\cos x}} dx$

Solution:

Assume x – cos x = t

$\Rightarrow$ d (x – cos x) = dt

$\Rightarrow$ (1 + sin x) dx = dt

∴ Substituting t and dt in given equation we get

$\Rightarrow \int \frac{1}{\sqrt{t}} dt$

$\Rightarrow \int t^{-1\backslash 2} . dt$

$\Rightarrow 2t^{1\backslash 2} + c$

But t = x – cos x.

$\Rightarrow 2(x - \cos x)^{1/2} + c.$

10. $\int \frac{1}{\sqrt{1-x^2}(\sin^{-1} x)^2} dx$

Solution:

Assume $\sin^{-1}x$ = t

$\Rightarrow$ d ($\sin^{-1}x$) = dt

$\Rightarrow \frac{dx}{\sqrt{1-x^2}} = dt$

∴ Substituting t and dt in the given equation we get

$\Rightarrow \int \frac{1}{t^2} dt$

$\Rightarrow \int t^{-2} . dt$

On integrating the above equation we get

$\Rightarrow \frac{t^{-1}}{-1} + c$

But t = sin $^{-1}$x

$\Rightarrow \frac{-1}{\sin^{-1} x} + c$

EXERCISE 19.10 PAGE NO: 19.65

1. $\int x^2\sqrt{x+2}\,dx$

Solution:

Let $I = \int x^2\sqrt{x+2}dx$

Substituting, x + 2 = t ⇒dx = dt,

$I = \int (t-2)^2\sqrt{t}dt$

$\Rightarrow I = \int (t^2 - 4t + 4)\sqrt{t}dt$

$\Rightarrow I = \int \left(t^{\frac{5}{2}} - 4t^{\frac{3}{2}} + 4t^{\frac{1}{2}}\right)dt$

$\Rightarrow I = \frac{2}{7}t^{\frac{7}{2}} - \frac{8}{5}t^{\frac{5}{2}} + \frac{8}{3}t^{\frac{3}{2}} + c$

$\Rightarrow I = \frac{2}{7}(x+2)^{\frac{7}{2}} - \frac{8}{5}(x+2)^{\frac{5}{2}} + \frac{8}{3}(x+2)^{\frac{3}{2}} + c$

Therefore, $\int x^2\sqrt{x+2}dx = \frac{2}{7}(x+2)^{\frac{7}{2}} - \frac{8}{5}(x+2)^{\frac{5}{2}} + \frac{8}{3}(x+2)^{\frac{3}{2}} + c$

2. $\int \frac{x^2}{\sqrt{x-1}}\,dx$

Solution:

Let $I = \int \frac{x^2}{\sqrt{x-1}}dx$

Substituting x - 1 = t ⇒ dx = dt,

Now substituting the values we get

$$\Rightarrow I = \int \frac{(t+1)^2}{\sqrt{t}} dt$$

Expanding using $(a+b)^2$ formula

$$\Rightarrow I = \int \frac{t^2 + 2t + 1}{\sqrt{t}} dt$$

On simplification

$$\Rightarrow I = \int \left(t^{\frac{3}{2}} + 2t^{\frac{1}{2}} + t^{-\frac{1}{2}}\right) dt$$

On integrating we get

$$\Rightarrow I = \frac{2}{5} t^{\frac{5}{2}} + 2t^{\frac{1}{2}} + \frac{4}{3} t^{\frac{3}{2}} + c$$

Again taking LCM

$$\Rightarrow I = \frac{\left(6t^{\frac{5}{2}} + 30t^{\frac{1}{2}} + 20t^{\frac{3}{2}}\right)}{15} + c$$

$$\Rightarrow I = \frac{2}{15} t^{\frac{1}{2}} (3t^2 + 15 + 10t) + c$$

Substituting the value of t we get

$$\Rightarrow I = \frac{2}{15} (x-1)^{\frac{1}{2}} (3(x-1)^2 + 15 + 10(x-1)) + c$$

$$\Rightarrow I = \frac{2}{15} (x-1)^{\frac{1}{2}} (3(x^2 - 2x + 1) + 15 + 10x - 10) + c$$

By simplifying we get

$$\Rightarrow I = \frac{2}{15} (x-1)^{\frac{1}{2}} (3x^2 + 4x + 8) + c$$

Therefore, $\int \frac{x^2}{\sqrt{x-1}} dx = \frac{2}{15} (x-1)^{\frac{1}{2}} (3x^2 + 4x + 8) + c$

3. $\int \frac{x^2}{\sqrt{3x+4}} dx$

Solution:

Let $I = \int \frac{x^2}{\sqrt{3x + 4}} dx$

Substituting 3 x + 4 = t ⇒ 3 dx = dt,

Substituting the values of x

$$\Rightarrow I = \int \frac{\left(\frac{t-4}{3}\right)^2}{3\sqrt{t}} dt$$

Expanding the above given function using $(a - b)^2$ formula

$$\Rightarrow I = \frac{1}{27}\int \frac{t^2 + 16 - 8t}{\sqrt{t}} dt$$

On simplifying, we get

$$\Rightarrow I = \frac{1}{27}\int \left(t^{\frac{3}{2}} - 8t^{\frac{1}{2}} + 16t^{-\frac{1}{2}}\right) dt$$

On integrating, we get

$$\Rightarrow I = \frac{1}{27}\left[\frac{2}{5}t^{\frac{5}{2}} - \frac{16}{3}t^{\frac{3}{2}} + 32t^{\frac{1}{2}}\right] + c$$

$$\Rightarrow I = \frac{1}{27}\left[\frac{2}{5}(3x + 4)^{\frac{5}{2}} - \frac{16}{3}(3x + 4)^{\frac{3}{2}} + 32(3x + 4)^{\frac{1}{2}}\right] + c$$

$$\Rightarrow I = \frac{2}{135}(3x + 4)^{\frac{5}{2}} - \frac{16}{81}(3x + 4)^{\frac{3}{2}} + \frac{32}{27}(3x + 4)^{\frac{1}{2}} + c$$

Therefore, $\int \frac{x^2}{\sqrt{3x + 4}} dx$

$$= \frac{2}{135}(3x + 4)^{\frac{5}{2}} - \frac{16}{81}(3x + 4)^{\frac{3}{2}} + \frac{32}{27}(3x + 4)^{\frac{1}{2}} + c$$

4. $\int \frac{2x-1}{(x-1)^2} dx$

Solution:

$$\text{Let } I = \int \frac{2x-1}{(x-1)^2}dx$$

Substituting x - 1 = t ⇒dx = dt

Substituting the values of x

$$\Rightarrow I = \int \frac{2(t+1)-1}{t^2}dt$$

Multiplying and simplifying we get

$$\Rightarrow I = \int \frac{2t+1}{t^2}dt$$

$$\Rightarrow I = \int \left(\frac{2}{t} + \frac{1}{t^2}\right)dt$$

On integration

$$\Rightarrow I = 2\log|t| - \frac{1}{t} + c$$

$$\Rightarrow I = 2\log|x-1| - \frac{1}{x-1} + c$$

$$\text{Therefore, } \int \frac{2x-1}{(x-1)^2}dx = 2\log|x-1| - \frac{1}{x-1} + c$$

5. $\int (2x^2+3)\sqrt{x+2}\,dx$

Solution:

$$\text{Let } I = \int (2x^2+3)\sqrt{x+2}\,dx$$

Substituting x + 2 = t ⇒ dx = dt

Substituting the values of x in given equation, we get

$\Rightarrow I = \int [2(t-2)^2 + 3]\sqrt{t}dt$

Expanding above equation using $(a-b)^2$ formula

$\Rightarrow I = \int [2t^2 - 8t + 8 + 3]\sqrt{t}dt$

On simplification

$\Rightarrow I = \int \left[2t^{\frac{5}{2}} - 8t^{\frac{3}{2}} + 11t^{\frac{1}{2}}\right]dt$

On integrating we get

$\Rightarrow I = \frac{4}{7}t^{\frac{7}{2}} - \frac{16}{5}t^{\frac{5}{2}} + \frac{22}{3}t^{\frac{3}{2}} + c$

$\Rightarrow I = \frac{4}{7}(x+2)^{\frac{7}{2}} - \frac{16}{5}(x+2)^{\frac{5}{2}} + \frac{22}{3}(x+2)^{\frac{3}{2}} + c$

$\therefore \int (2x^2 + 3)\sqrt{x+2}dx = \frac{4}{7}(x+2)^{\frac{7}{2}} - \frac{16}{5}(x+2)^{\frac{5}{2}} + \frac{22}{3}(x+2)^{\frac{3}{2}} + c$

EXERCISE 19.11 PAGE NO: 19.69

Evaluate the following integrals:

1. $\int \tan^3 x \sec^2 x \, dx$

Solution:

$\text{Let } I = \int \tan^3 x \sec^2 x \, dx$

Let tan x = t, then

$\Rightarrow \sec^2 x \, dx = dt$

Substituting the values of x

$\Rightarrow I = \int t^3 dt$

On integrating we get

$\Rightarrow I = \frac{t^4}{4} + c$

Substituting the value of t we get

$\Rightarrow I = \frac{\tan^4 x}{4} + c$

Therefore, $\int \tan^3 x \sec^2 x \, dx = \frac{\tan^4 x}{4} + c$

2. $\int \tan x \sec^4 x \, dx$

Solution:

$\text{Let } I = \int \tan x \sec^4 x \, dx$

The above equation can be written as

$$\Rightarrow I = \int \tan x \sec^2 x \sec^2 x \, dx$$

$$\Rightarrow I = \int \tan x (1 + \tan^2 x) \sec^2 x \, dx$$

$$\Rightarrow I = \int (\tan x + \tan^3 x) \sec^2 x \, dx$$

Let tan x = t, then

$\Rightarrow \sec^2 x \, dx = dt$

Substituting the values of x

$$\Rightarrow I = \int (t + t^3) dt$$

On integrating we get

$$\Rightarrow I = \frac{t^2}{2} + \frac{t^4}{4} + c$$

$$\Rightarrow I = \frac{\tan^2 x}{2} + \frac{\tan^4 x}{4} + c$$

Therefore, $\int \tan x \sec^4 x \, dx = \frac{\tan^2 x}{2} + \frac{\tan^4 x}{4} + c$

3. $\int \tan^5 x \sec^4 x \, dx$

Solution:

Let $I = \int \tan^5 x \sec^4 x \, dx$

The above equation can be written as

$$\Rightarrow I = \int \tan^5 x \sec^2 x \sec^2 x \, dx$$

Taking $\tan^5 x$ as common

$$\Rightarrow I = \int \tan^5 x\,(1 + \tan^2 x)\sec^2 x\,dx$$

On simplifying

$$\Rightarrow I = \int (\tan^5 x + \tan^7 x)\sec^2 x\,dx$$

Let tan x = t, then

$\Rightarrow \sec^2 x\,dx = dt$

Substituting the value of x

$$\Rightarrow I = \int (t^5 + t^7)dt$$

Integrating we get

$$\Rightarrow I = \frac{t^6}{6} + \frac{t^8}{8} + c$$

Substituting the values of t

$$\Rightarrow I = \frac{\tan^6 x}{6} + \frac{\tan^8 x}{8} + c$$

Therefore, $\int \tan^5 x \sec^4 x\,dx = \frac{\tan^6 x}{6} + \frac{\tan^8 x}{8} + c$

4. $\int \sec^6 x \tan x\, dx$

Solution:

Let $I = \int \sec^6 x \tan x\,dx$

The above equation can be written as

$$\Rightarrow I = \int \sec^5 x\,(\sec x \tan x)dx$$

Substituting, sec x = t ⇒ sec x tan x dx = dt

$$\Rightarrow I = \int t^5 dt$$

On integrating we get

$$\Rightarrow I = \frac{t^6}{6} + c$$

Now substituting the values of t we get

$$\Rightarrow I = \frac{\sec^6 x}{6} + c$$

Therefore, $\int \sec^6 x\,(\sec x \tan x)dx = \frac{\sec^6 x}{6} + c$

5. $\int \tan^5 x\,dx$

Solution:

Let $I = \int \tan^5 x\,dx$

The above equation can be written as

$$\Rightarrow I = \int \tan^2 x \tan^3 x\,dx$$

Using standard formula

$$\Rightarrow I = \int (\sec^2 x - 1)\tan^3 x\,dx$$

Splitting the above equation we get

$$\Rightarrow I = \int \tan^3 x \sec^2 x\,dx - \int \tan^3 x\,dx$$

$$\Rightarrow I = \int \tan^3 x \sec^2 x\,dx - \int (\sec^2 x - 1)\tan x\,dx$$

$$\Rightarrow I = \int \tan^3 x \sec^2 x\,dx - \int (\sec^2 x \tan x)dx + \int \tan x\,dx$$

Let tan x = t, then

$\Rightarrow \sec^2 x\, dx = dt$

$$\Rightarrow I = \int t^3 dt - \int t dt + \int \tan x\, dx$$

$$\Rightarrow I = \frac{t^4}{4} - \frac{t^2}{2} + \log|\sec x| + c$$

$$\Rightarrow I = \frac{\tan^4 x}{4} - \frac{\tan^2 x}{2} + \log|\sec x| + c$$

Therefore, $\int \tan^5 x\, dx = \frac{\tan^4 x}{4} - \frac{\tan^2 x}{2} + \log|\sec x| + c$

6. $\int \sqrt{\tan x} \sec^4 x\, dx$

Solution:

Let $I = \int \sqrt{\tan x} \sec^4 x\, dx$

The above equation can be written as

$$\Rightarrow I = \int \sqrt{\tan x} \sec^2 x \sec^2 x\, dx$$

Taking common

$$\Rightarrow I = \int \sqrt{\tan x}\,(1 + \tan^2 x) \sec^2 x\, dx$$

$$\Rightarrow I = \int (\tan^{\frac{1}{2}} x + \tan^{\frac{5}{2}} x) \sec^2 x\, dx$$

Let tan x = t, then

$\Rightarrow \sec^2 x\, dx = dt$

$$\Rightarrow I = \int \left(t^{\frac{1}{2}} + t^{\frac{5}{2}}\right) dt$$

On integrating we get

$$\Rightarrow I = \frac{2}{3}t^{\frac{3}{2}} + \frac{2}{7}t^{\frac{7}{2}} + c$$

Substituting the value of t

$$\Rightarrow I = \frac{2}{3}\tan^{\frac{3}{2}}x + \frac{2}{7}\tan^{\frac{7}{2}}x + c$$

Therefore, $\int \sqrt{\tan x}\sec^4 x\,dx = \frac{2}{3}\tan^{\frac{3}{2}}x + \frac{2}{7}\tan^{\frac{7}{2}}x + c$

EXERCISE 19.12 PAGE NO: 19.73

1. $\int \sin^4 x \cos^3 x \, dx$

Solution:

Let

$\sin x = t$

We know the Differentiation of $\sin x = \cos x$

$dt = d(\sin x) = \cos x dx$

So, $dx = \frac{dt}{\cos x}$

Substitute all in above equation,

$\int \sin^4 x \cos^3 x \, dx = \int t^4 \cos^3 x \frac{dt}{\cos x}$

$= \int t^4 \cos^2 x \, dt$

$= \int t^4 (1 - \sin^2 x) \, dt$

$= \int t^4 (1 - t^2) \, dt$

$= \int (t^4 - t^6) \, dt$

We know, basic integration formula, $\int x^n \, dx = \frac{x^{n+1}}{n+1} + c$ for any c≠-1

Hence, $\int (t^4 - t^6) dt = \frac{t^5}{5} - \frac{t^7}{7} + c$

Put back t = sin x

$\int \sin^4 x \cos^3 x \, dx = \frac{1}{5} \sin^5 x - \frac{1}{7} \sin^7 x + c$

2. $\int \sin^5 x \, dx$

Solution:

The given equation can be written as

$\int \sin^5 x\, dx = \int \sin^3 x \sin^2 x\, dx$

$= \int \sin^3 x(1 - \cos^2 x)dx$ {since $\sin^2 x + \cos^2 x = 1$}

$= \int (\sin^3 x - \sin^3 x \cos^2 x)\, dx$

$= \int (\sin x\, (\sin^2 x) - \sin^3 x \cos^2 x)\, dx$

$= \int (\sin x\, (1 - \cos^2 x) - \sin^3 x \cos^2 x)\, dx$ {since $\sin^2 x + \cos^2 x = 1$}

$= \int (\sin x - \sin x \cos^2 x - \sin^3 x \cos^2 x)dx$

$= \int \sin x\, dx - \int \sin x \cos^2 x\, dx - \int \sin^3 x \cos^2 x\, dx$ (separate the integrals)

We know, d (cos x) = -sin x dx

So put cos x = t and dt = -sin x dx in above integrals

$= \int \sin x\, dx - \int \sin x \cos^2 x\, dx - \int \sin^3 x \cos^2 x\, dx$

$= \int \sin x\, dx - \int t^2 (-dt) - \int (\sin^2 x \sin x)\, t^2\, dx$

$= \int \sin x\, dx - \int t^2 (-dt) - \int (1 - \cos^2 x)t^2\, (-dt)$

$= \int \sin x\, dx + \int (t^2 dt) + \int (1 - t^2)t^2\, dt$

$= \int \sin x\, dx + \int (t^2 dt) + \int (t^2 - t^4)dt$

$= -\cos x + \frac{t^3}{3} + \frac{t^3}{3} - \frac{t^5}{5} + c$ (since $\int x^n\, dx = \frac{x^{n+1}}{n+1} + c$ for any $c \neq -1$)

Put back t = cos x

$= -\cos x + \frac{t^3}{3} + \frac{t^3}{3} - \frac{t^5}{5} + c$

$= -\cos x + \frac{\cos^3 x}{3} + \frac{\cos^3 x}{3} - \frac{\cos^5 x}{5} + c$

$= -\cos x + \frac{2}{3}\cos^3 x - \frac{1}{5}\cos^5 x + c = -[\cos x - \frac{2}{3}\cos^3 x + \frac{1}{5}\cos^5 x] + c$

3. $\int \cos^5 x \, dx$

Solution:

The given question can be written as

$\int \cos^5 x \, dx = \int \cos^3 x \cos^2 x \, dx$

$= \int \cos^3 x(1 - \sin^2 x) dx$ {since $\sin^2 x + \cos^2 x = 1$}

$= \int (\cos^3 x - \cos^3 x \sin^2 x) \, dx$

$= \int (\cos x \, (\cos^2 x) - \cos^3 x \sin^2 x) \, dx$

$= (\cos x \, (1 - \sin^2 x) - \cos^3 x \sin^2 x) \, dx$ {since $\sin^2 x + \cos^2 x = 1$}

$= \int (\cos x - \cos x \sin^2 x - \cos^3 x \sin^2 x) dx$

$= \int \cos x \, dx - \int \cos x \sin^2 x \, dx - \int \cos^3 x \sin^2 x \, dx$ (separate the integrals)

We know, d (sin x) = cos x dx

So put sin x = t and dt = cos x dx in above integrals

$= \int \cos x \, dx - \int t^2 \, dt - \int \cos x \cos^2 x \sin^2 x \, dx$

$= \int \cos x \, dx - \int t^2 \, (dt) - \int (\cos^2 x \cos x) \, t^2 \, dx$

$= \int \cos x \, dx - \int t^2 \, (dt) - \int (1 - \sin^2 x) t^2 \, (dt)$

$= \int \cos x \, dx - \int (t^2 \, dt) - \int (1 - t^2) t^2 \, dt$

$= \int \cos x \, dx - \int (t^2 \, dt) - \int (t^2 - t^4) dt$

$= \sin x - \frac{t^3}{3} - \frac{t^3}{3} + \frac{t^5}{5} + c$

Put back t = sin x

$= \sin x - \frac{\sin^3 x}{3} - \frac{\sin^3 x}{3} + \frac{\cos^5 x}{5} + c$

$= \sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x + c$

4. $\int \sin^5 x \cos x \, dx$

Solution:

Let sin x = t

Then d (sin x) = dt = cos x dx

Put t = sin x and dt = cos x dx in given equation

$\int \sin^5 x \cos x \, dx = \int t^5 dt$

On integrating we get

$= \frac{t^6}{6} + c$

Substituting the value of t

$= \frac{\sin^6 x}{6} + c$

5. $\int \sin^3 x \cos^6 x \, dx$

Solution:

Since power of sin is odd, put cos x = t

Then dt = -sin x dx

Substitute these in above equation,

$\int \sin^3 x \cos^6 x \, dx = \int \sin x \sin^2 x \, t^6 \, dx$

$= \int (1 - \cos^2 x) \, t^6 \sin x \, dx$

$= \int -(1 - t^2)\ t^6 dt$

$= \int -(t^6 - t^8) dt$

On integrating we get

$= -\frac{t^7}{7} + \frac{t^9}{9} + c$

Put the value of t we get

$= \frac{-1}{7}\cos^7 x + \frac{1}{9}\cos^9 x + c$

EXERCISE 19.13 PAGE NO: 19.79

1. $\int \frac{x^2}{(a^2 - x^2)^{\frac{3}{2}}} \, dx$

Solution:

Given

$$\int \frac{x^2}{\left(a^2 - x^2\right)^{3/2}} dx$$

Put x = a sin θ, so dx = a cos θ dθ and θ = $\sin^{-1}(x/a)$

Above equation becomes,

$$= \int \frac{a^2 \sin^2\theta}{(a^2 - a^2\sin^2\theta)^{3/2}} (a\cos\theta \, d\theta) = \int \frac{a^2 \sin^2\theta}{(a^2)(a^2 - a^2\sin^2\theta)^{3/2}} (a\cos\theta \, d\theta)$$

By taking a^2 common we get

$$= \int \frac{a^2 \sin^2\theta}{(a^2)3/2(a^2 - a^2\sin^2\theta)^{3/2}} (a\cos\theta \, d\theta) = \int \sin^2\theta * \frac{\cos\theta}{\cos^3\theta} \, d\theta$$

$$= \int \frac{\sin^2\theta}{\cos^2\theta} \, d\theta = \int \tan^2\theta d\theta = \int (\sec^2\theta - 1) \, d\theta$$ $(\sec^2\theta - 1 = \tan^2\theta)$

$$= \int \sec^2\theta \, d\theta - \int 1.d\theta$$

$$= \tan\theta - \theta + c$$

Put θ = $\sin^{-1}(x/a)$

$$= \frac{x}{(\sqrt{a^2 - x^2})} - \sin^{-1}\frac{x}{a} + c$$

2. $\int \frac{x^7}{(a^2 - x^2)^5} \, dx$

Solution:

$$\text{Let I} = \int \frac{x^7}{(a^2 - x^2)^5} dx$$

$$\text{Let x} = a \sin\theta$$

On differentiating both sides we get

$$dx = a\cos\theta\, d\theta$$

$$\therefore I = \int \frac{a^8 \sin^7\theta \cos\theta}{(a^2 - a^2\sin^2\theta)^5} d\theta$$

$$= \int \frac{a^8 \sin^7\theta \cos\theta}{a^{10}(1 - \sin^2\theta)^5} d\theta$$

$$= \int \frac{\sin^7\theta}{a^2\cos^9\theta} d\theta$$

$$= \frac{1}{a^2} \int \tan^7\theta \sec^2\theta d\theta$$

Let

$$\tan\theta = t$$

Differentiating on both sides

$$\sec^2\theta\, d\theta = dt$$

$$\therefore I = \frac{1}{a^2} \int t^7 dt$$

$$= \frac{1}{a^2} \frac{t^8}{8} + c$$

$$= \frac{1}{8a^2} (\tan^8\theta) + c$$

$$= \frac{1}{8a^2} \left(\tan\left(\sin^{-1}\frac{x}{a}\right)\right)^8 + c$$

$$= \frac{1}{8a^2} \left(\tan\left(\tan^{-1}\frac{x}{\sqrt{a^2 - x^2}}\right)\right)^8 + c$$

$$= \frac{1}{8a^2} \left(\frac{x}{\sqrt{a^2 - x^2}}\right)^8 + c$$

$$= \frac{1}{8a^2} \frac{x^8}{(a^2 - x^2)^4} + c$$

$$Hence, \int \frac{x^7}{(a^2 - x^2)^5} dx = \frac{1}{8a^2} \frac{x^8}{(a^2 - x^2)^4} + c$$

EXERCISE 19.14 PAGE NO: 19.83

Evaluate the following integrals:

1. $\int \frac{1}{a^2 - b^2x^2}\, dx$

Solution:

Taking out b^2 as common from the given equation, we get

$$\frac{1}{b^2}\int \frac{1}{\left(\frac{a^2}{b^2}\right)-x^2}dx$$

$$= \frac{1}{b^2}\int \frac{1}{\left(\frac{a^2}{b^2}\right)-x^2}dx = \frac{1}{b^2}\int \frac{1}{\left(\frac{a}{b}\right)^2-x^2}dx$$

On integrating above equation using

$\int \frac{1}{a^2-x^2}dx = \frac{1}{2a}\log\left|\frac{x+a}{a-x}\right| + c$, we get

$$= \frac{1}{b^2} \times \frac{1}{2\left(\frac{a}{b}\right)}\log\left|\frac{\frac{a}{b}+x}{\frac{a}{b}-x}\right| + c$$

On simplification we get

$$= \frac{1}{2ab}\log\left|\frac{a+bx}{a-bx}\right| + c$$

2. $\int \frac{1}{a^2x^2 - b^2}\, dx$

Solution:

Taking out a^2 as common from the given equation, we get

$$= \frac{1}{a^2}\int \frac{1}{x^2-\frac{b^2}{a^2}}dx$$

On integrating above equation using

$$\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + c$$

$= \frac{1}{a^2}\int \frac{1}{x^2-(\frac{b}{a})^2}\,dx = \frac{1}{a^2} * \frac{1}{2(\frac{b}{a})}\log|\frac{x-(\frac{b}{a})}{x+\frac{b}{a}}| + c$

On simplification

$= \frac{1}{2ab}\log\frac{ax-b}{ax+b} + c$

3. $\int \frac{1}{a^2x^2+b^2}\,dx$

Solution:

Taking out a^2 as common from the given equation, we get

$= \frac{1}{a^2}\int \frac{1}{x^2+\frac{b^2}{a^2}}\,dx$

On integrating above equation using

$\int \frac{1}{x^2+a^2}dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$, we get

$= \frac{1}{a^2}\int \frac{1}{x^2+(\frac{b}{a})^2}\,dx = \frac{1}{a^2} * \frac{1}{(\frac{b}{a})}\tan^{-1}[\frac{x}{\frac{b}{a}}] + c$

By simplifying we get

$= \frac{1}{ab}\tan^{-1}\left(\frac{ax}{b}\right) + c$

4. $\int \frac{x^2-1}{x^2+4}\,dx$

Solution:

Add and subtract 4 in the numerator of given equation, we get

$= \int \frac{x^2+4-4-1}{x^2+4} = \int \frac{(x^2+4)-4-1}{x^2+4}\,dx$

Now separate the numerator terms, we get

$= \int \frac{(x^2+4)-5}{x^2+4}dx = \int \frac{(x^2+4)}{x^2+4}dx - \int \frac{5}{x^2+4}dx$

On computing we get

$= \int dx - \int \frac{5}{x^2+4}dx = \int dx - 5\int \frac{1}{x^2+4}dx$

We know $\int \frac{1}{x^2+a^2}dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$

$= \int dx - 5\int \frac{1}{x^2+2^2}dx = x - 5 \times \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + c$

On integrating we get

$= x - \frac{5}{2}\tan^{-1}\left(\frac{x}{2}\right) + c$

5. $\int \frac{1}{\sqrt{1+4x^2}}\,dx$

Solution:

Let I = $\int \frac{1}{\sqrt{1+4x^2}}dx$

The above equation can be written as

$= \int \frac{1}{\sqrt{1+(2x)^2}}dx$

Let t = 2x, then dt = 2dx or dx = dt/2

Therefore,

$\int \frac{1}{\sqrt{1+(2x)^2}}dx = \frac{1}{2}\int \frac{dt}{\sqrt{1+t^2}}$

We know $\int \frac{1}{\sqrt{(a^2+x^2)}}dx = \log|x + \sqrt{(a^2+x^2)}| + c$

$= \frac{1}{2}\log|t + \sqrt{1+t^2}| + c$

$= \frac{1}{2}\log|2x + \sqrt{1+4x^2}| + c$

EXERCISE 19.15 PAGE NO: 19.86

1. $\int \frac{1}{4x^2+12x+5} dx$

Solution:

Let

$I = \int \frac{1}{4x^2+12x+5} dx$

Taking out ¼ as common, then we get

$$= \frac{1}{4}\int \frac{1}{x^2+3x+\frac{5}{4}} dx$$

Adding and subtracting $(3/2)^2$ to the denominator

$$= \frac{1}{4}\int \frac{1}{x^2+2x\times\frac{3}{2}+\left(\frac{3}{2}\right)^2-\left(\frac{3}{2}\right)^2+\frac{5}{4}} dx$$

The above equation can be written as

$$= \frac{1}{4}\int \frac{1}{\left(x+\frac{3}{2}\right)^2-1} dx$$

Let

$\left(x+\frac{3}{2}\right) = t$ ..... (i)

$\Rightarrow dx = dt$

So, substituting the t values we get

$$I = \frac{1}{4}\int \frac{1}{t^2-(1)^2} dt$$

$$I = \frac{1}{4}\times\frac{1}{2\times 1}\log\left|\frac{t-1}{t+1}\right| + c$$

[since, $\int \frac{1}{x^2-(a)^2}dx = \frac{1}{2\times a}\log\left|\frac{x-a}{x+a}\right| + c$

$I = \frac{1}{8}\log\left|\frac{x+\frac{3}{2}-1}{x+\frac{3}{2}+1}\right| + c$ [Using (i)]

2. $\int \frac{1}{x^2-10x+34}dx$

Solution:

Let

$I = \int \frac{1}{x^2-10x+34}dx$

$I = \int \frac{1}{x^2-10x+34}dx$

Adding and subtracting 5^2 to both sides

$= \int \frac{1}{x^2-2x\times 5+(5)^2-(5)^2+34}dx$

The above equation can be written as

$= \int \frac{1}{(x-5)^2+9}dx$

Let $(x-5) = t$ ..... (i)

⇒ dx = dt

So, substituting the values of t we get

$I = \int \frac{1}{t^2+(3)^2}dt$

$I = \frac{1}{3}\tan^{-1}(\frac{t}{3}) + c$

[since, $\int \frac{1}{x^2 + (a)^2} dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$]

$I = \frac{1}{3}\tan^{-1}(\frac{x-5}{3}) + c$ [Using (i)]

$$I = \frac{1}{3}\tan^{-1}(\frac{x-5}{3}) + c$$

3. $\int \frac{1}{1+x-x^2} dx$

Solution:

Let $I = \int \frac{1}{1+x-x^2} dx = \int \frac{1}{-(x^2-x-1)} dx$

The above equation can be written as

$$= \int \frac{1}{-(x^2 - x - 1)} dx$$

Add and subtract ¼ to both sides

$$= \int \frac{1}{-(x^2 - x - \frac{1}{4} - 1 + \frac{1}{4})} dx$$

The above equation can be written as

$$= \int \frac{1}{-\left(\left(x - \frac{1}{2}\right)^2 - \frac{5}{4}\right)} dx$$

On computing we get

$$= \int \frac{1}{\left(\left(\frac{\sqrt{5}}{2}\right)^2 - \left(x - \frac{1}{2}\right)^2\right)} dx$$

$$I = \frac{1}{2 \times \frac{\sqrt{5}}{2}} \log\left|\frac{\frac{\sqrt{5}}{2} + (x - \frac{1}{2})}{\frac{\sqrt{5}}{2} - (x - \frac{1}{2})}\right| + c$$

By using, $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + c$

$$I = \frac{1}{\sqrt{5}}\log\left|\frac{\sqrt{5}+2x-1}{\sqrt{5}-2x+1}\right| + c$$

$$I = \frac{1}{\sqrt{5}}\log\left|\frac{\sqrt{5}-1+2x}{\sqrt{5}+1-2x}\right| + c$$

4. $\int \frac{1}{2x^2 - x - 1}dx$

Solution:

Let $I = \int \frac{1}{2x^2-x-1}dx$

Taking out ½ as common we get

$$= \frac{1}{2}\int \frac{1}{x^2 - \frac{x}{2} - \frac{1}{2}}dx$$

Again adding and subtracting $(¼)^2$ to the denominator we get

$$= \frac{1}{2}\int \frac{1}{x^2 - 2x \times \frac{1}{4} + \left(\frac{1}{4}\right)^2 - \left(\frac{1}{4}\right)^2 - \frac{1}{2}}dx$$

The above equation can be written as

$$= \frac{1}{2}\int \frac{1}{\left(x - \frac{1}{4}\right)^2 - \frac{9}{16}}dx$$

Let $\left(x - \frac{1}{4}\right) = t$ ...... (i)

$\Rightarrow dx = dt$

So, $I = \frac{1}{2}\int \frac{1}{t^2 - \left(\frac{3}{4}\right)^2}dt$

$$I = \frac{1}{2} \times \frac{1}{2 \times \frac{3}{4}} \log \left| \frac{t - \frac{3}{4}}{t + \frac{3}{4}} \right| + c$$

[since, $\int \frac{1}{x^2 - (a)^2} dx = \frac{1}{2 \times a} \log \left|\frac{x - a}{x + a}\right| + c$]

$$I = \frac{1}{3} \log \left| \frac{x - \frac{1}{4} - \frac{3}{4}}{x - \frac{1}{4} + \frac{3}{4}} \right| + c$$ [Using (i)]

$$I = \frac{1}{3} \log \left| \frac{2x - 2}{2x + 1} \right| + c$$

5. $\int \frac{1}{x^2 + 6x + 13} dx$

Solution:

In the denominator we have, and it can be written as

$x^2 + 6x + 13 = x^2 + 6x + 3^2 - 3^2 + 13$

The above equation can be written as

$= (x + 3)^2 + 4$

Substituting these values we get

So, $\int \frac{1}{x^2+6x+13} dx = \int \frac{1}{(x+3)^2+2^2} dx$

Let x+3 =t

Then dx = dt

$$\int \frac{1}{(t)^2 + 2^2} dt = \frac{1}{2} \tan^{-1} \frac{t}{2} + c$$

[since, $\int \frac{1}{x^2 + (a)^2} dx = \frac{1}{a} \tan^{-1} \left(\frac{x}{a}\right) + c$]

$$\frac{1}{2} \tan^{-1} \frac{x + 3}{2} + c$$

EXERCISE 19.16 PAGE NO: 19.90

Evaluate the following integrals:

1. $\int \frac{\sec^2 x}{1-\tan^2 x} dx$

Solution:

Let $I = \int \frac{\sec^2 x}{1-\tan^2 x} dx$

Let tan x = t..... (i)

$\Rightarrow \sec^2 x$ dx = dt

So, substituting these values in given equation we get

$$I = \int \frac{dt}{(1)^2 - t^2}$$

$$I = \frac{1}{2\times 1}\log\left|\frac{1+t}{1-t}\right| + c \text{ [since, } \int \frac{1}{a^2-(x)^2}dx = \frac{1}{2\times a}\log\left|\frac{a+x}{a-x}\right| + c]$$

$I = \frac{1}{2}\log\left|\frac{1+\tan x}{1-\tan x}\right| + c$ [Using (i)]

2. $\int \frac{e^x}{1+e^{2x}} dx$

Solution:

Let $I = \int \frac{e^x}{1+e^{2x}} dx$

Let e^x = t..... (i)

$\Rightarrow e^x$ dx = dt

So, substituting these values in given equation we get

$$I = \int \frac{dt}{(1)^2 + t^2}$$

$I = \tan^{-1} t + c$

$[\text{since}, \int \frac{1}{1+(x)^2} dx = \tan^{-1} x + c]$

$I = \tan^{-1}(e^x) + c$ [Using (i)]

3. $\int \frac{\cos x}{\sin^2 x + 4\sin x + 5} dx$

Solution:

Let $I = \int \frac{\cos x}{\sin^2 x+4\sin x+5} dx$

Let sin x = t..... (i)

⇒ Cos x dx = dt

So, $I = \int \frac{dt}{t^2+4t+5}$

Adding and subtracting 2^2 to the denominator we get

$= \int \frac{dt}{t^2 + (2t)(2) + 2^2 - 2^2 + 5}$

Above equation can be written as

$\int \frac{dt}{(t+2)^2 + 1}$

Again, let t + 2 = u .....(ii)

⇒ dt = du

$I = \int \frac{du}{u^2 + 1}$

$= \tan^{-1} u + c$

$[\text{since}, \int \frac{1}{1+(x)^2} dx = \tan^{-1} x + c]$

$= \tan^{-1}(\sin x + 2) + c$ [Using (i), (ii)]

4. $\int \frac{e^x}{e^{2x}+5e^x+6}\,dx$

Solution:

Let $I = \int \frac{e^x}{e^{2x}+5e^x+6}dx$

Let e^x = t..... (i)

$\Rightarrow e^x$ dx = dt

$= \int \frac{1}{t^2+5t+6}dt$

$= \int \frac{1}{t^2+2t\times\frac{5}{2}+\left(\frac{5}{2}\right)^2-\left(\frac{5}{2}\right)^2+6}dt$

$= \int \frac{1}{\left(t+\frac{5}{2}\right)^2-\frac{1}{4}}dt$

Let $t+\frac{5}{2}=u$ ..... (ii)

⇒ dt = du

So, substituting these values we get

$I = \int \frac{1}{u^2-\left(\frac{1}{2}\right)^2}du$

$I = \frac{1}{2\times\frac{1}{2}}\log\left|\frac{u-\frac{1}{2}}{u+\frac{1}{2}}\right|+c$

[since, $\int \frac{1}{x^2-(a)^2}dx = \frac{1}{2\times a}\log\left|\frac{x-a}{x+a}\right|+c$]

$$I = \log\left|\frac{2u-1}{2u+1}\right| + c$$

$$I = \log\left|\frac{2(t+\frac{5}{2})-1}{2(t+\frac{5}{2})+1}\right| + c \text{ [Using (ii)]}$$

$$I = \log\left|\frac{e^x+2}{e^x+3}\right| + c \text{ [Using (i)]}$$

5. $\int \frac{e^{3x}}{4e^{6x}-9}\,dx$

Solution:

Let $I = \int \frac{e^{3x}}{4e^{6x}-9}\,dx$

Let $e^{3x} = t$..... (i)

$\Rightarrow 3e^{3x}\,dx = dt$

$$I = \frac{1}{3}\int \frac{1}{4t^2-9}\,dt$$

Taking (¼) as common we get

$$= \frac{1}{12}\int \frac{1}{t^2-\frac{9}{4}}\,dt$$

The above equation can be written as

$$I = \frac{1}{12}\int \frac{1}{t^2-\left(\frac{3}{2}\right)^2}\,dt$$

$$I = \frac{1}{36}\log\left|\frac{t-\frac{3}{2}}{t+\frac{3}{2}}\right| + c$$

$$\left[\text{since, } \int \frac{1}{x^2-(a)^2}\,dx = \frac{1}{2\times a}\log\left|\frac{x-a}{x+a}\right| + c\right]$$

$$I = 1/36\ \log\left|\frac{2t-3}{2t+3}\right| + c$$

$$I = 1/36\ \log\left|\frac{2e^{3x}-3}{2e^{3x}+3}\right| + c \quad \text{[Using (i)]}$$

EXERCISE 19.17 PAGE NO: 19.93

Evaluate the following integrals:

1. $\int \frac{1}{\sqrt{2x - x^2}} dx$

Solution:

Let $I = \int \frac{1}{\sqrt{2x-x^2}} dx$

The above equation can be written as

$$= \int \frac{1}{\sqrt{-(x^2 - 2x)}} dx$$

Now by adding and subtracting 1^2 to the denominator we get

$$= \int \frac{1}{\sqrt{-[x^2 - 2x(1) + 1^2 - 1^2]}} dx$$

On simplifying

$$= \int \frac{1}{\sqrt{-[(x-1)^2 - 1]}} dx$$

The above equation becomes

$$= \int \frac{1}{\sqrt{1 - (x-1)^2}} dx$$

Let (x - 1) = t and dx = dt

So, $I = \int \frac{1}{\sqrt{1-t^2}} dt$

$$= \sin^{-1} t + c \text{ [since } \int \frac{1}{\sqrt{1 - x^2}} dx = \sin^{-1} x + c \text{]}$$

$$I = \sin^{-1}(x-1) + c$$

2. $\int \frac{1}{\sqrt{8+3x-x^2}}\,dx$

Solution:

The denominator of given question $8 + 3x – x^2$ by adding and subtracting (9/4) can be written as

$$8-\left(x^2-3x+\frac{9}{4}-\frac{9}{4}\right)$$

Therefore

$$8-\left(x^2-3x+\frac{9}{4}-\frac{9}{4}\right)$$

The above equation can be written as

$$=\frac{41}{4}-\left(x-\frac{3}{2}\right)^2$$

Substituting these values in given question we get

$$\int \frac{1}{\sqrt{8+3x-x^2}}dx=\int \frac{1}{\sqrt{\frac{41}{4}-\left(x-\frac{3}{2}\right)^2}}dx$$

Let x-3/2=t

dx = dt

$$\int \frac{1}{\sqrt{\frac{41}{4}-\left(x-\frac{3}{2}\right)^2}}dx=\int \frac{1}{\sqrt{\left(\frac{\sqrt{41}}{2}\right)^2-t^2}}dt$$

$$=\sin^{-1}\left(\frac{t}{\frac{\sqrt{41}}{2}}\right)+c$$

[since $\int \frac{1}{\sqrt{a^2-x^2}}dx=\sin^{-1}\left(\frac{x}{a}\right)+c$]

$$= \sin^{-1}\left(\frac{x - \frac{3}{2}}{\frac{\sqrt{41}}{2}}\right) + c$$

$$= \sin^{-1}\left(\frac{2x - 3}{\sqrt{41}}\right) + c$$

3. $\int \frac{1}{\sqrt{5 - 4x - 2x^2}}\,dx$

Solution:

Let I = $\int \frac{1}{\sqrt{5-4x-2x^2}}dx$

Now taking out 2 as common from the denominator we get

$$= \int \frac{1}{\sqrt{-2\left[x^2 + 2x - \frac{5}{2}\right]}}dx$$

By adding and subtracting 1^2 to the denominator we get

$$= \frac{1}{\sqrt{2}}\int \frac{1}{\sqrt{-\left[x^2 + 2x + (1)^2 - (1)^2 - \frac{5}{2}\right]}}dx$$

By computing

$$= \frac{1}{\sqrt{2}}\int \frac{1}{\sqrt{-\left[(x+1)^2 - \frac{7}{2}\right]}}dx$$

$$= \frac{1}{\sqrt{2}}\int \frac{1}{\sqrt{\frac{7}{2} - (x+1)^2}}dx$$

Let (x + 1) = t

Differentiating both sides, we get, dx = dt

So, $I = \frac{1}{\sqrt{2}}\int \frac{1}{\sqrt{\left(\sqrt{\left(\frac{7}{2}\right)}\right)^2 - t^2}} dt$

$$= \frac{1}{\sqrt{2}}\sin^{-1}\left(\frac{t}{\sqrt{\frac{7}{2}}}\right) + c$$

[since $\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\left(\frac{x}{a}\right) + c$]

$$I = \frac{1}{\sqrt{2}}\sin^{-1}\left(\sqrt{\frac{2}{7}} \times (x+1)\right) + c$$

4. $\int \frac{1}{\sqrt{3x^2 + 5x + 7}} dx$

Solution:

Let $I = \int \frac{1}{\sqrt{3x^2+5x+7}} dx$

Taking 1/√3 as common from the denominator we get

$$= \frac{1}{\sqrt{3}}\int \frac{1}{\sqrt{x^2 + \frac{5}{3}x + \frac{7}{3}}} dx$$

Now by adding and subtracting $(5/6)^2$ to the denominator complete perfect square, we get

$$= \frac{1}{\sqrt{3}}\int \frac{1}{\sqrt{x^2 + 2x\left(\frac{5}{6}\right) + \left(\frac{5}{6}\right)^2 - \left(\frac{5}{6}\right)^2 + \frac{7}{3}}} dx$$

The above equation can be written as

$$= \frac{1}{\sqrt{3}} \int \frac{1}{\sqrt{\left(x + \frac{5}{6}\right)^2 + \frac{59}{36}}} dx$$

let $\left(x + \frac{5}{6}\right) = t$

dx = dt

$$I = \frac{1}{\sqrt{3}} \int \frac{1}{\sqrt{t^2 + \left(\frac{\sqrt{59}}{6}\right)^2}} dt$$

$$= \frac{1}{\sqrt{3}} \log\left|t + \sqrt{t^2 + \left(\frac{\sqrt{59}}{6}\right)}\right| + c \text{ [since } \int \frac{1}{\sqrt{x^2 + a^2}} dx = \log\left|x + \sqrt{x^2 + a^2}\right| + c$$

On simplification we get

$$I = \frac{1}{\sqrt{3}} \log\left|x + \frac{5}{6} + \sqrt{\left(x + \frac{5}{6}\right)^2 + \left(\frac{\sqrt{59}}{6}\right)^2}\right| + c$$

$$I = \frac{1}{\sqrt{3}} \log\left|x + \frac{5}{6} + \sqrt{x^2 + \frac{5x}{3} + \frac{7}{3}}\right| + c$$

EXERCISE 19.18 PAGE NO: 19.98

Evaluate the following integrals:

1. $\int \frac{x}{\sqrt{x^4 + a^4}} dx$

Solution:

The given equation can be written as

$$\int \frac{x}{\sqrt{x^4 + a^4}} dx = \int \frac{x}{\sqrt{(x^2)^2 + (a^2)^2}} dx$$

Let $x^2 = t$, so $2x\, dx = dt$

Or, $x\, dx = dt/2$

Hence, $\int \frac{x}{\sqrt{(x^2)^2 + (a^2)^2}} dx = \int \frac{1}{\sqrt{t^2 + (a^2)^2}} \frac{dt}{2} = \frac{1}{2} \int \frac{1}{\sqrt{t^2 + (a^2)^2}} dt$

Since, $\int \frac{1}{\sqrt{(x^2 + a^2)}} dx = \log|x + \sqrt{(x^2 + a^2)}| + c$

Hence, $\frac{1}{2} \int \frac{1}{\sqrt{t^2 + (a^2)^2}} dt = \frac{1}{2} \log|t + \sqrt{t^2 + (a^2)^2}| + c$

Put $t = x^2$

$= \frac{1}{2} \log| x^2 + \sqrt{(x^2)^2 + (a^2)^2}| + c$

$= \frac{1}{2} \log| x^2 + \sqrt{x^4 + a^4}| + c$

2. $\int \frac{\sec^2 x}{\sqrt{4 + \tan^2 x}} dx$

Solution:

Let tan x = t

Then $dt = \sec^2 x\, dx$

Therefore, $\int \frac{\sec^2 x}{\sqrt{4+\tan^2 x}} dx = \int \frac{dt}{\sqrt{2^2+t^2}}$

Since, $\int \frac{1}{\sqrt{(x^2+a^2)}} dx = \log|x + \sqrt{(x^2 + a^2)}| + c$

Hence, $\int \frac{dt}{\sqrt{2^2+t^2}} = \log|t + \sqrt{t^2 + 2^2}| + c$

$= \log|\tan x + \sqrt{\tan^2 x + 4}| + c$

3. $\int \frac{e^x}{\sqrt{16 - e^{2x}}} dx$

Solution:

Let $e^x = t$

Then we have, $e^x\, dx = dt$

Substituting these values,

Therefore, $\int \frac{e^x}{\sqrt{16-e^{2x}}} dx = \int \frac{dt}{\sqrt{4^2-t^2}}$

Since we have, $\int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1}\left(\frac{x}{a}\right) + c$

Hence, $\int \frac{dt}{\sqrt{4^2-t^2}} = \sin^{-1}\left(\frac{e^x}{4}\right) + c$

4. $\int \frac{\cos x}{\sqrt{4 + \sin^2 x}} dx$

Solution:

Let sin x = t

Then $dt = \cos x\, dx$

Now substituting these values we get

Hence, $\int \frac{\cos x}{\sqrt{4+\sin^2 x}} dx = \int \frac{dt}{\sqrt{2^2+t^2}}$

Since we have, $\int \frac{1}{\sqrt{(x^2+a^2)}} dx = \log[x + \sqrt{(x^2 + a^2)}] + c$

Therefore, $\int \frac{dt}{\sqrt{2^2+t^2}} = \log\left| t + \sqrt{t^2 + 2^2} \right| + c$

$= \log\left| t + \sqrt{t^2 + 2^2} \right| + c = \log\left| \sin x + \sqrt{\sin^2 x + 4} \right| + c$

5. $\int \frac{\sin x}{\sqrt{4\cos^2 x - 1}} dx$

Solution:

Let

$2\cos x = t$

Then $dt = -2\sin x\, dx$

Or, $\sin x\, dx = -\frac{dt}{2}$

Then substituting these values we get,

Therefore, $\int \frac{\sin x}{\sqrt{4\cos^2 x - 1}} dx = \int -\frac{dt}{2\sqrt{(t^2-1^2)}}$

Since, $\int \frac{1}{\sqrt{(x^2-a^2)}} dx = \log[x + \sqrt{(x^2 - a^2)}] + c$

Therefore, $\int -\frac{dt}{2\sqrt{(t^2-1^2)}} = -\frac{1}{2}\log\left|t + \sqrt{t^2 - 1}\right| + c$

On integrating we get

$= -\frac{1}{2}\log\left|2\cos x + \sqrt{4\cos^2 x - 1}\right| + c$

6. $\int \frac{x}{\sqrt{4-x^4}}\,dx$

Solution:

Let $x^2 = t$

$2x\,dx = dt$ or $x\,dx = dt/2$

Now substituting these values in the given equation we get

Hence, $\int \frac{x}{\sqrt{4-x^4}}dx = \int \frac{dt}{2\left(\sqrt{2^2-t^2}\right)}$

Since we have, $\int \frac{1}{\sqrt{a^2-x^2}}dx = \sin^{-1}\left(\frac{x}{a}\right) + c$

So, $\int \frac{dt}{2\left(\sqrt{2^2-t^2}\right)} = \frac{1}{2}\sin^{-1}\left(\frac{t}{2}\right) + c$

Put $t = x^2$

$= \frac{1}{2}\sin^{-1}\left(\frac{t}{2}\right) + c = \frac{1}{2}\sin^{-1}\left(\frac{x^2}{2}\right) + c$

7. $\int \frac{1}{x\sqrt{4-9(\log x)^2}}\,dx$

Solution:

Let 3 log x = t

We have $d(\log x) = 1/x$

Hence, $d(3\log x) = dt = 3/x\,dx$

Or $1/x\,dx = dt/3$

Hence, $\int \frac{1}{x\sqrt{4-9(\log x)^2}}dx = \int \frac{1}{3}\frac{dt}{\sqrt{2^2-t^2}}$

Since we have, $\int \frac{1}{\sqrt{a^2-x^2}}dx = \sin^{-1}\left(\frac{x}{a}\right) + c$

Hence, $\int \frac{1}{3} \frac{dt}{\sqrt{2^2 - t^2}} = \frac{1}{3} \sin^{-1}\left(\frac{t}{2}\right) + c$

Put t = 3 log x

$= \frac{1}{3} \sin^{-1}\left(\frac{t}{2}\right) + c = \frac{1}{3} \sin^{-1}\left(\frac{3\log x}{2}\right) + c$

8. $\int \frac{\sin 8x}{\sqrt{9 + \sin^4 4x}} dx$

Solution:

dt = 2sin 4x cos 4x × 4 dx

We know sin 2x = 2 sin 2x cos 2x

Therefore, dt = 4 sin 8x dx

Or, sin 8x dx = dt/4

$$\int \frac{\sin 8x}{\sqrt{9 + \sin^4 4x}} dx = \frac{1}{4} \int \frac{dt}{\sqrt{3^2 + t^2}}$$

Since we have, $\int \frac{1}{\sqrt{(x^2 + a^2)}} dx = \log|x + \sqrt{(x^2 + a^2)}| + c$

$= \frac{1}{4} \int \frac{dt}{\sqrt{3^2 + t^2}} = \frac{1}{4} \log|t + \sqrt{t^2 + 3^2}| + c$

$= \frac{1}{4} \log[\sin^2 4x + \sqrt{9 + \sin^4 4x} + c$

9. $\int \frac{\cos 2x}{\sqrt{\sin^2 2x + 8}} dx$

Solution:

Let t = sin2x

dt = 2 cos 2x dx

Cos 2x dx = dt/2

$\int \frac{\cos 2x}{\sqrt{\sin^2 2x + 8}} dx = \frac{1}{2} \int dt/\sqrt{(t^2 + (2\sqrt{2})^2}$

Since we have, $\int \frac{1}{\sqrt{(x^2 + a^2)}} dx = \log|x + \sqrt{(x^2 + a^2)}| + c$

$= \frac{1}{2} \int dt/\sqrt{(t^2 + (2\sqrt{2})^2} = \frac{1}{2}\log\left|t + \sqrt{t^2 + 8}\right| + c$

$= \frac{1}{2}\log|t + \sqrt{t^2 + 8}| + c = \frac{1}{2}\log\left|\sin 2x + \sqrt{\sin^2 2x + 8}\right| + c$

EXERCISE 19.19 PAGE NO: 19.104

Evaluate the following integrals:

1. $\int \frac{x}{x^2+3x+2} dx$

Solution:

Let

$I = \int \frac{x}{x^2+3x+2} dx$

As we can see that there is a term of x in numerator and derivative of x^2 is also 2x. So there is a chance that we can make substitution for $x^2 + 3x + 2$ and it can be reduced to a fundamental integration.

As, $\frac{d}{dx}(x^2+3x+2) = 2x+3$

∴ Let, x = A (2x + 3) + B

⇒ x = 2Ax + 3A + B

On comparing both sides

We have, 2A = 1 ⇒ A = 1/2

3A + B = 0 ⇒ B = –3A = –3/2

Hence,

$I = \int \frac{\frac{1}{2}(2x+3)-\frac{3}{2}}{x^2+3x+2} dx$

$\therefore I = \frac{1}{2}\int \frac{2x+3}{x^2+3x+2} dx - \frac{3}{2}\int \frac{1}{x^2+3x+2} dx$

Let, $I_1 = \frac{1}{2}\int \frac{2x+3}{x^2+3x+2} dx$ and $I_2 = \frac{3}{2}\int \frac{1}{x^2+3x+2} dx$

Now, $I = I_1 - I_2$ ....equation 1

We will solve I_1 and I_2 individually.

As, $I_1 = \frac{1}{2}\int \frac{2x+3}{x^2+3x+2} dx$

Let $u = x^2 + 3x + 2 \Rightarrow du = (2x + 3)\,dx$

$\therefore I_1$ reduces to $\frac{1}{2}\int \frac{du}{u}$

Hence,

$I_1 = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\log|u| + C$

On substituting value of u, we have:

$I_1 = \frac{1}{2}\log|x^2 + 3x + 2| + C$ .... Equation 2

As, $I_2 = \frac{3}{2}\int \frac{1}{x^2+3x+2} dx$ and we don't have any derivative of function present in denominator. ∴ we will use some special integrals to solve the problem.

As denominator doesn't have any square root term. So one of the following two integrals will use to solve the problem.

i) $\int \frac{1}{x^2 - a^2} dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C$ ii) $\int \frac{1}{x^2 + a^2} dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$

Now we have to reduce I_2 such that it matches with any of above two forms.

We will make to create a complete square so that no individual term of x is seen in denominator.

$\therefore I_2 = \frac{3}{2}\int \frac{1}{x^2+3x+2} dx$

$\Rightarrow I_2 = \frac{3}{2}\int \frac{1}{\{x^2+2\left(\frac{3}{2}\right)x+\left(\frac{3}{2}\right)^2\}+2-\left(\frac{3}{2}\right)^2} dx$

Using: $a^2 + 2ab + b^2 = (a + b)^2$

We have:

$I_2 = \frac{3}{2}\int \frac{1}{\left(x+\frac{3}{2}\right)^2-\left(\frac{1}{2}\right)^2} dx$

I_2 matches with $\int \frac{1}{x^2-a^2}dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C$

$\therefore I_2 = \frac{3}{2}\left\{\frac{1}{2(\frac{1}{2})}\log\left|\frac{(x+\frac{3}{2})-\frac{1}{2}}{(x+\frac{3}{2})+\frac{1}{2}}\right| + C\right\}$

$\Rightarrow I_2 = \frac{3}{2}\log\left|\frac{2x+3-1}{2x+3+1}\right| + C$

$\Rightarrow I_2 = \frac{3}{2}\log\left|\frac{2x+2}{2x+4}\right| + C = \frac{3}{2}\log\left|\frac{x+1}{x+2}\right| + C$ ... equation 3

From equation 1:

$I = I_1 - I_2$

Using equation 2 and equation 3:

$I = \frac{1}{2}\log|x^2 + 3x + 2| - \frac{3}{2}\log\left|\frac{x+1}{x+2}\right| + C$

2. $\int \frac{x+1}{x^2+x+3}dx$

Solution:

$I = \int \frac{x+1}{x^2+x+3}dx$

As we can see that there is a term of x in numerator and derivative of x^2 is also 2x. So there is a chance that we can make substitution for $x^2 + x + 3$ and it can be reduced to a fundamental integration.

As, $\frac{d}{dx}(x^2 + x + 3) = 2x + 1$

$\therefore$ Let, $x = A(2x + 1) + B$

$\Rightarrow x = 2Ax + A + B$

On comparing both sides

We have,

$2A = 1 \Rightarrow A = 1/2$

$A + B = 0 \Rightarrow B = -A = -1/2$

Hence,

$I = \int \frac{\frac{1}{2}(2x+1)-\frac{1}{2}}{x^2+x+3} dx$

$\therefore I = \frac{1}{2}\int \frac{2x+1}{x^2+x+3} dx - \frac{1}{2}\int \frac{1}{x^2+x+3} dx$

Let, $I_1 = \frac{1}{2}\int \frac{2x+1}{x^2+x+3} dx$ and $I_2 = \frac{1}{2}\int \frac{1}{x^2+x+3} dx$

Now, $I = I_1 - I_2$ .... Equation 1

We will solve I_1 and I_2 individually.

As $I_1 = \frac{1}{2}\int \frac{2x+1}{x^2+x+3} dx$

Let $u = x^2 + x + 3 \Rightarrow du = (2x + 1)\, dx$

$\therefore I_1$ reduces to $\frac{1}{2}\int \frac{du}{u}$

Hence,

$I_1 = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\log|u| + C$

On substituting the value of u, we have:

$I_1 = \frac{1}{2}\log|x^2 + x + 3| + C$ ....equation 2

As, $I_2 = \frac{1}{2}\int \frac{1}{x^2+x+3} dx$ and we don't have any derivative of function present in denominator. ∴ we will use some special integrals to solve the problem.

As denominator doesn't have any square root term. So one of the following two integrals will help to solve the problem.

i) $\int \frac{1}{x^2 - a^2} dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C$ ii) $\int \frac{1}{x^2 + a^2} dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$

Now we have to reduce I_2 such that it matches with any of above two forms.

We will make to create a complete square so that no individual term of x is seen in denominator.

$\therefore I_2 = \frac{1}{2}\int \frac{1}{x^2+x+3}dx$

$\Rightarrow I_2 = \frac{1}{2}\int \frac{1}{\{x^2+2(\frac{1}{2})x+(\frac{1}{2})^2\}+3-(\frac{1}{2})^2}\,dx$

Using $a^2 + 2ab + b^2 = (a + b)^2$

We have

$I_2 = \frac{1}{2}\int \frac{1}{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{11}}{2}\right)^2}\,dx$

I_2 matches with $\int \frac{1}{x^2+a^2}dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+C$

$\therefore I_2 = \frac{1}{2}\left\{\frac{1}{\left(\frac{\sqrt{11}}{2}\right)}\tan^{-1}\left(\frac{x+\frac{1}{2}}{\frac{\sqrt{11}}{2}}\right)+C\right\}$

$\Rightarrow I_2 = \frac{1}{\sqrt{11}}\tan^{-1}\left(\frac{2x+1}{\sqrt{11}}\right)+C$ ... equation 3

From equation 1 we have

$I = I_1 - I_2$

Using equation 2 and equation 3:

$I = \frac{1}{2}\log|x^2+x+3| - \frac{1}{\sqrt{11}}\tan^{-1}\left(\frac{2x+1}{\sqrt{11}}\right)+C$

3. $\int \frac{x-3}{x^2+2x-4}\,dx$

Solution:

Let $I = \int \frac{x-3}{x^2+2x-4}dx$

As we can see that there is a term of x in numerator and derivative of x^2 is also 2x. So there is a chance that we can make substitution for $x^2 + 2x - 4$ and it can be reduced to a fundamental integration.

As, $\frac{d}{dx}(x^2 + 2x - 4) = 2x + 2$

∴ Let, $x - 3 = A(2x + 2) + B$

$\Rightarrow x - 3 = 2Ax + 2A + B$

On comparing both sides we have, $2A = 1 \Rightarrow A = 1/2$

$2A + B = -3 \Rightarrow B = -3 - 2A = -4$

Hence, $I = \int \frac{\frac{1}{2}(2x+2)-4}{x^2+2x-4}dx$

$\therefore I = \frac{1}{2}\int \frac{2x+2}{x^2+2x-4}dx - 4\int \frac{1}{x^2+2x-4}dx$

Let, $I_1 = \frac{1}{2}\int \frac{2x+2}{x^2+2x-4}dx$ and $I_2 = \int \frac{1}{x^2+2x-4}dx$

Now, $I = I_1 - 4I_2$ ....equation 1

We will solve I_1 and I_2 individually.

As, $I_1 = \frac{1}{2}\int \frac{2x+2}{x^2+2x-4}dx$

Let $u = x^2 + 2x - 4 \Rightarrow du = (2x + 2)\,dx$

$\therefore I_1$ reduces to $\frac{1}{2}\int \frac{du}{u}$

Hence, $I_1 = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\log|u| + C$

On substituting value of u, we have:

$I_1 = \frac{1}{2}\log|x^2 + 2x - 4| + C$ .... Equation 2

As, $I_2 = \int \frac{1}{x^2+2x-4}dx$ and we don't have any derivative of function present in denominator. ∴ we will use some special integrals to solve the problem.

As denominator doesn't have any square root term. So one of the following two integrals will solve the problem.

i) $\int \frac{1}{x^2 - a^2} dx = \frac{1}{2a} \log\left|\frac{x-a}{x+a}\right| + C$ ii) $\int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + C$

Now we have to reduce I_2 such that it matches with any of above two forms.

We will make to create a complete square so that no individual term of x is seen in denominator.

$\therefore I_2 = \int \frac{1}{x^2+2x-4} dx \Rightarrow I_2 = \int \frac{1}{\{x^2+2(1)x+(1)^2\} - 4 - (1)^2} dx$

Using $a^2 + 2ab + b^2 = (a + b)^2$

We have:

$I_2 = \int \frac{1}{(x+1)^2 - (\sqrt{5})^2} dx$

I_2 matches with $\int \frac{1}{x^2-a^2} dx = \frac{1}{2a} \log\left|\frac{x-a}{x+a}\right| + C$

$\therefore I_2 = \frac{1}{2\sqrt{5}} \log\left|\frac{x+1-\sqrt{5}}{x+1+\sqrt{5}}\right| + C$ ... equation 3

From equation 1 we have

$I = I_1 - 4I_2$

Using equation 2 and equation 3:

$I = \frac{1}{2} \log|x^2 + 2x - 4| - 4\left(\frac{1}{2\sqrt{5}} \log\left|\frac{x+1-\sqrt{5}}{x+1+\sqrt{5}}\right|\right) + C$

$I = \frac{1}{2} \log|x^2 + 2x - 4| - \frac{2}{\sqrt{5}} \log\left|\frac{x+1-\sqrt{5}}{x+1+\sqrt{5}}\right| + C$

4. $\int \frac{2x - 3}{x^2 + 6x + 13} dx$

Solution:

Let $I = \int \frac{2x-3}{x^2+6x+13} dx$

As we can see that there is a term of x in numerator and derivative of x^2 is also 2x. So there is a chance that we can make a substitution for $x^2 + 6x + 13$ and It can be reduced to a fundamental integration.

As $\frac{d}{dx}(x^2 + 6x + 13) = 2x + 6$

∴ Let, $2x - 3 = A(2x + 6) + B$

⇒ $2x - 3 = 2Ax + 6A + B$

On comparing both sides

We have, $2A = 2 \Rightarrow A = 1$

$6A + B = -3 \Rightarrow B = -3-6A = -9$

Hence, $I = \int \frac{(2x+6)-9}{x^2+6x+13} dx$

∴ $I = \int \frac{2x+6}{x^2+6x+13} dx - 9\int \frac{1}{x^2+6x+13} dx$

Let, $I_1 = \int \frac{2x+6}{x^2+6x+13} dx$ and $I_2 = \int \frac{1}{x^2+6x+13} dx$

Now, $I = I_1 - 9I_2$ .... Equation 1

We will solve I_1 and I_2 individually.

As, $I_1 = \int \frac{2x+6}{x^2+6x+13} dx$

Let $u = x^2 + 6x + 13 \Rightarrow du = (2x + 6)\, dx$

∴ I_1 reduces to $\int \frac{du}{u}$

Hence, $I_1 = \int \frac{du}{u} = \log|u| + C$

On substituting value of u, we have

$I_1 = \log|x^2 + 6x + 13| + C$ ....equation 2

As, $I_2 = \int \frac{1}{x^2+6x+13} dx$ and we don't have any derivative of function present in denominator. ∴ we will use some special integrals to solve the problem.

As denominator doesn't have any square root term. So one of the following two integrals will solve the problem.

i) $\int \frac{1}{x^2-a^2}dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C$ ii) $\int \frac{1}{x^2+a^2}dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$

Now we have to reduce I_2 such that it matches with any of above two forms.

We will make to create a complete square so that no individual term of x is seen in denominator.

$\therefore I_2 = \int \frac{1}{x^2+6x+13}dx$

$\Rightarrow I_2 = \int \frac{1}{\{x^2+2(3)x+(3)^2\}+13-(3)^2}dx$

Using $a^2 + 2ab + b^2 = (a + b)^2$

We have $I_2 = \int \frac{1}{(x+3)^2+(2)^2}dx$

I_2 matches with $\int \frac{1}{x^2+a^2}dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$

$\therefore I_2 = \frac{1}{2}\tan^{-1}\left(\frac{x+3}{2}\right) + C$ ... equation 3

From equation 1

$I = I_1 - 9I_2$

Using equation 2 and equation 3:

$I = \log|x^2+6x+13| - 9\,\frac{1}{2}\tan^{-1}\left(\frac{x+3}{2}\right) + C$

$I = \log|x^2+6x+13| - \frac{9}{2}\tan^{-1}\left(\frac{x+3}{2}\right) + C$

5. $\int \frac{x-1}{3x^2-4x+3}dx$

Solution:

Let I = $\int \frac{x-1}{3x^2-4x+3} dx$

As we can see that there is a term of x in numerator and derivative of x^2 is also 2x. So there is a chance that we can make substitution for $3x^2 - 4x + 3$ and I can be reduced to a fundamental integration.

As, $\frac{d}{dx}(3x^2 - 4x + 3) = 6x - 4$

∴ Let, x – 1 = A (6x – 4) + B

⇒ x – 1 = 6Ax – 4A + B

On comparing both sides

We have, 6A = 1 ⇒ A = 1/6

–4A + B = –1 ⇒ B = –1+4A = –2/6 = –1/3

Hence, I = $\int \frac{\frac{1}{6}(6x-4)-\frac{1}{3}}{3x^2-4x+3} dx$

∴ I = $\frac{1}{6}\int \frac{6x-4}{3x^2-4x+3} dx - \frac{1}{3}\int \frac{1}{3x^2-4x+3} dx$

Let, $I_1 = \frac{1}{6}\int \frac{6x-4}{3x^2-4x+3} dx$ and $I_2 = \frac{1}{3}\int \frac{1}{3x^2-4x+3} dx$

Now, $I = I_1 - I_2$ ....equation 1

We will solve I_1 and I_2 individually.

As, $I_1 = \frac{1}{6}\int \frac{6x-4}{3x^2-4x+3} dx$

Let $u = 3x^2 - 4x + 3 \Rightarrow du = (6x - 4)\, dx$

∴ I_1 reduces to $\frac{1}{6}\int \frac{du}{u}$

Hence,

$I_1 = \frac{1}{6}\int \frac{du}{u} = \frac{1}{6}\log|u| + C$

On substituting value of u, we have:

$I_1 = \frac{1}{6}\log|3x^2 - 4x + 3| + C$ ....equation 2

As, $I_2 = \frac{1}{3}\int \frac{1}{3x^2-4x+3}dx$ and we don't have any derivative of function present in denominator. ∴ we will use some special integrals to solve the problem.

As denominator doesn't have any square root term. So one of the following two integrals will solve the problem.

i) $\int \frac{1}{x^2 - a^2}dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C$ ii) $\int \frac{1}{x^2 + a^2}dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$

Now we have to reduce I_2 such that it matches with any of above two forms.

We will make to create a complete square so that no individual term of x is seen in the denominator

$\therefore I_2 = \frac{1}{9}\int \frac{1}{x^2 - \frac{4}{3}x + 1}dx$ {on taking 3 common from denominator}

$\Rightarrow I_2 = \frac{1}{9}\int \frac{1}{\{x^2 - 2\left(\frac{2}{3}\right)x + \left(\frac{2}{3}\right)^2\} + 1 - \left(\frac{2}{3}\right)^2}dx$

Using $a^2 + 2ab + b^2 = (a + b)^2$

We have $I_2 = \frac{1}{9}\int \frac{1}{\left(x - \frac{2}{3}\right)^2 + \left(\frac{\sqrt{5}}{3}\right)^2}dx$

I_2 matches with $\int \frac{1}{x^2+a^2}dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$

$\therefore I_2 = \frac{1}{9}\frac{1}{\frac{\sqrt{5}}{3}}\tan^{-1}\left(\frac{x - \frac{2}{3}}{\frac{\sqrt{5}}{3}}\right) + C$

$\therefore I_2 = \frac{3}{9\sqrt{5}}\tan^{-1}\left(\frac{3x-2}{\sqrt{5}}\right) + C = \frac{1}{3\sqrt{5}}\tan^{-1}\left(\frac{3x-2}{\sqrt{5}}\right) + C$ ...equation 3

From equation 1:

$I = I_1 - I_2$

Using equation 2 and equation 3:

$I = \frac{1}{6}\log|3x^2 - 4x + 3| - \frac{\sqrt{5}}{15}\tan^{-1}\left(\frac{3x-2}{\sqrt{5}}\right) + C$

EXERCISE 19.20 PAGE NO: 19.106

Evaluate the following integrals:

1. $\int \frac{x^2+x+1}{x^2-x} dx$

Solution:

Given $I = \int \frac{x^2+x+1}{x^2-x} dx$

Expressing the integral $\int \frac{P(x)}{ax^2+bx+c} dx = \int Q(x)\,dx + \int \frac{R(x)}{ax^2+bx+c} dx$

$\Rightarrow \int \frac{x^2+x+1}{(x-1)x} dx$

$\Rightarrow \int (\frac{2x+1}{(x-1)x} + 1) dx$

$\Rightarrow \int \frac{2x+1}{(x-1)x} dx + \int 1\ dx$

Consider $\int \frac{2x+1}{(x-1)x} dx$

By partial fraction decomposition,

$\Rightarrow \frac{2x+1}{(x-1)x} = \frac{A}{x-1} + \frac{B}{x}$

$\Rightarrow 2x + 1 = Ax + B(x - 1)$

$\Rightarrow 2x + 1 = Ax + Bx - B$

$\Rightarrow 2x + 1 = (A + B)\,x - B$

$\therefore B = -1$ and $A + B = 2$

$\therefore A = 2 + 1 = 3$

Thus, $\Rightarrow \frac{2x+1}{(x-1)x} = \frac{3}{x-1} - \frac{1}{x}$

$\Rightarrow \int (\frac{3}{x-1} - \frac{1}{x})dx$

$\Rightarrow 3\int \frac{1}{x-1}dx - \int \frac{1}{x}\ dx$

Consider $\int \frac{1}{x-1}dx$

Substitute u = x – 1 → dx = du.

$\Rightarrow \int \frac{1}{x-1}dx\ = \int \frac{1}{u}du$

We know that $\int \frac{1}{x}dx = \log|x| + c$

$\therefore \int \frac{1}{u}du\ = \log|u| = \log|x-1|$

Then,

$\Rightarrow 3\int \frac{1}{x-1}dx - \int \frac{1}{x}\ dx = 3(\log|x-1|) - \int \frac{1}{x}\ dx$

$= 3(\log|x-1|) - \log|x|$

$\therefore \int \frac{2x+1}{(x-1)x}dx = 3(\log|x-1|) - \log|x|$

Then,

$\Rightarrow \int \frac{2x+1}{(x-1)x}dx + \int 1\ dx = 3(\log|x-1|) - \log|x| + \int 1\ dx$

We know that $\int 1\ dx = x + c$

$\Rightarrow \int \frac{2x+1}{(x-1)x}dx + \int 1\ dx = 3(\log|x-1|) - \log|x| + x + c$

$\therefore I = \int \frac{x^2+x+1}{x^2-x}dx = -\log|x| + x + 3(\log|x-1|) + c$

2. $\int \frac{x^2+x-1}{x^2+x-6}dx$

Solution:

Consider $I=\int \frac{x^2+x-1}{x^2+x-6}dx$

Expressing the integral $\int \frac{P(x)}{ax^2+bx+c}dx=\int Q(x)\,dx+\int \frac{R(x)}{ax^2+bx+c}dx$

Let $x^2 + x - 1 = x^2 + x - 6 + 5$

$$\Rightarrow \int \frac{x^2+x-1}{x^2+x-6}dx=\int \left(\frac{x^2+x-6}{x^2+x-6}+\frac{5}{x^2+x-6}\right)dx$$

$$=\int \left(\frac{5}{x^2+x-6}+1\right)dx$$

$$=5\int \left(\frac{1}{x^2+x-6}\right)dx+\int 1\,dx$$

Consider $\int \frac{1}{x^2+x-6}dx$

Factorizing the denominator,

$$\Rightarrow \int \frac{1}{x^2+x-6}dx=\int \frac{1}{(x-2)(x+3)}dx$$

By partial fraction decomposition,

$$\Rightarrow \frac{1}{(x-2)(x+3)}=\frac{A}{x-2}+\frac{B}{x+3}$$

$\Rightarrow 1 = A(x + 3) + B(x - 2)$

$\Rightarrow 1 = Ax + 3A + Bx - 2B$

$\Rightarrow 1 = (A + B)x + (3A - 2B)$

$\Rightarrow$ Then $A + B = 0$ … (1)

And $3A - 2B = 1$ … (2)

Solving (1) and (2),

$2 \times (1) \rightarrow 2A + 2B = 0$

$1 \times (2) \rightarrow 3A - 2B = 1$

$5A = 1$

$\therefore A = 1/5$

Substituting A value in (1),

⇒ A + B = 0

⇒ 1/5 + B = 0

∴ B = -1/5

Thus, $\frac{1}{(x-2)(x+3)}=\frac{1}{5(x-2)}-\frac{1}{5(x+3)}$

$=\frac{1}{5}\int\frac{1}{x-2}dx-\frac{1}{5}\int\frac{1}{x+3}dx$

Let x – 2 = u → dx = du

And x + 3 = v → dx = dv.

$\Rightarrow\frac{1}{5}\int\frac{1}{u}du-\frac{1}{5}\int\frac{1}{v}dv$

We know that $\int\frac{1}{x}dx=\log|x|+c$

$\Rightarrow\frac{1}{5}\log|u|-\frac{1}{5}\log|v|$

$\Rightarrow\frac{1}{5}\log|x-2|-\frac{1}{5}\log|x+3|$

$\Rightarrow\frac{1}{5}(\log|x-2|-\log|x+3|)$

Then,

$\Rightarrow 5\int\left(\frac{1}{x^2+x-6}\right)dx+\int 1\,dx=5\left(\frac{1}{5}(\log|x-2|-\log|x+3|)\right)+\int 1\,dx$

We know that $\int 1\ dx=x+c$

$\Rightarrow(\log|x-2|-\log|x+3|)+x+c$

$\therefore I=\int\frac{x^2+x-1}{x^2+x-6}dx=-\log|x+3|+x+\log|x-2|+c$

Or I = log|(x - 2)/(x + 3)| + x + c

3. $\int \frac{(1-x^2)}{x(1-2x)} dx$

Solution:

Given $I = \int \frac{1-x^2}{(1-2x)x} dx$

Rewriting, we get $\int \frac{x^2-1}{x(2x-1)} dx$

Expressing the integral $\int \frac{P(x)}{ax^2+bx+c} dx = \int Q(x)\,dx + \int \frac{R(x)}{ax^2+bx+c} dx$

$$\Rightarrow \int \frac{x^2-1}{x(2x-1)} dx = \int \left(\frac{x-2}{2x(2x-1)} + \frac{1}{2}\right) dx$$

$$= \frac{1}{2}\int \frac{x-2}{x(2x-1)} dx + \frac{1}{2}\int 1dx$$

Consider $\int \frac{x-2}{x(2x-1)} dx$

By partial fraction decomposition,

$$\Rightarrow \frac{x-2}{x(2x-1)} = \frac{A}{x} + \frac{B}{2x-1}$$

$\Rightarrow x - 2 = A(2x - 1) + Bx$

$\Rightarrow x - 2 = 2Ax - A + Bx$

$\Rightarrow x - 2 = (2A + B)x - A$

$\therefore A = 2$ and $2A + B = 1$

$\therefore B = 1 - 4 = -3$

Thus, $\Rightarrow \frac{x-2}{x(2x-1)} = \frac{2}{x} - \frac{3}{2x-1}$

$$\Rightarrow \int \left(\frac{2}{x} - \frac{3}{2x-1}\right) dx$$

$$\Rightarrow 2\int \frac{1}{x} dx - 3\int \frac{1}{2x-1} dx$$

Consider $\int \frac{1}{x} dx$

We know that $\int \frac{1}{x} dx = \log|x| + c$

$$\Rightarrow \int \frac{1}{x} dx = \log|x|$$

And consider $\int \frac{1}{2x-1} dx$

Let u = 2x – 1 → dx = 1/2 du

$$\Rightarrow \int \frac{1}{2x-1} dx = \frac{1}{2} \int \frac{1}{u} du$$

We know that $\int \frac{1}{x} dx = \log|x| + c$

$$\Rightarrow \frac{1}{2} \int \frac{1}{u} du = \frac{\log|u|}{2} = \frac{\log|2x-1|}{2}$$

Then,

$$\Rightarrow \int \frac{x-2}{x(2x-1)} dx = 2 \int \frac{1}{x} dx - 3 \int \frac{1}{2x-1} dx$$

$$= 2(\log|x|) - 3\left(\frac{\log|2x-1|}{2}\right)$$

Then,

$$\Rightarrow \int \frac{x^2-1}{x(2x-1)} dx = \frac{1}{2} \int \frac{x-2}{x(2x-1)} dx + \frac{1}{2} \int 1 dx$$

$$= \frac{1}{2}\left(2(\log|x|) - 3\left(\frac{\log|2x-1|}{2}\right)\right) + \frac{1}{2} \int 1 \, dx$$

We know that $\int 1 \, dx = x + c$

$$\Rightarrow \log|x| - \frac{3\log|2x-1|}{4} + \frac{x}{2} + c$$

$$\therefore I = \int \frac{1-x^2}{(1-2x)x} dx = -\frac{3\log|2x-1|}{4} + \log|x| + \frac{x}{2} + c$$

4. $\int \frac{x^2+1}{x^2-5x+6}\,dx$

Solution:

Let $u = x^2 - 5x + 6 \rightarrow dx = \frac{1}{2x-5}du$

$\Rightarrow \int \frac{2x-5}{(x^2-5x+6)}dx = \int \frac{2x-5}{u}\frac{1}{2x-5}du$

$= \int \frac{1}{u}du$

We know that $\int \frac{1}{x}dx = \log|x| + c$

$\Rightarrow \int \frac{1}{u}du = \log|u| = \log|x^2-5x+6|$

Now consider $\int \frac{1}{x^2-5x+6}dx$

$\Rightarrow \int \frac{1}{x^2-5x+6}dx = \int \frac{1}{(x-3)(x-2)}dx$

By partial fraction decomposition,

$\Rightarrow \frac{1}{(x-3)(x-2)} = \frac{A}{x-3} + \frac{B}{x-2}$

$\Rightarrow 1 = A(x-2) + B(x-3)$

$\Rightarrow 1 = Ax - 2A + Bx - 3B$

$\Rightarrow 1 = (A + B)x - (2A + 3B)$

$\Rightarrow A + B = 0$ and $2A + 3B = -1$

Solving the two equations,

$\Rightarrow \quad A + B = 0$

2A + 3B = -1

-B = 1

∴ B = -1 and A = 1

$$\Rightarrow \int \frac{1}{(x-3)(x-2)}dx = \int \left(\frac{1}{x-3} - \frac{1}{x-2}\right)dx$$

$$= \int \frac{1}{x-3}dx - \int \frac{1}{x-2}dx$$

Consider $\int \frac{1}{x-3}dx$

Let u = x – 3 → dx = du

$$\Rightarrow \int \frac{1}{x-3}dx = \int \frac{1}{u}du$$

We know that $\int \frac{1}{x}dx = \log|x| + c$

$$\Rightarrow \int \frac{1}{u}du = \log|u| = \log|x-3|$$

Similarly $\int \frac{1}{x-2}dx$

Let u = x – 2 → dx = du

$$\Rightarrow \int \frac{1}{x-2}dx = \int \frac{1}{u}du$$

We know that $\int \frac{1}{x}dx = \log|x| + c$

$$\Rightarrow \int \frac{1}{u}du = \log|u| = \log|x-2|$$

Then,

$$\Rightarrow \int \frac{1}{x^2-5x+6}dx = \int \frac{1}{(x-3)(x-2)}dx = \int \frac{1}{x-3}dx - \int \frac{1}{x-2}dx$$

$$= \log|x-3| - \log|x-2|$$

$$\Rightarrow \int \frac{x-1}{x^2-5x+6}dx = \frac{1}{2}\int \frac{2x-5}{(x^2-5x+6)}dx + \frac{3}{2}\int \frac{1}{x^2-5x+6}dx$$

$$= \frac{1}{2}(\log|x^2-5x+6|) + \frac{3}{2}(\log|x-3| - \log|x-2|)$$

$$= \frac{\log|x^2-5x+6|}{2} + \frac{3\log|x-3|}{2} - \frac{3\log|x-2|}{2}$$

Then,

$$\Rightarrow \int \frac{x^2+1}{x^2-5x+6}dx = 5\int \frac{x-1}{x^2-5x+6}dx + \int 1\, dx$$

We know that $\int 1\ dx = x + c$

$$\Rightarrow 5\int \frac{x-1}{x^2-5x+6}dx + \int 1\, dx$$

$$= \frac{5\log|x^2-5x+6|}{2} + \frac{15\log|x-3|}{2} - \frac{15\log|x-2|}{2} + x + c$$

$$= \frac{5\log|x-2|\log|x-3|}{2} + \frac{15\log|x-3|}{2} - \frac{15\log|x-2|}{2} + x + c$$

$$= x - 5\log|x-2| + 10\log|x-3| + c$$

$$\therefore I = \int \frac{x^2+1}{x^2-5x+6}dx = x - 5\log|x-2| + 10\log|x-3| + c$$

5. $\int \frac{x^2}{x^2+7x+10}dx$

Solution:

Given $I = \int \frac{x^2}{x^2+7x+10}dx$

Expressing the integral $\int \frac{P(x)}{ax^2+bx+c}dx = \int Q(x)\,dx + \int \frac{R(x)}{ax^2+bx+c}dx$

$$\Rightarrow \int \frac{x^2}{x^2+7x+10}dx = \int \left(\frac{-7x-10}{x^2+7x+10} + 1\right)dx$$

$$= -\int \frac{7x + 10}{x^2 + 7x + 10} dx + \int 1\, dx$$

Consider $\int \frac{7x+10}{x^2+7x+10} dx$

Let $7x + 10 = \frac{7}{2}(2x + 7) - \frac{29}{2}$ and split,

$$\Rightarrow \int \frac{7x + 10}{x^2 + 7x + 10} dx = \int \left(\frac{7(2x + 7)}{2(x^2 + 7x + 10)} - \frac{29}{2(x^2 + 7x + 10)}\right) dx$$

$$= \frac{7}{2}\int \frac{2x + 7}{x^2 + 7x + 10} dx - \frac{29}{2}\int \frac{1}{x^2 + 7x + 10} dx$$

Consider $\int \frac{2x+7}{x^2+7x+10} dx$

Let $u = x^2 + 7x + 10 \to dx = \frac{1}{2x+7} du$

$$\Rightarrow \int \frac{2x + 7}{(x^2 + 7x + 10)} dx = \int \frac{2x + 7}{u} \frac{1}{2x + 7} du$$

$$= \int \frac{1}{u} du$$

We know that $\int \frac{1}{x} dx = \log|x| + c$

$$\Rightarrow \int \frac{1}{u} du = \log|u| = \log|x^2 + 7x + 10|$$

Now consider $\int \frac{1}{x^2+7x+10} dx$

$$\Rightarrow \int \frac{1}{x^2 + 7x + 10} dx = \int \frac{1}{(x + 2)(x + 5)} dx$$

By partial fraction decomposition,

$$\Rightarrow \frac{1}{(x + 2)(x + 5)} = \frac{A}{x + 2} + \frac{B}{x + 5}$$

$\Rightarrow 1 = A(x + 2) + B(x + 5)$

$\Rightarrow 1 = Ax + 2A + Bx + 5B$

$\Rightarrow$ A + B = 0 and 2A + 5B = 1

Solving the two equations,

$\Rightarrow$ 2A + 2B = 0

2A + 5B = 1

-3B = -1

$\therefore$ B = 1/3 and A = -1/3

$$\Rightarrow \int \frac{1}{(x+2)(x+5)} dx = \int \left(\frac{-1}{3(x+2)} + \frac{1}{3(x+5)}\right) dx$$

$$= -\frac{1}{3}\int \frac{1}{x+2} dx + \frac{1}{3}\int \frac{1}{x+5} dx$$

Consider $\int \frac{1}{x+2} dx$

Let u = x + 2 → dx = du

$$\Rightarrow \int \frac{1}{x+2} dx = \int \frac{1}{u} du$$

We know that $\int \frac{1}{x} dx = \log|x| + c$

$$\Rightarrow \int \frac{1}{u} du = \log|u| = \log|x+2|$$

Similarly $\int \frac{1}{x+5} dx$

Let u = x + 5 → dx = du

$$\Rightarrow \int \frac{1}{x+5} dx = \int \frac{1}{u} du$$

We know that $\int \frac{1}{x} dx = \log|x| + c$

$$\Rightarrow \int \frac{1}{u} du = \log|u| = \log|x+5|$$

Then,

$$\Rightarrow \int \frac{1}{x^2+7x+10}dx = \int \frac{1}{(x+2)(x+5)}dx = -\frac{1}{3}\int \frac{1}{x+2}dx + \frac{1}{3}\int \frac{1}{x+5}dx$$

$$= \frac{-\log|x+2|}{3} + \frac{\log|x+5|}{3}$$

Then,

$$\Rightarrow \int \frac{7x+10}{x^2+7x+10}dx = \frac{7}{2}\int \frac{2x+7}{x^2+7x+10}dx - \frac{29}{2}\int \frac{1}{x^2+7x+10}dx$$

$$= \frac{7}{2}(\log|x^2+7x+10|) - \frac{29}{2}(\frac{-\log|x+2|}{3} + \frac{\log|x+5|}{3})$$

$$= \frac{7\log|x^2+7x+10|}{2} + \frac{29\log|x+2|}{6} - \frac{29\log|x+5|}{6}$$

Then,

$$\Rightarrow \int \frac{x^2}{x^2+7x+10}dx = -\int \frac{7x+10}{x^2+7x+10}dx + \int 1\,dx$$

We know that $\int 1 \ dx = x + c$

$$\Rightarrow -\int \frac{7x+10}{x^2+7x+10}dx + \int 1\,dx$$

$$= \frac{-7\log|x^2+7x+10|}{2} - \frac{29\log|x+2|}{6} + \frac{29\log|x+5|}{6} + x + c$$

Hence,

$$I = x - \frac{7}{2}\log\left|x^2+7x+10\right| + \frac{29}{6}\log\left|\frac{x+2}{x+5}\right| + c$$

EXERCISE 19.21 PAGE NO: 19.110

Evaluate the following integrals:

1. $\int \frac{x}{\sqrt{x^2+6x+10}} dx$

Solution:

Given $I = \int \frac{x}{\sqrt{x^2+6x+10}} dx$

Integral is of form $\int \frac{px+q}{\sqrt{ax^2+bx+c}} dx$

Writing numerator as $px + q = \lambda\left\{\frac{d}{dx}(ax^2 + bx + c)\right\} + \mu$

$\Rightarrow p x + q = \lambda (2ax + b) + \mu$

$\Rightarrow x = \lambda (2x + 6) + \mu$

$\therefore \lambda = 1/2$ and $\mu = -3$

Let $x = 1/2(2x + 6) - 3$ and split,

$$\Rightarrow \int \frac{x}{\sqrt{x^2+6x+10}} dx = \int \left(\frac{2x+6}{2\sqrt{x^2+6x+10}} - \frac{3}{\sqrt{x^2+6x+10}}\right) dx$$

$$= \int \frac{x+3}{\sqrt{x^2+6x+10}} dx - 3\int \frac{1}{\sqrt{x^2+6x+10}} dx$$

Consider $\int \frac{x+3}{\sqrt{x^2+6x+10}} dx$

Let $u = x^2 + 6x + 10 \rightarrow dx = \frac{1}{2x+6} du$

$$\Rightarrow \int \frac{x+3}{\sqrt{x^2+6x+10}} dx = \int \frac{1}{2\sqrt{u}} du$$

$$= \frac{1}{2}\int \frac{1}{\sqrt{u}} du$$

We know that $\int x^n\,dx = \frac{x^{n+1}}{n+1} + c$

$$\Rightarrow \frac{1}{2}\int \frac{1}{\sqrt{u}}\,du = \frac{1}{2}(2\sqrt{u})$$

$$= \sqrt{u} = \sqrt{x^2 + 6x + 10}$$

Consider $\int \frac{1}{\sqrt{x^2+6x+10}}\,dx$

$$\Rightarrow \int \frac{1}{\sqrt{x^2 + 6x + 10}}\,dx = \int \frac{1}{\sqrt{(x+3)^2 + 1}}\,dx$$

Let u = x + 3 → dx = du

$$\Rightarrow \int \frac{1}{\sqrt{(x+3)^2 + 1}}\,dx = \int \frac{1}{\sqrt{(u)^2 + 1}}\,du$$

We know that $\int \frac{1}{\sqrt{x^2+1}}\,dx = \sinh^{-1} x + c$

$$\Rightarrow \int \frac{1}{\sqrt{u^2 + 1}}\,du = \sinh^{-1}(u)$$

$$= \sinh^{-1}(x + 3)$$

Then,

$$\Rightarrow \int \frac{x}{\sqrt{x^2 + 6x + 10}}\,dx = \int \frac{x+3}{\sqrt{x^2 + 6x + 10}}\,dx - 3\int \frac{1}{\sqrt{x^2 + 6x + 10}}\,dx$$

$$= \sqrt{x^2 + 6x + 10} - 3\sinh^{-1}(x + 3) + c$$

$$\therefore I = \int \frac{x}{\sqrt{x^2 + 6x + 10}}\,dx = \sqrt{x^2 + 6x + 10} - 3\sinh^{-1}(x + 3) + c$$

2. $\int \frac{2x + 1}{\sqrt{x^2 + 2x - 1}}\,dx$

Solution:

Given $I=\int \frac{2x+1}{\sqrt{x^2+2x-1}}dx$

Integral is of form $\int \frac{px+q}{\sqrt{ax^2+bx+c}}dx$

Writing numerator as $px+q=\lambda\left\{\frac{d}{dx}(ax^2+bx+c)\right\}+\mu$

$\Rightarrow$ p x + q = λ (2ax + b) + μ

$\Rightarrow$ 2x + 1 = λ (2x + 2) + μ

∴ λ = 1 and μ = -1

Let 2x + 1 = 2x + 2 – 1 and split,

$$\Rightarrow \int \frac{2x+1}{\sqrt{x^2+2x-1}}dx=\int\left(\frac{2x+2}{\sqrt{x^2+2x-1}}-\frac{1}{\sqrt{x^2+2x-1}}\right)dx$$

$$=2\int \frac{x+1}{\sqrt{x^2+2x-1}}dx-\int \frac{1}{\sqrt{x^2+2x-1}}dx$$

Consider $\int \frac{x+1}{\sqrt{x^2+2x-1}}dx$

Let $u=x^2+2x-1 \rightarrow dx=\frac{1}{2x+2}du$

$$\Rightarrow \int \frac{x+1}{\sqrt{x^2+2x-1}}dx=\int \frac{1}{2\sqrt{u}}du$$

$$=\frac{1}{2}\int \frac{1}{\sqrt{u}}du$$

We know that $\int x^n\,dx=\frac{x^{n+1}}{n+1}+c$

$$\Rightarrow \frac{1}{2}\int \frac{1}{\sqrt{u}}du=\frac{1}{2}\left(2\sqrt{u}\right)$$

$$=\sqrt{u}=\sqrt{x^2+2x-1}$$

Consider $\int \frac{1}{\sqrt{x^2+2x-1}}dx$

$$\Rightarrow \int \frac{1}{\sqrt{x^2+2x-1}}dx = \int \frac{1}{\sqrt{(x+1)^2-2}}dx$$

Let $u = \frac{x+1}{\sqrt{2}} \rightarrow dx = \sqrt{2}du$

$$\Rightarrow \int \frac{1}{\sqrt{(x+1)^2-2}}dx = \int \frac{\sqrt{2}}{\sqrt{2u^2-2}}du$$

$$= \int \frac{1}{\sqrt{u^2-1}}du$$

We know that $\int \frac{1}{\sqrt{x^2-1}}dx = \cosh^{-1}x + c$

$$\Rightarrow \int \frac{1}{\sqrt{u^2-1}}du = \cosh^{-1}(u)$$

$$= \cosh^{-1}\left(\frac{x+1}{\sqrt{2}}\right)$$

Then, $\Rightarrow \int \frac{2x+1}{\sqrt{x^2+2x-1}}dx = 2\int \frac{x+1}{\sqrt{x^2+2x-1}}dx - \int \frac{1}{\sqrt{x^2+2x-1}}dx$

$$= 2\sqrt{x^2+2x-1} - \cosh^{-1}\left(\frac{x+1}{\sqrt{2}}\right) + c$$

$$\therefore I = \int \frac{2x+1}{\sqrt{x^2+2x-1}}dx = 2\sqrt{x^2+2x-1} - \cosh^{-1}\left(\frac{x+1}{\sqrt{2}}\right) + c$$

3. $\int \frac{x+1}{\sqrt{4+5x-x^2}}dx$

Solution:

Given $I = \int \frac{x+1}{\sqrt{4+5x-x^2}}dx$

Integral is of form $\int \frac{px+q}{\sqrt{ax^2+bx+c}}dx$

Writing numerator as $px+q=\lambda\left\{\frac{d}{dx}(ax^2+bx+c)\right\}+\mu$

⇒ p x + q = λ (2ax + b) + μ

⇒ x + 1 = λ (-2x + 5) + μ

∴ λ = -1/2 and μ = 7/2

Let x + 1 = – 1/2(–2x + 5) + 7/2

$$\Rightarrow \int \frac{x+1}{\sqrt{-x^2+5x+4}}dx = \int\left(\frac{-2x+5}{2\sqrt{-x^2+5x+4}}+\frac{7}{2\sqrt{-x^2+5x+4}}\right)dx$$

$$=\frac{-1}{2}\int \frac{-2x+5}{\sqrt{-x^2+5x+4}}dx+\frac{7}{2}\int\frac{1}{\sqrt{-x^2+5x+4}}dx$$

Consider $\int \frac{-2x+5}{\sqrt{-x^2+5x+4}}dx$

Let $u=-x^2+5x+4 \rightarrow dx=\frac{1}{-2x+5}du$

$$\Rightarrow \int \frac{-2x+5}{\sqrt{-x^2+5x+4}}dx = -\int\frac{1}{\sqrt{u}}du$$

We know that $\int x^n\,dx=\frac{x^{n+1}}{n+1}+c$

$$\Rightarrow -\int\frac{1}{\sqrt{u}}du=-(2\sqrt{u})$$

$$=-2\sqrt{-x^2+5x+4}$$

Consider $\int \frac{1}{\sqrt{-x^2+5x+4}}dx$

$$\Rightarrow \int\frac{1}{\sqrt{-x^2+5x+4}}dx=\int\frac{1}{\sqrt{-\left(x-\frac{5}{2}\right)^2+\frac{41}{4}}}dx$$

Let $u=\frac{2x-5}{\sqrt{41}} \rightarrow dx=\frac{\sqrt{41}}{2}du$

$$\Rightarrow \int \frac{1}{\sqrt{-\left(x-\frac{5}{2}\right)^2+\frac{41}{4}}}dx = \int \frac{\sqrt{41}}{\sqrt{41-41u^2}}du$$

$$= \int \frac{1}{\sqrt{1-u^2}}du$$

We know that $\int \frac{1}{\sqrt{1-x^2}}dx = \sin^{-1}(x) + c$

$$\Rightarrow \int \frac{1}{\sqrt{1-u^2}}du = \sin^{-1}\left(\frac{2x-5}{\sqrt{41}}\right)$$

Then,

$$\Rightarrow \int \frac{x+1}{\sqrt{-x^2+5x+4}}dx = \frac{1}{2}\int \frac{-2x+5}{\sqrt{-x^2+5x+4}}dx + \frac{7}{2}\int \frac{1}{\sqrt{-x^2+5x+4}}dx$$

$$= -\sqrt{-x^2+5x+4} + \frac{7}{2}\left(\sin^{-1}\left(\frac{2x-5}{\sqrt{41}}\right)\right) + c$$

$$\therefore I = \int \frac{x+1}{\sqrt{-x^2+5x+4}}dx = -\sqrt{-x^2+5x+4} + \frac{7}{2}\left(\sin^{-1}\left(\frac{2x-5}{\sqrt{41}}\right)\right) + c$$

4. $\int \frac{6x-5}{\sqrt{3x^2-5x+1}}dx$

Solution:

Given $I = \int \frac{6x-5}{\sqrt{3x^2-5x+1}}dx$

Integral is of form $\int \frac{px+q}{\sqrt{ax^2+bx+c}}dx$

Writing numerator as $px + q = \lambda\left\{\frac{d}{dx}(ax^2+bx+c)\right\} + \mu$

$\Rightarrow px + q = \lambda(2ax + b) + \mu$

$\Rightarrow 6x - 5 = \lambda(6x - 5) + \mu$

$\therefore \lambda = 1$ and $\mu = 0$

Let $u = 3x^2 - 5x + 1 \rightarrow dx = \frac{1}{6x-5}du$

$$\Rightarrow \int \frac{6x-5}{\sqrt{3x^2-5x+1}}dx = \int \frac{1}{\sqrt{u}}du$$

We know that $\int x^n dx = \frac{x^{n+1}}{n+1} + c$

$$\Rightarrow \int \frac{1}{\sqrt{u}}du = (2\sqrt{u}) + c$$

$$= 2\sqrt{3x^2-5x+1} + c$$

$$\therefore I = \int \frac{6x-5}{\sqrt{3x^2-5x+1}}dx = 2\sqrt{3x^2-5x+1} + c$$

5. $\int \frac{3x+1}{\sqrt{5-2x-x^2}}dx$

Solution:

Given $I = \int \frac{3x+1}{\sqrt{-x^2-2x+5}}dx$

Integral is of form $\int \frac{px+q}{\sqrt{ax^2+bx+c}}dx$

Writing numerator as $px + q = \lambda\left\{\frac{d}{dx}(ax^2+bx+c)\right\} + \mu$

$\Rightarrow p x + q = \lambda (2ax + b) + \mu$

$\Rightarrow 3x + 1 = \lambda (-2x - 2) + \mu$

$\therefore \lambda = -3/2$ and $\mu = -2$

Let $3x + 1 = -(3/2)(-2x - 2) - 2$

$$\Rightarrow \int \frac{3x+1}{\sqrt{-x^2-2x+5}}dx = \int \left(\frac{-3(-2x-2)}{2\sqrt{-x^2-2x+5}} - \frac{2}{\sqrt{-x^2-2x+5}}\right)dx$$

$$= 3\int \frac{x+1}{\sqrt{-x^2-2x+5}}dx - 2\int \frac{1}{\sqrt{-x^2-2x+5}}dx$$

Consider $\int \frac{x+1}{\sqrt{-x^2-2x+5}} dx$

Let $u = -x^2 - 2x + 5 \to dx = \frac{1}{-2x-2} du$

$$\Rightarrow \int \frac{x+1}{\sqrt{-x^2-2x+5}} dx = \int -\frac{1}{2\sqrt{u}} du$$

$$= -\frac{1}{2}\int \frac{1}{\sqrt{u}} du$$

We know that $\int x^n\, dx = \frac{x^{n+1}}{n+1} + c$

$$\Rightarrow -\frac{1}{2}\int \frac{1}{\sqrt{u}} du = -(\sqrt{u})$$

$$= -\sqrt{-x^2-2x+5}$$

Consider $\int \frac{1}{\sqrt{-x^2-2x+5}} dx$

$$\Rightarrow \int \frac{1}{\sqrt{-x^2-2x+5}} dx = \int \frac{1}{\sqrt{6-(x+1)^2}} dx$$

Let $u = \frac{x+1}{\sqrt{6}} \to dx = \sqrt{6} du$

$$\Rightarrow \int \frac{1}{\sqrt{6-(x+1)^2}} dx = \int \frac{\sqrt{6}}{\sqrt{6-6u^2}} du$$

$$= \int \frac{1}{\sqrt{1-u^2}} du$$

We know that $\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1}(x) + c$

$$\Rightarrow \int \frac{1}{\sqrt{1-u^2}} du = \sin^{-1}\left(\frac{x+1}{\sqrt{6}}\right)$$

Then,

$$\Rightarrow \int \frac{3x+1}{\sqrt{-x^2-2x+5}}dx = 3\int \frac{x+1}{\sqrt{-x^2-2x+5}}dx - 2\int \frac{1}{\sqrt{-x^2-2x+5}}dx$$

$$= -3\sqrt{-x^2-2x+5} - 2\left(\sin^{-1}\left(\frac{x+1}{\sqrt{6}}\right)\right) + c$$

$$\therefore I = \int \frac{3x+1}{\sqrt{-x^2-2x+5}}dx = -3\sqrt{-x^2-2x+5} - 2\sin^{-1}\left(\frac{x+1}{\sqrt{6}}\right) + c$$

EXERCISE 19.22 PAGE NO: 19.114

Evaluate the following integrals:

1. $\int \frac{1}{4\cos^2 x + 9\sin^2 x}\,dx$

Solution:

Given $I = \int \frac{1}{4\cos^2 x + 9\sin^2 x}\,dx$

Dividing the numerator and denominator of the given integrand by $\cos^2 x$, we get

$$\Rightarrow I = \int \frac{1}{4\cos^2 x + 9\sin^2 x}\,dx = \int \frac{\sec^2 x}{4 + 9\tan^2 x}\,dx$$

Putting tan x = t and $\sec^2 x\,dx = dt$, we get

$$\Rightarrow I = \int \frac{dt}{4 + 9t^2} = \frac{1}{9}\int \frac{dt}{\frac{4}{9} + t^2}$$

We know that $\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$

$$\Rightarrow \frac{1}{9}\int \frac{dt}{\frac{4}{9} + t^2} = \frac{1}{9} \times \frac{1}{\frac{2}{3}}\tan^{-1}\left(\frac{t}{\frac{2}{3}}\right) + c$$

$$= \frac{1}{6}\tan^{-1}\left(\frac{3t}{2}\right) + c$$

$$= \frac{1}{6}\tan^{-1}\left(\frac{3\tan x}{2}\right) + c$$

$$\therefore I = \int \frac{1}{4\cos^2 x + 9\sin^2 x}\,dx = \frac{1}{6}\tan^{-1}\left(\frac{3\tan x}{2}\right) + c$$

2. $\int \frac{1}{4\sin^2 x + 5\cos^2 x}\,dx$

Solution:

Given $I=\int \frac{1}{4\sin^2 x+5\cos^2 x}dx$

Dividing the numerator and denominator of the given integrand by $\cos^2 x$, we get

$$\Rightarrow I=\int \frac{1}{4\sin^2 x+5\cos^2 x}dx=\int \frac{\sec^2 x}{4\tan^2 x+5}dx$$

Putting tan x = t and $\sec^2 x$ dx = dt, we get

$$\Rightarrow I=\int \frac{dt}{4t^2+5}=\frac{1}{4}\int \frac{dt}{t^2+\left(\frac{5}{4}\right)}$$

We know that $\int \frac{1}{a^2+x^2}dx=\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+c$

$$\Rightarrow \frac{1}{4}\int \frac{dt}{t^2+\left(\frac{5}{4}\right)}=\frac{1}{4}\times\frac{1}{\frac{\sqrt{5}}{2}}\tan^{-1}\left(\frac{t}{\frac{\sqrt{5}}{2}}\right)+c$$

$$=\frac{1}{2\sqrt{5}}\tan^{-1}\left(\frac{2t}{\sqrt{5}}\right)+c$$

$$=\frac{1}{2\sqrt{5}}\tan^{-1}\left(\frac{2\tan x}{\sqrt{5}}\right)+c$$

$$\therefore I=\int \frac{1}{4\sin^2 x+5\cos^2 x}dx=\frac{1}{2\sqrt{5}}\tan^{-1}\left(\frac{2\tan x}{\sqrt{5}}\right)+c$$

3. $\int \frac{2}{2+\sin 2x}\,dx$

Solution:

Given $I=\int \frac{2}{2+\sin 2x}dx$

We know that sin 2x = 2 sin x cos x

$$\Rightarrow \int \frac{2}{2+\sin 2x}dx=\int \frac{2}{2+2\sin x\,\cos x}dx$$

$$= \int \frac{1}{1 + \sin x \cos x} dx$$

Dividing the numerator and denominator by $\cos^2 x$,

$$\Rightarrow \int \frac{1}{1 + \sin x \cos x} dx = \int \frac{\sec^2 x}{\sec^2 x + \tan x} dx$$

Replacing $\sec^2 x$ in denominator by $1 + \tan^2 x$,

$$\Rightarrow \int \frac{\sec^2 x}{\sec^2 x + \tan x} dx = \int \frac{\sec^2 x}{1 + \tan^2 x + \tan x} dx$$

Putting $\tan x = t$ so that $\sec^2 x\, dx = dt$,

$$\Rightarrow \int \frac{\sec^2 x}{\tan^2 x + \tan x + 1} dx = \int \frac{dt}{t^2 + t + 1}$$

$$= \int \frac{dt}{\left(t + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

We know that $\int \frac{1}{a^2+x^2} dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + c$

$$\Rightarrow \int \frac{dt}{\left(t + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \frac{1}{\frac{\sqrt{3}}{2}} \tan^{-1}\left(\frac{t + \frac{1}{2}}{\frac{\sqrt{3}}{2}}\right) + c$$

$$= \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{2t + 1}{\sqrt{3}}\right) + c$$

$$\therefore I = \int \frac{2}{2 + \sin 2x} dx = \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{2t + 1}{\sqrt{3}}\right) + c$$

$$= \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{2 \tan x + 1}{\sqrt{3}}\right) + C$$

4. $\int \frac{\cos x}{\cos 3x} dx$

Solution:

Given $I = \int \frac{\cos x}{\cos 3x} dx$

$$\Rightarrow \int \frac{\cos x}{\cos 3x} dx = \int \frac{\cos x}{4\cos^3 x - 3\cos x} dx$$

$$= \int \frac{1}{4\cos^2 x - 3} dx$$

Dividing numerator and denominator by $\cos^2 x$,

$$\Rightarrow \int \frac{1}{4\cos^2 x - 3} dx = \int \frac{\sec^2 x}{4 - 3\sec^2 x} dx$$

Replacing $\sec^2 x$ by $1 + \tan^2 x$ in denominator,

$$\Rightarrow \int \frac{\sec^2 x}{4 - 3\sec^2 x} dx = \int \frac{\sec^2 x}{4 - 3 - 3\tan^2 x} dx$$

$$= \int \frac{\sec^2 x}{1 - 3\tan^2 x} dx$$

Putting $\tan x = t$ and $\sec^2 x\, dx = dt$, we get

$$I = \int \frac{dt}{1 - 3t^2} = \frac{1}{3}\int \frac{1}{\frac{1}{3} - t^2} dt$$

We know that $\int \frac{1}{a^2 - x^2} dx = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + c$

$$\Rightarrow \frac{1}{3}\int \frac{1}{\frac{1}{3} - t^2} dt = \frac{1}{3} \times \frac{1}{2\left(\frac{1}{\sqrt{3}}\right)}\log\left|\frac{\frac{1}{\sqrt{3}} + t}{\frac{1}{\sqrt{3}} - t}\right| + c$$

$$= \frac{1}{2\sqrt{3}}\log\left|\frac{1 + \sqrt{3}t}{1 - \sqrt{3}t}\right| + c$$

$$= \frac{1}{2\sqrt{3}}\log\left|\frac{1 + \sqrt{3}\tan x}{1 - \sqrt{3}\tan x}\right| + c$$

$$\therefore I = \int \frac{\cos x}{\cos 3x} dx = \frac{1}{2\sqrt{3}}\log\left|\frac{1 + \sqrt{3}\tan x}{1 - \sqrt{3}\tan x}\right| + c$$

EXERCISE 19.23 PAGE NO: 19.117

Evaluate the following integrals:

1. $\int \frac{1}{5+4\cos x}\,dx$

Solution:

Given $I = \int \frac{1}{5+4\cos x}dx$

We know that $\cos x = \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}$

$$\Rightarrow \int \frac{1}{5+4\cos x}dx = \int \frac{1}{5+4\left(\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right)}dx$$

$$= \int \frac{1+\tan^2\frac{x}{2}}{5\left(1+\tan^2\frac{x}{2}\right)+4\left(1-\tan^2\frac{x}{2}\right)}dx$$

Replacing 1 + $\tan^2 x/2$ in numerator by $\sec^2 x/2$,

$$\Rightarrow \int \frac{1+\tan^2\frac{x}{2}}{5\left(1+\tan^2\frac{x}{2}\right)+4\left(1-\tan^2\frac{x}{2}\right)}dx = \int \frac{\sec^2\frac{x}{2}}{\tan^2\frac{x}{2}+9}dx$$

Putting tan x/2 = t and $\sec^2(x/2)\,dx = 2dt$,

$$\Rightarrow \int \frac{\sec^2\frac{x}{2}}{\tan^2\frac{x}{2}+9}dx = \int \frac{2dt}{t^2+9}$$

$$= 2\int \frac{1}{t^2+9}dt$$

We know that $\int \frac{1}{a^2+x^2}dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+c$

$$\Rightarrow 2\int \frac{1}{t^2+9}dt = 2\left(\frac{1}{3}\right)\tan^{-1}\left(\frac{t}{3}\right)+c$$

$$= \frac{2}{3}\tan^{-1}\left(\frac{\tan x/2}{3}\right) + c$$

2. $\int \frac{1}{5 - 4\sin x}\, dx$

Solution:

Given $I = \int \frac{1}{5-4\sin x} dx$

We know that $\sin x = \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}$

$$\Rightarrow \int \frac{1}{5-4\sin x} dx = \int \frac{1}{5-4\left(\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right)} dx$$

$$= \int \frac{1+\tan^2\frac{x}{2}}{5\left(1+\tan^2\frac{x}{2}\right) - 4\left(2\tan\frac{x}{2}\right)} dx$$

Replacing $1 + \tan^2 x/2$ in numerator by $\sec^2 x/2$,

$$\Rightarrow \int \frac{1+\tan^2\frac{x}{2}}{5\left(1+\tan^2\frac{x}{2}\right) - 4\left(2\tan\frac{x}{2}\right)} dx = \int \frac{\sec^2\frac{x}{2}}{5+5\tan^2\frac{x}{2} - 8\tan\frac{x}{2}} dx$$

Putting $\tan x/2 = t$ and $\sec^2(x/2)\, dx = 2dt$,

$$\Rightarrow \int \frac{\sec^2\frac{x}{2}}{5+5\tan^2\frac{x}{2} - 8\tan\frac{x}{2}} dx = \int \frac{2dt}{5+5t^2-8t}$$

$$= \frac{2}{5}\int \frac{1}{t^2 - \frac{8}{5}t + 1} dt$$

$$= \frac{2}{5}\int \frac{1}{\left(t-\frac{4}{5}\right)^2 + \left(\frac{3}{5}\right)^2} dt$$

We know that $\int \frac{1}{a^2+x^2}dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$

$$\Rightarrow \frac{2}{5}\int \frac{1}{\left(t-\frac{4}{5}\right)^2 + \left(\frac{3}{5}\right)^2}dt = \frac{2}{5}\left(\frac{1}{\frac{3}{5}}\right)\tan^{-1}\left(\frac{t-\frac{4}{5}}{\frac{3}{5}}\right) + c$$

$$= \frac{2}{3}\tan^{-1}\left(\frac{5\tan x/2 - 4}{3}\right) + c$$

3. $\int \frac{1}{1-2\sin x}dx$

Solution:

Given $I = \int \frac{1}{1-2\sin x}dx$

We know that $\sin x = \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}$

$$\Rightarrow \int \frac{1}{1-2\sin x}dx = \int \frac{1}{1-2\left(\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right)}dx$$

$$= \int \frac{1+\tan^2\frac{x}{2}}{1\left(1+\tan^2\frac{x}{2}\right) - 2\left(2\tan\frac{x}{2}\right)}dx$$

Replacing 1 + tan²x/2 in numerator by sec²x/2,

$$\Rightarrow \int \frac{1+\tan^2\frac{x}{2}}{1\left(1+\tan^2\frac{x}{2}\right) - 2\left(2\tan\frac{x}{2}\right)}dx = \int \frac{\sec^2\frac{x}{2}}{1+\tan^2\frac{x}{2} - 4\tan\frac{x}{2}}dx$$

Putting tan x/2 = t and sec²(x/2) dx = 2dt,

$$\Rightarrow \int \frac{\sec^2\frac{x}{2}}{1+\tan^2\frac{x}{2} - 4\tan\frac{x}{2}}dx = \int \frac{2dt}{1+t^2-4t}$$

$$= 2\int \frac{1}{t^2-4t+1}dt$$

$$= 2\int \frac{1}{(t-2)^2 - (\sqrt{3})^2} dt$$

We know that $\int \frac{1}{x^2-a^2} dx = \frac{1}{2a} \log\left|\frac{x-a}{x+a}\right| + c$

$$\Rightarrow 2\int \frac{1}{(t-2)^2 - (\sqrt{3})^2} dt = 2\left(\frac{1}{2\sqrt{3}}\right) \log\left|\left(\frac{t-2-\sqrt{3}}{t-2+\sqrt{3}}\right)\right| + c$$

$$= \frac{1}{\sqrt{3}} \log\left|\left(\frac{\tan x - (2+\sqrt{3})}{\tan x + (2+\sqrt{3})}\right)\right| + c$$

$$= \frac{1}{\sqrt{3}} \log\left|\frac{\tan \frac{x}{2} - 2 - \sqrt{3}}{\tan \frac{x}{2} - 2 + \sqrt{3}}\right| + c$$

4. $\int \frac{1}{4\cos x - 1} dx$

Solution:

Given $I = \int \frac{1}{4\cos x - 1} dx$

We know that $\cos x = \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}$

$$\Rightarrow \int \frac{1}{-1+4\cos x} dx = \int \frac{1}{-1+4\left(\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right)} dx$$

$$= \int \frac{1+\tan^2\frac{x}{2}}{-1\left(1+\tan^2\frac{x}{2}\right) + 4(1-\tan^2\frac{x}{2})} dx$$

Replacing $1 + \tan^2 x/2$ in numerator by $\sec^2 x/2$,

$$\Rightarrow \int \frac{1+\tan^2\frac{x}{2}}{-1\left(1+\tan^2\frac{x}{2}\right) + 4(1-\tan^2\frac{x}{2})} dx = \int \frac{\sec^2\frac{x}{2}}{-5\tan^2\frac{x}{2} + 3} dx$$

Putting $\tan\frac{x}{2} = t$ and $\sec^2(\frac{x}{2})dx = dt$,

$$\Rightarrow \int \frac{\sec^2\frac{x}{2}}{-5\tan^2\frac{x}{2}+3}dx = \int \frac{2\,dt}{3-5t^2}$$

$$= \frac{2}{5}\int \frac{1}{\frac{3}{5}-t^2}dt$$

We know that $\int \frac{1}{a^2-x^2}dx = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + c$

$$\Rightarrow \frac{2}{5}\int \frac{1}{\frac{3}{5}-t^2}dt = \frac{2}{5}\left(\frac{1}{2\sqrt{\frac{3}{5}}}\right)\log\left|\frac{\sqrt{\frac{3}{5}}+t}{\sqrt{\frac{3}{5}}-t}\right| + c$$

$$= \frac{1}{\sqrt{15}}\log\left|\frac{\sqrt{3}+\sqrt{5}\tan\frac{x}{2}}{\sqrt{3}-\sqrt{5}\tan\frac{x}{2}}\right| + c$$

$$\therefore I = \int \frac{1}{4\cos x - 1}dx = \frac{1}{\sqrt{15}}\log\left|\frac{\sqrt{3}+\sqrt{5}\tan\frac{x}{2}}{\sqrt{3}-\sqrt{5}\tan\frac{x}{2}}\right| + c$$

5. $\int \frac{1}{1-\sin x+\cos x}dx$

Solution:

Given $I = \int \frac{1}{1-\sin x+\cos x}dx$

We know that $\sin x = \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}$ and $\cos x = \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}$

$$\Rightarrow \int \frac{1}{1-\sin x+\cos x}dx = \int \frac{1}{1-\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}+\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}}dx$$

$$= \int \frac{1+\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}-2\tan\frac{x}{2}+1-\tan^2\frac{x}{2}}dx$$

Replacing $1 + \tan^2 x/2$ in numerator by $\sec^2 x/2$ and putting $\tan x/2 = t$ and $\sec^2 x/2\, dx = 2dt$,

$$\Rightarrow \int \frac{1+\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}-2\tan\frac{x}{2}+1-\tan^2\frac{x}{2}}dx = \int \frac{\sec^2\frac{x}{2}}{2-2\tan\frac{x}{2}}dx$$

$$= \int \frac{2dt}{2-2t}$$

$$= \int \frac{1}{1-t}dt$$

We know that $\int \frac{1}{x}dx = \log|x| + c$

$$\Rightarrow \int \frac{1}{1-t}dt = -\log|1-t| + c$$

$$= -\log\left|1-\tan\frac{x}{2}\right| + c$$

$$\therefore I = \int \frac{1}{1-\sin x+\cos x}dx = -\log\left|1-\tan\frac{x}{2}\right| + c$$

EXERCISE 19.24 PAGE NO: 19.122

Evaluate the following integrals:

1. $\int \frac{1}{1-\cot x} dx$

Solution:

Let, I = $\int \frac{1}{1-\cot x} dx$

To solve such integrals involving trigonometric terms in numerator and denominators. If I has the form $\int \frac{a\sin x+b\cos x+c}{d\sin x+e\cos x+f} dx$

Then substitute numerator as

$$a\sin x + b\cos x + c = A\frac{d}{dx}(d\sin x + e\cos x + f) + B(d\sin x + e\cos x + f) + C$$

Where A, B and C are constants

We have, I = $\int \frac{1}{1-\cot x} dx = \int \frac{1}{1-\frac{\cos x}{\sin x}} dx = \int \frac{\sin x}{\sin x-\cos x} dx$

As I matches with the form described above, so we will take the steps as described.

$\therefore \sin x = A\frac{d}{dx}(\sin x - \cos x) + B(\sin x - \cos x) + C$

$\Rightarrow \sin x = A(\cos x + \sin x) + B(\sin x - \cos x) + C \left\{\because \frac{d}{dx}\cos x = -\sin x\right\}$

$\Rightarrow \sin x = \sin x (B + A) + \cos x (A - B) + C$

Comparing both sides we have:

C = 0

A – B = 0 ⇒ A = B

B + A = 1 ⇒ 2A = 1 ⇒ A = ½

∴ A = B = ½

Thus I can be expressed as:

I = $\int \frac{\frac{1}{2}(\cos x+\sin x)+\frac{1}{2}(\sin x-\cos x)}{\sin x-\cos x} dx$

I = $\int \frac{\frac{1}{2}(\cos x+\sin x)}{\sin x-\cos x} dx + \int \frac{\frac{1}{2}(\sin x-\cos x)}{\sin x-\cos x} dx$

∴ Let $I_1 = \frac{1}{2}\int \frac{(\cos x+\sin x)}{\sin x-\cos x} dx$ and $I_2 = \frac{1}{2}\int \frac{(\sin x-\cos x)}{\sin x-\cos x} dx$

⇒ $I = I_1 + I_2$ ....equation 1

$I_1 = \frac{1}{2}\int \frac{(\cos x+\sin x)}{\sin x-\cos x} dx$

Let, u = sin x – cos x ⇒ du = (cos x + sin x) dx

So, I_1 reduces to:

$I_1 = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\log|u| + C_1$

∴ $I_1 = \frac{1}{2}\log|\sin x - \cos x| + C_1$ ..... Equation 2

As, $I_2 = \frac{1}{2}\int \frac{(\sin x-\cos x)}{\sin x-\cos x} dx = \frac{1}{2}\int dx$

∴ $I_2 = \frac{x}{2} + C_2$ ..... Equation 3

From equation 1, 2 and 3 we have:

$I = \frac{1}{2}\log|\sin x - \cos x| + C_1 + \frac{x}{2} + C_2$

∴ $I = \frac{1}{2}\log|\sin x - \cos x| + \frac{x}{2} + C$

2. $\int \frac{1}{1-\tan x} dx$

Solution:

Let, I = $\int \frac{1}{1-\tan x}dx$

To solve such integrals involving trigonometric terms in numerator and denominators. If I has the form $\int \frac{a\sin x+b\cos x+c}{d\sin x+e\cos x+f}dx$

Then substitute numerator as

$$a\sin x+b\cos x+c = A\frac{d}{dx}(d\sin x+e\cos x+f)+B(d\sin x+e\cos x+f)+C$$

Where A, B and C are constants

We have, I = $\int \frac{1}{1-\tan x}dx = \int \frac{1}{1-\frac{\sin x}{\cos x}}dx = \int \frac{\cos x}{\cos x-\sin x}dx$

As I matches with the form described above, so we will take the steps as described.

$\therefore \cos x = A\frac{d}{dx}(\cos x-\sin x)+B(\cos x-\sin x)+C$

$\Rightarrow \cos x = A(-\sin x-\cos x)+B(\cos x-\sin x)+C \left\{\because \frac{d}{dx}\cos x = -\sin x\right\}$

$\Rightarrow \cos x = -\sin x(B+A)+\cos x(B-A)+C$

Comparing both sides we have:

C = 0

B – A = 1 ⇒ A = B - 1

B + A = 0 ⇒ 2B - 1 = 0 ⇒ B = ½

∴ A = B - 1 = - ½

Thus I can be expressed as:

$$I = \int \frac{-\frac{1}{2}(-\sin x-\cos x)+\frac{1}{2}(\cos x-\sin x)}{(\cos x-\sin x)}dx$$

$$I = \int \frac{\frac{1}{2}(\cos x+\sin x)+\frac{1}{2}(\cos x-\sin x)}{(\cos x-\sin x)}dx$$

$$I = \int \frac{\frac{1}{2}(\cos x+\sin x)}{(\cos x-\sin x)}dx+\int \frac{\frac{1}{2}(\cos x-\sin x)}{(\cos x-\sin x)}dx$$

∴ Let $I_1 = \frac{1}{2}\int \frac{(\cos x+\sin x)}{(\cos x-\sin x)}dx$ and $I_2 = \frac{1}{2}\int \frac{(\cos x-\sin x)}{(\cos x-\sin x)}dx$

⇒ $I = I_1 + I_2$ ....equation 1

$I_1 = \frac{1}{2}\int \frac{(\cos x+\sin x)}{(\cos x-\sin x)}dx$

Let, $u = \cos x - \sin x \Rightarrow du = -(\cos x + \sin x)\,dx$

So, I_1 reduces to:

$I_1 = -\frac{1}{2}\int \frac{du}{u} = -\frac{1}{2}\log|u| + C_1$

∴ $I_1 = -\frac{1}{2}\log|\cos x - \sin x| + C_1$ ..... Equation 2

As, $I_2 = \frac{1}{2}\int \frac{(\cos x-\sin x)}{(\cos x-\sin x)}dx = \frac{1}{2}\int dx$

∴ $I_2 = \frac{x}{2} + C_2$ ..... Equation 3

From equation 1, 2 and 3 we have:

$I = -\frac{1}{2}\log|\cos x - \sin x| + C_1 + \frac{x}{2} + C_2$

∴ $I = -\frac{1}{2}\log|\cos x - \sin x| + \frac{x}{2} + C$

3. $\int \frac{3 + 2\cos x + 4\sin x}{2\sin x + \cos x + 3}dx$

Solution:

Let, $I = \int \frac{3+2\cos x+4\sin x}{2\sin x+\cos x+3}dx$

To solve such integrals involving trigonometric terms in numerator and denominators. If I has the form $\int \frac{a\sin x+b\cos x+c}{d\sin x+e\cos x+f}dx$

Then substitute numerator as

$$a\sin x + b\cos x + c = A\frac{d}{dx}(d\sin x + e\cos x + f) + B(d\sin x + e\cos x + f) + C$$

Where A, B and C are constants

We have, I = $\int \frac{3+2\cos x+4\sin x}{2\sin x+\cos x+3}dx$

As I matches with the form described above, so we will take the steps as described.

$\therefore 3 + 2\cos x + 4\sin x = A\frac{d}{dx}(2\sin x + \cos x + 3) + B(2\sin x + \cos x + 3) + C$

$\Rightarrow 3 + 2\cos x + 4\sin x = A(2\cos x - \sin x) + B(2\sin x + \cos x + 3) + C$

$\{\because \frac{d}{dx}\cos x = -\sin x\}$

$\Rightarrow 3 + 2\cos x + 4\sin x = \sin x\,(2B - A) + \cos x\,(B + 2A) + 3B + C$

Comparing both sides we have:

3B+ C = 3

B + 2A = 2

2B - A = 4

On solving for A, B and C we have:

A = 0, B = 2 and C = -3

Thus I can be expressed as:

I = $\int \frac{2(2\sin x+\cos x+3)-3}{2\sin x+\cos x+3}dx$

I = $\int \frac{2(2\sin x+\cos x+3)}{2\sin x+\cos x+3}dx + \int \frac{-3}{2\sin x+\cos x+3}dx$

$\therefore$ Let $I_1 = 2\int \frac{(2\sin x+\cos x+3)}{2\sin x+\cos x+3}dx$ and $I_2 = -3\int \frac{1}{2\sin x+\cos x+3}dx$

$\Rightarrow I = I_1 + I_2$ ....equation 1

$I_1 = 2\int \frac{(2\sin x+\cos x+3)}{2\sin x+\cos x+3}dx$

So, I_1 reduces to:

$I_1 = 2\int dx = 2x + C_1$ ..... Equation 2

As, $I_2 = -3\int \frac{1}{2\sin x+\cos x+3}dx$

To solve the integrals of the form $\int \frac{1}{a\sin x+b\cos x+c}dx$

To apply substitution method we take following procedure.

We substitute:

$$\sin x = \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}} \text{ and } \cos x = \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}$$

$\therefore I_2 = -3\int \frac{1}{2\sin x+\cos x+3}dx$

$\Rightarrow I_2 = -3\int \frac{1}{2\left(\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right)+3\left(\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right)+3}dx$

$\Rightarrow I_2 = -3\int \frac{1+\tan^2\frac{x}{2}}{4\tan\frac{x}{2}+1-\tan^2\frac{x}{2}+3(1+\tan^2\frac{x}{2})}dx$

$\Rightarrow I_2 = -3\int \frac{\sec^2\frac{x}{2}}{2(2\tan\frac{x}{2}+2+1\tan^2\frac{x}{2})}dx$

Let, $t = \tan\frac{x}{2} \Rightarrow dt = \frac{1}{2}\sec^2\frac{x}{2}\,dx$

$\therefore I_2 = -3\int \frac{1}{(2t+2+t^2)}dt$

As, the denominator is polynomial without any square root term. So one of the special integral will be used to solve I_2.

$I_2 = -3\int \frac{1}{(2t+2+t^2)}dt$

$\Rightarrow I_2 = -3\int \frac{1}{(t^2+2(1)t+1)+1}\,dt$

$\therefore I_2 = -3\int \frac{1}{(t+1)^2+1}\,dt$ {∵ $a^2 + 2ab + b^2 = (a + b)^2$}

As, I_2 matches with the special integral form

$$\int \frac{1}{x^2+a^2}dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$$

$I_2 = -3\tan^{-1}(t+1)$

Putting value of t we have:

$\therefore I_2 = -3\tan^{-1}\left(\tan\frac{x}{2} + 1\right) + C_2$ ......equation 3

From equation 1, 2 and 3:

$I = 2x + C_1 - 3\tan^{-1}\left(\tan\frac{x}{2} + 1\right) + C_2$

$\therefore I = 2x - 3\tan^{-1}\left(\tan\frac{x}{2} + 1\right) + C$

4. $\int \frac{1}{p + q\tan x}\,dx$

Solution:

Let, $I = \int \frac{1}{p+q\tan x}dx$

To solve such integrals involving trigonometric terms in numerator and denominators. If I has the form $\int \frac{a\sin x+b\cos x+c}{d\sin x+e\cos x+f}\,dx$

Then substitute numerator as

$$a\sin x + b\cos x + c = A\frac{d}{dx}(d\sin x + e\cos x + f) + B(d\sin x + e\cos x + f) + C$$

Where A, B and C are constants

We have, I = $\int \frac{1}{p+q\tan x}dx = \int \frac{1}{p+q\frac{\sin x}{\cos x}}dx = \int \frac{\cos x}{p\cos x+q\sin x}dx$

As I matches with the form described above, so we will take the steps as described.

$\therefore \cos x = A\frac{d}{dx}(p\cos x + q\sin x) + B(p\cos x + q\sin x) + C$

$\Rightarrow \cos x = A(-p\sin x + q\cos x) + B(p\cos x + q\sin x) + C \{\because \frac{d}{dx}\cos x = -\sin x\}$

$\Rightarrow \cos x = \sin x\,(Bq - Ap) + \cos x\,(Bp + Aq) + C$

Comparing both sides we have:

C = 0

B p + A q = 1

B q − A p = 0

On solving above equations, we have:

A = $\frac{q}{p^2+q^2}$ B = $\frac{p}{p^2+q^2}$ and C = 0

Thus I can be expressed as:

I = $\int \frac{\frac{q}{p^2+q^2}(-p\sin x+q\cos x)+\frac{p}{p^2+q^2}(p\cos x+q\sin x)}{(p\cos x+q\sin x)}dx$

I = $\int \frac{\frac{q}{p^2+q^2}(-p\sin x+q\cos x)}{(p\cos x+q\sin x)}dx + \int \frac{\frac{p}{p^2+q^2}(p\cos x+q\sin x)}{(p\cos x+q\sin x)}dx$

∴ Let $I_1 = \frac{q}{p^2+q^2}\int \frac{(-p\sin x+q\cos x)}{(p\cos x+q\sin x)}dx$ and $I_2 = \frac{p}{p^2+q^2}\int \frac{(p\cos x+q\sin x)}{(p\cos x+q\sin x)}dx$

$\Rightarrow I = I_1 + I_2$ ….equation 1

$I_1 = \frac{q}{p^2+q^2}\int \frac{(-p\sin x+q\cos x)}{(p\cos x+q\sin x)}dx$

Let, u = p cos x + q sin x ⇒ du = (-p sin x + q cos x) dx

So, I_1 reduces to:

$I_1 = \frac{q}{p^2+q^2}\int\frac{du}{u} = \frac{q}{p^2+q^2}\log|u| + C_1$

$\therefore I_1 = \frac{q}{p^2+q^2}\log|(p\cos x + q\sin x)| + C_1$ ..... Equation 2

As, $I_2 = \frac{p}{p^2+q^2}\int\frac{(p\cos x+q\sin x)}{(p\cos x+q\sin x)}dx = \frac{p}{p^2+q^2}\int dx$

$\therefore I_2 = \frac{px}{p^2+q^2} + C_2$ ..... Equation 3

From equation 1, 2 and 3 we have:

$I = \frac{q}{p^2+q^2}\log|(p\cos x + q\sin x)| + C_1 + \frac{px}{p^2+q^2} + C_2$

$\therefore I = \frac{q}{p^2+q^2}\log|(p\cos x + q\sin x)| + \frac{px}{p^2+q^2} + C$

5. $\int \frac{5\cos x + 6}{2\cos x + \sin x + 3}\,dx$

Solution:

Let, $I = \int\frac{5\cos x+6}{2\cos x+\sin x+3}dx$

To solve such integrals involving trigonometric terms in numerator and denominators. If I has the form $\int\frac{a\sin x+b\cos x+c}{d\sin x+e\cos x+f}dx$

Then substitute numerator as

$$a\sin x + b\cos x + c = A\frac{d}{dx}(d\sin x + e\cos x + f) + B(d\sin x + e\cos x + f) + C$$

Where A, B and C are constants

We have, $I = \int\frac{5\cos x+6}{2\cos x+\sin x+3}dx$

As I matches with the form described above, so we will take the steps as described.

$\therefore 5\cos x + 6 = A\frac{d}{dx}(2\cos x + \sin x + 3) + B(2\cos x + \sin x + 3) + C$

$\Rightarrow 5\cos x + 6 = A(-2\sin x + \cos x) + B(2\cos x + \sin x + 3) + C$

$\because \frac{d}{dx}\cos x = -\sin x \}$

$\Rightarrow 5\cos x + 6 = \sin x\,(B - 2A) + \cos x\,(2B + A) + 3B + C$

Comparing both sides we have:

3B+ C = 6

2B + A = 5

B - 2A = 0

On solving for A, B and C we have:

A = 1, B = 2 and C = 0

Thus I can be expressed as:

$I = \int \frac{(-2\sin x+\cos x)+2(2\cos x+\sin x+3)}{2\cos x+\sin x+3}dx$

$I = \int \frac{(-2\sin x+\cos x)}{2\cos x+\sin x+3}dx + \int \frac{2(2\cos x+\sin x+3)}{2\cos x+\sin x+3}dx$

$\therefore$ Let $I_1 = \int \frac{(-2\sin x+\cos x)}{2\cos x+\sin x+3}dx$ and $I_2 = \int \frac{2(2\cos x+\sin x+3)}{2\cos x+\sin x+3}dx$

$\Rightarrow I = I_1 + I_2$ ....equation 1

$I_1 = \int \frac{(-2\sin x+\cos x)}{2\cos x+\sin x+3}dx$

Let, 2 cos x + sin x + 3 = u

$\Rightarrow$ (-2sin x + cos x) dx = du

So, I_1 reduces to:

$I_1 = \int \frac{du}{u} = \log|u| + C_1$

$\therefore I_1 = \log|2\cos x + \sin x + 3| + C_1$ ..... Equation 2

As, $I_2 = \int \frac{2(2\cos x+\sin x+3)}{2\cos x+\sin x+3}dx$

$\Rightarrow I_2 = 2\int dx = 2x + C_2$ ..... Equation 3

From equation 1, 2 and 3 we have:

$I = \log|2\cos x + \sin x + 3| + C_1 + 2x + C_2$

$\therefore I = \log|2\cos x + \sin x + 3| + 2x + C$

EXERCISE 19.25

Evaluate the following integrals:

1. $\int x \cos x \, dx$

Solution:

Let $I = \int x \cos x \, dx$

We know that, $\int u\, v\, dx = u \int v\, dx - \int \left(\frac{du}{dx} \int v\, dx\right) dx$

Using integration by parts,

$$I = x \int \cos x \, dx - \int \left(\frac{d}{dx} x \int \cos x \, dx\right) dx$$

We have, $\int \sin x = -\cos x, \int \cos x = \sin x$

$$= x \times \sin x - \int \sin x \, dx$$

$$= x\sin x + \cos x + c$$

2. $\int \log(x+1) \, dx$

Solution:

Let $I = \int \log(x+1) \, dx$

That is,

$$I = \int 1 \times \log(x+1) \, dx$$

Using integration by parts,

$$I = \log(x+1) \int 1 \, dx - \int \frac{d}{dx} \log(x+1) \int 1 \, dx$$

We know that, $\int 1 \, dx = x$ and $\int \log x = \frac{1}{x}$

$$= \log(x+1) \times x - \int \frac{1}{x+1} \times x\, dx$$

Now,

$$\frac{x}{x+1} = 1 - \frac{1}{x+1}$$

$$= x\log(x+1) - \int \left(1 - \frac{1}{x+1}\right) dx$$

$$= x\log(x+1) - x + \log(x+1) + c$$

3. $\int x^3 \log x\, dx$

Solution:

Let $I = \int x^3 \log x\, dx$

Using integration by parts,

$$I = \log x \int x^3\, dx - \int \frac{d}{dx} \log x \int x^3\, dx$$

We have, $\int x^n dx = \frac{x^{n+1}}{n+1}$ and $\int \log x = \frac{1}{x}$

$$= \log x \times \frac{x^4}{4} - \int \frac{1}{x} \times \frac{x^4}{4}\, dx$$

$$= \log x \times \frac{x^4}{4} - \frac{1}{4}\int x^3 dx$$

$$= \frac{x^4}{4}\log x - \frac{1}{4} \times \frac{x^4}{4}$$

$$= \frac{x^4}{4}\log x - \frac{x^4}{16} + c$$

4. $\int x e^x\, dx$

Solution:

Let $I = \int xe^{x}\,dx$

Using integration by parts,

$$I = x\int e^{x}\,dx - \int\left(\frac{d}{dx}x\int e^{x}dx\right)dx$$

We know that, $\int e^{x}\,dx = e^{x}$ and $\frac{d}{dx}x = 1$

$$= xe^{x} - \int e^{x}\,dx$$

$$= xe^{x} - e^{x} + c$$

5. $\int xe^{2x}\,dx$

Solution:

Let $I = \int xe^{2x}dx$

Using integration by parts,

$$I = x\int e^{2x}dx - \int\left(\frac{d}{dx}x\int e^{2x}dx\right)dx$$

We know that, $\int e^{nx}\,dx = \frac{e^{x}}{n}$ and $\frac{d}{dx}x = 1$

$$= \frac{xe^{2x}}{2} - \int \frac{e^{2x}}{2}dx$$

$$= \frac{xe^{2x}}{2} - \frac{e^{2x}}{4} + c$$

$$I = \left(\frac{x}{2} - \frac{1}{4}\right)e^{2x} + c$$

EXERCISE 19.26 PAGE NO: 19.143

Evaluate the following integrals:

1. $\int e^x(\cos x - \sin x)\, dx$

Solution:

Let $I = \int e^x(\cos x - \sin x)dx$

Using integration by parts,

$$= \int e^x \cos x\, dx - \int e^x \sin x\, dx$$

We know that, $\frac{d}{dx}\cos x = -\sin x$

$$= \cos x \int e^x - \int \frac{d}{dx}\cos x \int e^x dx - \int e^x \sin x\, dx$$

$$= e^x \cos x + \int e^x \sin x\, dx - \int e^x \sin x\, dx$$

$$= e^x \cos x + c$$

2. $\int e^x \left(\frac{1}{x^2} - \frac{2}{x^3}\right) dx$

Solution:

Let $I = \int e^x\left(\frac{1}{x^2} - \frac{2}{x^3}\right)dx$

$$= \int e^x x^{-2} dx - 2\int e^x x^{-3} dx$$

Integrating by parts

$$= x^{-2}\int e^x dx - \int \frac{d}{dx}x^{-2}\int e^x dx - 2\int e^x x^{-3} dx$$

We know that,

$$\int x^n dx = \frac{x^{n+1}}{n+1}$$

$$= e^x x^{-2} + 2\int e^x x^{-3} dx - 2\int e^x x^{-3} dx$$

$$= \frac{e^x}{x^2} + c$$

3. $\int e^x \left(\frac{1+\sin x}{1+\cos x}\right) dx$

Solution:

Let $I = \int e^x \left(\frac{1+\sin x}{1+\cos x}\right) dx$

We know that, $\sin^2 x + \cos^2 x = 1$ and $\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}$

$$= e^x\left(\frac{\sin^2\frac{x}{2} + \cos^2\frac{x}{2} + 2\sin\frac{x}{2}\cos\frac{x}{2}}{2\cos^2\frac{x}{2}}\right)$$

$$= \frac{e^x\left(\sin\frac{x}{2} + \cos\frac{x}{2}\right)^2}{2\cos^2\frac{x}{2}}$$

$$= \frac{1}{2}e^x\left(\frac{\sin\frac{x}{2} + \cos\frac{x}{2}}{\cos\frac{x}{2}}\right)^2$$

$$= \frac{1}{2}e^x\left[\tan\frac{x}{2} + 1\right]^2$$

$$= \frac{1}{2}e^x\left[1 + \tan\frac{x}{2}\right]^2$$

$$= \frac{1}{2}e^x\left[1 + \tan^2\frac{x}{2} + 2\tan\frac{x}{2}\right]$$

$$= \frac{1}{2} e^x \left[\sec^2 \frac{x}{2} + 2 \tan \frac{x}{2}\right]$$

$$= e^x \left[\frac{1}{2} \sec^2 \frac{x}{2} + \tan \frac{x}{2}\right] \dots \dots (1)$$

Let $\tan \frac{x}{2} = f(x)$

$$f'(x) = \frac{1}{2} \sec^2 \frac{x}{2}$$

We know that,

$$\int e^x \{f(x) + f'(x)\} dx = e^x f(x) + c$$

From equation (1), we obtain

$$\int e^x \left(\frac{1 + \sin x}{1 + \cos x}\right) dx = e^x \tan \frac{x}{2} + c$$

4. $\int e^x (\cot x - cosec^2 x)\, dx$

Solution:

Let $I = \int e^x (\cot x - \operatorname{cosec}^2 x) dx$

$$= \int e^x \cot x dx - \int e^x \operatorname{cosec}^2 x dx$$

Integrating by parts,

$$= \cot x \int e^x dx - \int \frac{d}{dx} \cot x \int e^x dx - \int e^x \operatorname{cosec}^2 x dx$$

$$= \cot x\, e^x + \int e^x \operatorname{cosec}^2 x dx - \int e^x \operatorname{cosec}^2 x dx$$

$$= e^x \cot x + c$$

5. $\int e^x \left(\frac{x - 1}{2x^2}\right) dx$

Solution:

Given

$$\int e^x\left(\frac{x-1}{2x^2}\right)dx$$

Let $I=\int e^x \frac{1}{2x}dx-\int e^x \frac{1}{2x^2}dx$

Integrating by parts,

$$=\frac{e^x}{2x}-\int e^x\left(\frac{d}{dx}\left(\frac{1}{2x}\right)\right)dx-\int \frac{e^x}{2x^2}dx$$

$$=\frac{e^x}{2x}+\int \frac{e^x}{2x^2}dx-\int \frac{e^x}{2x^2}dx$$

$$=\frac{e^x}{2x}+c$$

EXERCISE 19.27 PAGE NO: 19.149

Evaluate the following integrals:

1. $\int e^{ax} \cos bx \, dx$

Solution:

Let $I = e^{ax} \cos bx \, dx$

Integrating by parts, we get

$$I = e^{ax} \frac{\sin bx}{b} - a \int e^{ax} \frac{\sin bx}{b} \, dx$$

Taking 1/b as common and a/b as common we get

$$= \frac{1}{b} e^{ax} \sin bx - \frac{a}{b} \int e^{ax} \sin bx \, dx$$

Now again by using integration by parts, we get

$$= \frac{1}{b} e^{ax} \sin bx - \frac{a}{b} \left[-e^{ax} \frac{\cos bx}{b} + a \int e^{ax} \frac{\cos bx}{b} \, dx \right]$$

$$= \frac{1}{b} e^{ax} \sin bx + \frac{a}{b^2} e^{ax} \cos bx - \frac{a^2}{b^2} \int e^{ax} \cos bx \, dx$$

By computing,

$$I = \frac{e^{ax}}{b^2} [b \sin bx + a \cos bx] - \frac{a^2}{b^2} I + c$$

$$= \frac{e^{ax}}{a^2 + b^2} [b \sin bx + a \cos bx] + c$$

2. $\int e^{ax} \sin(bx + c) \, dx$

Solution:

Let $I = \int e^{ax}\sin(bx+c)\,dx$

$$= -e^{ax}\frac{\cos(bx+c)}{b} + \int ae^{ax}\frac{\cos(bx+c)}{b}dx$$

Now taking common

$$= -\frac{1}{b}e^{ax}\cos(bx+c) + \frac{a}{b}\int e^{ax}\cos(bx+c)$$

On integrating we get

$$I = \frac{e^{ax}}{b^2}\{a\sin(bx+c) - b\cos(bx+c)\} - \frac{a^2}{b^2}I + c_1$$

By computing the above equation can be written as

$$I = \left\{\frac{a^2+b^2}{b^2}\right\} - \frac{e^{ax}}{b^2}\{a\sin(bx+c) - b\cos(bx+c)\} + c_1$$

$$= \frac{e^{ax}}{a^2+b^2}\{a\sin(bx+c) - b\cos(bx+c)\} + c_1$$

3. $\int \cos(\log x)\,dx$

Solution:

Let $I = \int \cos(\log x)\,dx$

Let log x=t

$$\frac{1}{x}dx = dt$$

dx = x dt

$$= \int e^t \cos t\,dt$$

We know that, $\int e^{ax}\cos x\,dx = \frac{e^{ax}}{a^2+b^2}$ **{a cos bx + b sin bx}**

Hence, a=1, b=1

So, $I = \frac{e^t}{2}[\cos t + \sin t] + c$

Hence,

$$\int \cos(\log x)\,dx = \frac{e^{\log x}}{2}\{\cos(\log x) + \sin(\log x)\} + c$$

$$I = \frac{x}{2}\{\cos(\log x) + \sin(\log x)\} + c$$

4. $\int e^{2x}\cos(3x+4)\,dx$

Solution:

Let $I = \int e^{2x}\cos(3x+4)\,dx$

Integrating by parts

$$I = e^{2x}\frac{\sin(3x+4)}{3} - \int 2e^{2x}\frac{\sin(3x+4)}{3}dx$$

$$= \frac{1}{3}e^{2x}\sin(3x+4) - \frac{2}{3}\int e^{2x}\sin(3x+4)\,dx$$

$$= \frac{1}{3}e^{2x}\sin(3x+4) - \frac{2}{3}\left\{-e^{2x}\frac{\cos(3x+4)}{3} + \int 2e^{2x}\frac{\cos(3x+4)}{3}dx\right\}$$

$$I = \frac{1}{3}e^{2x}\sin(3x+4) + \frac{2}{9}e^{2x}\cos(3x+4) - \frac{4}{9}I$$

Hence,

$$I = \frac{e^{2x}}{13}[2\cos(3x+4) + 3\sin(3x+4)] + c$$

5. $\int e^{2x}\sin x\cos x\,dx$

Solution:

Let $I = \int e^{2x} \sin x \cos x dx$

$$= \frac{1}{2}\int e^{2x} 2 \sin x \cos x dx$$

$$= \frac{1}{2}\int e^{2x} \sin 2x \, dx$$

We know that,

$$\int e^{ax} \sin bx dx = \frac{e^{ax}}{a^2 + b^2}\{a \sin bx - b \cos bx\} + c$$

$$= \frac{e^{2x}}{8}\{2 \sin 2x - 2 \cos 2x\} + c$$

$$I = \frac{1}{2}\frac{e^{2x}}{8}\{2 \sin 2x - 2 \cos 2x\} + c$$

$$I = \frac{e^{2x}}{8}\{\sin 2x - \cos 2x\} + c$$

EXERCISE 19.28 PAGE NO: 19.154

Evaluate the following integrals:

1. $\int \sqrt{3+2x-x^2}\, dx$

Solution:

Let, I = $\int \sqrt{3+2x-x^2}\, dx$

$\therefore$ I = $\int \sqrt{3-(x^2-2(1)x)}dx = \int \sqrt{3-(x^2-2(1)x+1)+1}\, dx$

Using $a^2 - 2ab + b^2 = (a - b)^2$

We have:

I = $\int \sqrt{4-(x-1)^2}\, dx = \int \sqrt{2^2-(x-1)^2}dx$

As I match with the form:

$\int \sqrt{a^2-x^2}\, dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + C$

By using above form and simplifying we get

$\therefore$ I = $\frac{x-1}{2}\sqrt{4-(x-1)^2} + \frac{4}{2}\sin^{-1}(\frac{x-1}{2}) + C$

$\Rightarrow$ I = $\frac{1}{2}(x-1)\sqrt{3+2x-x^2} + 2\sin^{-1}\left(\frac{x-1}{2}\right) + C$

2. $\int \sqrt{x^2+x+1}\, dx$

Solution:

Let, I = $\int \sqrt{(x^2+x+1)}\, dx$

$\therefore$ I = $\int \sqrt{x^2+2\left(\frac{1}{2}\right)x+\left(\frac{1}{2}\right)^2+1-\left(\frac{1}{2}\right)^2}\, dx$

Using $a^2 + 2ab + b^2 = (a + b)^2$

We have:

I = $\int \sqrt{\left(x+\frac{1}{2}\right)^2 + 1 - \frac{1}{4}}\, dx = \int \sqrt{\left(x+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}\, dx$

As I match with the form:

$$\int \sqrt{x^2 + a^2}\, dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log\left|x + \sqrt{x^2+a^2}\right| + C$$

∴ I = $\frac{\left(x+\frac{1}{2}\right)}{2}\sqrt{\left(x+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} + \frac{\left(\frac{\sqrt{3}}{2}\right)^2}{2}\log\left|\left(x+\frac{1}{2}\right) + \sqrt{\left(x+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}\right| + C$

⇒ I = $\frac{1}{4}(2x+1)\sqrt{x^2+x+1} + \frac{3}{8}\log\left|\left(x+\frac{1}{2}\right) + \sqrt{\left(x+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}\right| + C$

⇒ I = $\frac{1}{4}(2x+1)\sqrt{x^2+x+1} + \frac{3}{8}\log\left|\left(x+\frac{1}{2}\right) + \sqrt{x^2+x+1}\right| + C$

3. $\int \sqrt{x - x^2}\, dx$

Solution:

Let, I = $\int \sqrt{x - x^2}\, dx$

∴ I = $\int \sqrt{-\left(x^2 - 2\left(\frac{1}{2}\right)x\right)}\, dx = \int \sqrt{\frac{1}{4} - \left(x^2 - 2\left(\frac{1}{2}\right)x + \left(\frac{1}{2}\right)^2\right)}\, dx$

Using $a^2 - 2ab + b^2 = (a - b)^2$

We have:

I = $\int \sqrt{\frac{1}{4} - \left(x - \frac{1}{2}\right)^2}\, dx = \int \sqrt{\left(\frac{1}{2}\right)^2 - \left(x - \frac{1}{2}\right)^2}\, dx$

As I match with the form: $\int \sqrt{a^2 - x^2}\, dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + C$

∴ I = $\frac{x-\frac{1}{2}}{2}\sqrt{\left(\frac{1}{2}\right)^2 - \left(x-\frac{1}{2}\right)^2} + \frac{\frac{1}{4}}{2}\sin^{-1}\left(\frac{x-\frac{1}{2}}{\frac{1}{2}}\right) + C$

$\Rightarrow I = \frac{1}{4}(2x - 1)\sqrt{x - x^2} + \frac{1}{8}\sin^{-1}(2x - 1) + C$

4. $\int \sqrt{1 + x - 2x^2}\, dx$

Solution:

Let, $I = \int \sqrt{1 + x - 2x^2}\, dx$

$\therefore I = \int \sqrt{1 - 2\left(x^2 - 2\left(\frac{1}{4}\right)x\right)}\, dx = \int \sqrt{1 - 2\left(x^2 - 2\left(\frac{1}{4}\right)x + \left(\frac{1}{4}\right)^2\right) + 2\left(\frac{1}{4}\right)^2}\, dx$

Using $a^2 - 2ab + b^2 = (a - b)^2$

We have:

$I = \int \sqrt{\frac{9}{8} - 2\left(x - \frac{1}{4}\right)^2}\, dx = \int \sqrt{2}\sqrt{\left(\frac{3}{4}\right)^2 - \left(x - \frac{1}{4}\right)^2}\, dx$

As I match with the form: $\int \sqrt{a^2 - x^2}\, dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + C$

$\therefore I = \sqrt{2}\left\{\frac{x - \frac{1}{4}}{2}\sqrt{\left(\frac{3}{4}\right)^2 - \left(x - \frac{1}{4}\right)^2} + \frac{\frac{9}{16}}{2}\sin^{-1}\left(\frac{x - \frac{1}{4}}{\frac{3}{4}}\right)\right\} + C$

$\Rightarrow I = \frac{1}{8}(4x - 1)\sqrt{2\left\{\left(\frac{3}{4}\right)^2 - \left(x - \frac{1}{4}\right)^2\right\}} + \frac{9\sqrt{2}}{32}\sin^{-1}\left(\frac{4x-1}{3}\right) + C$

$\Rightarrow I = \frac{1}{8}(4x - 1)\sqrt{1 + x - 2x^2} + \frac{9\sqrt{2}}{32}\sin^{-1}\left(\frac{4x-1}{3}\right) + C$

5. $\int \cos x\sqrt{4 - \sin^2 x}\, dx$

Solution:

Let, $I = \int \cos x\sqrt{4 - \sin^2 x}\, dx$

Let, sin x = t

Differentiating both sides:

$\Rightarrow$ cos x dx = dt

Substituting sin x with t, we have:

$\therefore$ I = $\int \sqrt{4 - t^2}dt = \int \sqrt{2^2 - t^2}dt$

As I match with the form: $\int \sqrt{a^2 - x^2}\,dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + C$

$\therefore$ I = $\frac{t}{2}\sqrt{4 - (t)^2} + \frac{4}{2}\sin^{-1}(\frac{t}{2}) + C$

Putting the value of t i.e. t = sin x

$\Rightarrow$ I = $\frac{1}{2}\sin x\sqrt{4 - \sin^2 x} + 2\sin^{-1}\left(\frac{\sin x}{2}\right) + C$

EXERCISE 19.29

Evaluate the following integrals:

1. $\int (x+1)\sqrt{x^2-x+1}\,dx$

Solution:

Let us assume $x+1=\lambda\frac{d}{dx}(x^2-x+1)+\mu$

$\Rightarrow x+1=\lambda\left[\frac{d}{dx}(x^2)-\frac{d}{dx}(x)+\frac{d}{dx}(1)\right]+\mu$

We know $\frac{d}{dx}(x^n)=nx^{n-1}$ and derivative of a constant is 0.

$\Rightarrow x + 1 = \lambda\,(2x^{2-1} - 1 + 0) + \mu$

$\Rightarrow x + 1 = \lambda\,(2x - 1) + \mu$

$\Rightarrow x + 1 = 2\lambda x + \mu - \lambda$

Comparing the coefficient of x on both sides, we get

$2\lambda = 1 \Rightarrow \lambda=\frac{1}{2}$

Comparing the constant on both sides, we get

$\mu - \lambda = 1$

$\Rightarrow \mu-\frac{1}{2}=1$

$\therefore \mu=\frac{3}{2}$

Hence, we have $x+1=\frac{1}{2}(2x-1)+\frac{3}{2}$

Substituting this value in I, we can write the integral as

$I=\int\left[\frac{1}{2}(2x-1)+\frac{3}{2}\right]\sqrt{x^2-x+1}\,dx$

$\Rightarrow I = \int \left[\frac{1}{2}(2x-1)\sqrt{x^2-x+1} + \frac{3}{2}\sqrt{x^2-x+1}\right] dx$

$\Rightarrow I = \int \frac{1}{2}(2x-1)\sqrt{x^2-x+1}dx + \int \frac{3}{2}\sqrt{x^2-x+1}dx$

$\Rightarrow I = \frac{1}{2}\int (2x-1)\sqrt{x^2-x+1}dx + \frac{3}{2}\int \sqrt{x^2-x+1}dx$

Let $I_1 = \frac{1}{2}\int (2x-1)\sqrt{x^2-x+1}dx$

Now, put $x^2 - x + 1 = t$

$\Rightarrow (2x - 1)\, dx = dt$ (Differentiating both sides)

Substituting this value in I_1, we can write

$I_1 = \frac{1}{2}\int \sqrt{t}dt$

$\Rightarrow I_1 = \frac{1}{2}\int t^{\frac{1}{2}}dt$

We know that $\int x^n dx = \frac{x^{n+1}}{n+1} + c$

$\Rightarrow I_1 = \frac{1}{2}\left(\frac{t^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right) + c$

$\Rightarrow I_1 = \frac{1}{2}\left(\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right) + c$

$\Rightarrow I_1 = \frac{1}{2} \times \frac{2}{3} t^{\frac{3}{2}} + c$

$\Rightarrow I_1 = \frac{1}{3} t^{\frac{3}{2}} + c$

$\therefore I_1 = \frac{1}{3}(x^2 - x + 1)^{\frac{3}{2}} + c$

Let $I_2 = \frac{3}{2}\int \sqrt{x^2-x+1}dx$

We can write $x^2 - x + 1 = x^2 - 2(x)\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2 + 1$

$$\Rightarrow x^2 - x + 1 = \left(x - \frac{1}{2}\right)^2 - \frac{1}{4} + 1$$

$$\Rightarrow x^2 - x + 1 = \left(x - \frac{1}{2}\right)^2 + \frac{3}{4}$$

$$\Rightarrow x^2 - x + 1 = \left(x - \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2$$

Hence, we can write I_2 as

$$I_2 = \frac{3}{2}\int \sqrt{\left(x - \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}\, dx$$

We know that $\int \sqrt{x^2 + a^2}\, dx = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2}\ln\left|x + \sqrt{x^2 + a^2}\right| + c$

$$\Rightarrow I_2 = \frac{3}{2}\left[\frac{\left(x - \frac{1}{2}\right)}{2}\sqrt{\left(x - \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} + \frac{\left(\frac{\sqrt{3}}{2}\right)^2}{2}\ln\left|\left(x - \frac{1}{2}\right) + \sqrt{\left(x - \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}\right|\right] + c$$

$$\Rightarrow I_2 = \frac{3}{2}\left[\frac{2x - 1}{4}\sqrt{x^2 - x + 1} + \frac{3}{8}\ln\left|x - \frac{1}{2} + \sqrt{x^2 - x + 1}\right|\right] + c$$

$$\therefore I_2 = \frac{3}{8}(2x - 1)\sqrt{x^2 - x + 1} + \frac{9}{16}\ln\left|x - \frac{1}{2} + \sqrt{x^2 - x + 1}\right| + c$$

Substituting I_1 and I_2 in I, we get

$$I = \frac{1}{3}(x^2 - x + 1)^{\frac{3}{2}} + \frac{3}{8}(2x - 1)\sqrt{x^2 - x + 1} + \frac{9}{16}\ln\left|x - \frac{1}{2} + \sqrt{x^2 - x + 1}\right| + c$$

Thus,

$$\int (x + 1)\sqrt{x^2 - x + 1}\, dx = \frac{1}{3}(x^2 - x + 1)^{\frac{3}{2}} + \frac{3}{8}(2x - 1)\sqrt{x^2 - x + 1} + \frac{9}{16}\ln\left|x - \frac{1}{2} + \sqrt{x^2 - x + 1}\right| + c$$

2. $\int (x+1)\sqrt{2x^2+3}\,dx$

Solution:

Let $I = \int (x+1)\sqrt{2x^2+3}dx$

Let us assume $x+1 = \lambda\frac{d}{dx}(2x^2+3)+\mu$

$\Rightarrow x+1 = \lambda\left[\frac{d}{dx}(2x^2)+\frac{d}{dx}(3)\right]+\mu$

$\Rightarrow x+1 = \lambda\left[2\frac{d}{dx}(x^2)+\frac{d}{dx}(3)\right]+\mu$

We know $\frac{d}{dx}(x^n) = nx^{n-1}$ and derivative of a constant is 0.

$\Rightarrow x + 1 = \lambda (2 \times 2x^{2-1} + 0) + \mu$

$\Rightarrow x + 1 = \lambda (4x) + \mu$

$\Rightarrow x + 1 = 4\lambda x + \mu$

Comparing the coefficient of x on both sides, we get

$4\lambda = 1 \Rightarrow \lambda = \frac{1}{4}$

Comparing the constant on both sides, we get

$4\lambda = 1 \Rightarrow \lambda = \frac{1}{4}$

Comparing the constant on both sides, we get

$\mu = 1$

Hence, we have $x+1 = \frac{1}{4}(4x)+1$

Substituting this value in I, we can write the integral as

$I = \int \left[\frac{1}{4}(4x)+1\right]\sqrt{2x^2+3}dx$

$\Rightarrow I = \int \left[\frac{1}{4}(4x)\sqrt{2x^2+3}+\sqrt{2x^2+3}\right]dx$

$\Rightarrow I = \int \frac{1}{4}(4x)\sqrt{2x^2+3}dx + \int \sqrt{2x^2+3}dx$

$\Rightarrow I = \frac{1}{4}\int (4x)\sqrt{2x^2+3}dx + \int \sqrt{2x^2+3}dx$

Let $I_1 = \frac{1}{4}\int (4x)\sqrt{2x^2+3}dx$

Now, put $2x^2 + 3 = t$

$\Rightarrow$ (4x) dx = dt (Differentiating both sides)

Substituting this value in I_1, we can write

$I_1 = \frac{1}{4}\int \sqrt{t}dt$

$\Rightarrow I_1 = \frac{1}{4}\int t^{\frac{1}{2}}dt$

We know that $\int x^n dx = \frac{x^{n+1}}{n+1} + c$

$\Rightarrow I_1 = \frac{1}{4}\left(\frac{t^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right) + c$

$\Rightarrow I_1 = \frac{1}{4}\left(\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right) + c$

$\Rightarrow I_1 = \frac{1}{4} \times \frac{2}{3}t^{\frac{3}{2}} + c$

$\Rightarrow I_1 = \frac{1}{6}t^{\frac{3}{2}} + c$

$\therefore I_1 = \frac{1}{6}(2x^2+3)^{\frac{3}{2}} + c$

Let $I_2 = \int \sqrt{2x^2+3}dx$

We can write $2x^2 + 3 = 2\left(x^2 + \frac{3}{2}\right)$

$$\Rightarrow 2x^2+3=2\left[x^2+\left(\sqrt{\frac{3}{2}}\right)^2\right]$$

Hence, we can write I_2 as

$$I_2=\int\sqrt{2\left[x^2+\left(\sqrt{\frac{3}{2}}\right)^2\right]}dx \Rightarrow I_2=\sqrt{2}\int\sqrt{x^2+\left(\sqrt{\frac{3}{2}}\right)^2}\,dx$$

We know that $\int\sqrt{x^2+a^2}dx=\frac{x}{2}\sqrt{x^2+a^2}+\frac{a^2}{2}\ln\left|x+\sqrt{x^2+a^2}\right|+c$

$$\Rightarrow I_2=\sqrt{2}\left[\frac{x}{2}\sqrt{x^2+\left(\sqrt{\frac{3}{2}}\right)^2}+\frac{\left(\sqrt{\frac{3}{2}}\right)^2}{2}\ln\left|x+\sqrt{x^2+\left(\sqrt{\frac{3}{2}}\right)^2}\right|\right]+c$$

$$\Rightarrow I_2=\sqrt{2}\left[\frac{x}{2}\sqrt{x^2+\frac{3}{2}}+\frac{3}{4}\ln\left|x+\sqrt{x^2+\frac{3}{2}}\right|\right]+c$$

$$\Rightarrow I_2=\sqrt{2}\left[\frac{x}{2\sqrt{2}}\sqrt{2x^2+3}+\frac{3}{2\times 2}\ln\left|x+\sqrt{x^2+\frac{3}{2}}\right|\right]+c$$

$$\therefore I_2=\frac{x}{2}\sqrt{2x^2+3}+\frac{3}{2\sqrt{2}}\ln\left|x+\sqrt{x^2+\frac{3}{2}}\right|+c$$

Substituting I_1 and I_2 in I, we get

$$I=\frac{1}{6}(2x^2+3)^{\frac{3}{2}}+\frac{x}{2}\sqrt{2x^2+3}+\frac{3}{2\sqrt{2}}\ln\left|x+\sqrt{x^2+\frac{3}{2}}\right|+c$$

Thus,

$$\int(x+1)\sqrt{2x^2+3}dx=\frac{1}{6}(2x^2+3)^{\frac{3}{2}}+\frac{x}{2}\sqrt{2x^2+3}+\frac{3}{2\sqrt{2}}\ln\left|x+\sqrt{x^2+\frac{3}{2}}\right|+c$$

3. $\int (2x-5)\sqrt{2+3x-x^2}\,dx$

Solution:

Let $I = \int (2x-5)\sqrt{2+3x-x^2}dx$

Let us assume $2x-5 = \lambda \frac{d}{dx}(2+3x-x^2)+\mu$

$\Rightarrow 2x-5 = \lambda\left[\frac{d}{dx}(2)+\frac{d}{dx}(3x)-\frac{d}{dx}(x^2)\right]+\mu$

$\Rightarrow 2x-5 = \lambda\left[\frac{d}{dx}(2)+3\frac{d}{dx}(x)-\frac{d}{dx}(x^2)\right]+\mu$

We know $\frac{d}{dx}(x^n) = nx^{n-1}$ and derivative of a constant is 0.

$\Rightarrow 2x - 5 = \lambda (0 + 3 - 2x^{2-1}) + \mu$

$\Rightarrow 2x - 5 = \lambda (3 - 2x) + \mu$

$\Rightarrow 2x - 5 = -2\lambda x + 3\lambda + \mu$

Comparing the coefficient of x on both sides, we get

$-2\lambda = 2 \Rightarrow \lambda = -1$

Comparing the constant on both sides, we get

$3\lambda + \mu = -5$

$\Rightarrow 3(-1) + \mu = -5$

$\Rightarrow -3 + \mu = -5$

$\therefore \mu = -2$

Hence, we have $2x-5 = -(3-2x)-2$

Substituting this value in I, we can write the integral as

$I = \int [-(3-2x)-2]\sqrt{2+3x-x^2}dx$

$\Rightarrow I = \int \left[-(3-2x)\sqrt{2+3x-x^2}-2\sqrt{2+3x-x^2}\right]dx$

$$\Rightarrow I = -\int (3-2x)\sqrt{2+3x-x^2}dx - \int 2\sqrt{2+3x-x^2}dx$$

$$\Rightarrow I = -\int (3-2x)\sqrt{2+3x-x^2}dx - 2\int \sqrt{2+3x-x^2}dx$$

Let $I_1 = -\int (3-2x)\sqrt{2+3x-x^2}dx$

Now, put $2 + 3x - x^2 = t$

$\Rightarrow (3 - 2x)\, dx = dt$ (Differentiating both sides)

Substituting this value in I_1, we can write

$$I_1 = -\int \sqrt{t}dt$$

$$\Rightarrow I_1 = -\int t^{\frac{1}{2}}dt$$

We know that $\int x^n dx = \frac{x^{n+1}}{n+1} + c$

$$\Rightarrow I_1 = -\frac{t^{\frac{1}{2}+1}}{\frac{1}{2}+1} + c$$

$$\Rightarrow I_1 = -\frac{t^{\frac{3}{2}}}{\frac{3}{2}} + c$$

$$\Rightarrow I_1 = -\frac{2}{3}t^{\frac{3}{2}} + c$$

$$\Rightarrow I_1 = -\frac{2}{3}t^{\frac{3}{2}} + c$$

$$\therefore I_1 = -\frac{2}{3}(2+3x-x^2)^{\frac{3}{2}} + c$$

Let $I_2 = -2\int \sqrt{2+3x-x^2}dx$

We can write $2 + 3x - x^2 = -(x^2 - 3x - 2)$

$$\Rightarrow 2+3x-x^2=-\left[x^2-2(x)\left(\frac{3}{2}\right)+\left(\frac{3}{2}\right)^2-\left(\frac{3}{2}\right)^2-2\right]$$

$$\Rightarrow 2+3x-x^2=-\left[\left(x-\frac{3}{2}\right)^2-\frac{9}{4}-2\right]$$

$$\Rightarrow 2+3x-x^2=-\left[\left(x-\frac{3}{2}\right)^2-\frac{17}{4}\right]$$

$$\Rightarrow 2+3x-x^2=\frac{17}{4}-\left(x-\frac{3}{2}\right)^2$$

$$\Rightarrow 2+3x-x^2=\left(\frac{\sqrt{17}}{2}\right)^2-\left(x-\frac{3}{2}\right)^2$$

Hence, we can write I_2 as

$$I_2=-2\int\sqrt{\left(\frac{\sqrt{17}}{2}\right)^2-\left(x-\frac{3}{2}\right)^2}\,dx$$

We have $\int\sqrt{a^2-x^2}dx=\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}+c$

$$\Rightarrow I_2=-2\left[\frac{\left(x-\frac{3}{2}\right)}{2}\sqrt{\left(\frac{\sqrt{17}}{2}\right)^2-\left(x-\frac{3}{2}\right)^2}+\frac{\left(\frac{\sqrt{17}}{2}\right)^2}{2}\sin^{-1}\left(\frac{x-\frac{3}{2}}{\frac{\sqrt{17}}{2}}\right)\right]+c$$

$$\Rightarrow I_2=-2\left[\frac{2x-3}{4}\sqrt{2+3x-x^2}+\frac{17}{8}\sin^{-1}\left(\frac{2x-3}{\sqrt{17}}\right)\right]+c$$

$$\therefore I_2=-\frac{1}{2}(2x-3)\sqrt{2+3x-x^2}-\frac{17}{4}\sin^{-1}\left(\frac{2x-3}{\sqrt{17}}\right)+c$$

Substituting I_1 and I_2 in I, we get

$$I=-\frac{2}{3}(2+3x-x^2)^{\frac{3}{2}}-\frac{1}{2}(2x-3)\sqrt{2+3x-x^2}-\frac{17}{4}\sin^{-1}\left(\frac{2x-3}{\sqrt{17}}\right)+c$$

Thus,

$\int(2x-5)\sqrt{2+3x-x^2}dx = -\frac{2}{3}(2+3x-x^2)^{\frac{3}{2}} - \frac{1}{2}(2x-3)\sqrt{2+3x-x^2} - \frac{17}{4}\sin^{-1}\left(\frac{2x-3}{\sqrt{17}}\right) + c$

4. $\int (x+2)\sqrt{x^2+x+1}\, dx$

Solution:

Let $I = \int(x+2)\sqrt{x^2+x+1}dx$

Let us assume $x+2 = \lambda\frac{d}{dx}(x^2+x+1) + \mu$

$\Rightarrow x+2 = \lambda\left[\frac{d}{dx}(x^2) + \frac{d}{dx}(x) + \frac{d}{dx}(1)\right] + \mu$

We know $\frac{d}{dx}(x^n) = nx^{n-1}$ and derivative of a constant is 0.

$\Rightarrow x + 2 = \lambda\,(2x^{2-1} + 1 + 0) + \mu$

$\Rightarrow x + 2 = \lambda\,(2x + 1) + \mu$

$\Rightarrow x + 2 = 2\lambda x + \lambda + \mu$

Comparing the coefficient of x on both sides, we get

$2\lambda = 1 \Rightarrow \lambda = \frac{1}{2}$

Comparing the constant on both sides, we get

$\lambda + \mu = 2$

$\Rightarrow \frac{1}{2} + \mu = 2$

$\therefore \mu = \frac{3}{2}$

Hence, we have $x+2 = \frac{1}{2}(2x+1) + \frac{3}{2}$

Substituting this value in I, we can write the integral as

$$I = \int \left[\frac{1}{2}(2x+1) + \frac{3}{2}\right] \sqrt{x^2+x+1}dx$$

$$\Rightarrow I = \int \left[\frac{1}{2}(2x+1)\sqrt{x^2+x+1} + \frac{3}{2}\sqrt{x^2+x+1}\right] dx$$

$$\Rightarrow I = \int \frac{1}{2}(2x+1)\sqrt{x^2+x+1}dx + \int \frac{3}{2}\sqrt{x^2+x+1}dx$$

$$\Rightarrow I = \frac{1}{2}\int (2x+1)\sqrt{x^2+x+1}dx + \frac{3}{2}\int \sqrt{x^2+x+1}dx$$

Let $I_1 = \frac{1}{2}\int (2x+1)\sqrt{x^2+x+1}dx$

Now, put $x^2 + x + 1 = t$

$\Rightarrow (2x + 1)\, dx = dt$ (Differentiating both sides)

Substituting this value in I_1, we can write

$$I_1 = \frac{1}{2}\int \sqrt{t}dt$$

$$\Rightarrow I_1 = \frac{1}{2}\int t^{\frac{1}{2}}dt$$

We know that $\int x^n dx = \frac{x^{n+1}}{n+1} + c$

$$\Rightarrow I_1 = \frac{1}{2}\left(\frac{t^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right) + c$$

$$\Rightarrow I_1 = \frac{1}{2}\left(\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right) + c$$

$$\Rightarrow I_1 = \frac{1}{2} \times \frac{2}{3} t^{\frac{3}{2}} + c$$

$$\Rightarrow I_1 = \frac{1}{3} t^{\frac{3}{2}} + c$$

$$\therefore I_1 = \frac{1}{3}(x^2+x+1)^{\frac{3}{2}} + c$$

Let $I_2 = \frac{3}{2}\int \sqrt{x^2+x+1}dx$

We can write $x^2+x+1 = x^2+2(x)\left(\frac{1}{2}\right)+\left(\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2+1$

$$\Rightarrow x^2+x+1 = \left(x+\frac{1}{2}\right)^2-\frac{1}{4}+1$$

$$\Rightarrow x^2+x+1 = \left(x+\frac{1}{2}\right)^2+\frac{3}{4}$$

$$\Rightarrow x^2+x+1 = \left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2$$

Hence, we can write I_2 as

$$I_2 = \frac{3}{2}\int \sqrt{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\,dx$$

We know that $\int \sqrt{x^2+a^2}dx = \frac{x}{2}\sqrt{x^2+a^2}+\frac{a^2}{2}\ln\left|x+\sqrt{x^2+a^2}\right|+c$

$$\Rightarrow I_2 = \frac{3}{2}\left[\frac{\left(x+\frac{1}{2}\right)}{2}\sqrt{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2} + \frac{\left(\frac{\sqrt{3}}{2}\right)^2}{2}\ln\left|\left(x+\frac{1}{2}\right)+\sqrt{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right|\right]+c$$

$$\Rightarrow I_2 = \frac{3}{2}\left[\frac{2x+1}{4}\sqrt{x^2+x+1}+\frac{3}{8}\ln\left|x+\frac{1}{2}+\sqrt{x^2+x+1}\right|\right]+c$$

$$\therefore I_2 = \frac{3}{8}(2x+1)\sqrt{x^2+x+1}+\frac{9}{16}\ln\left|x+\frac{1}{2}+\sqrt{x^2+x+1}\right|+c$$

Substituting I_1 and I_2 in I, we get

$$I = \frac{1}{3}(x^2 + x + 1)^{\frac{3}{2}} + \frac{3}{8}(2x + 1)\sqrt{x^2 + x + 1} + \frac{9}{16}\ln\left|x + \frac{1}{2} + \sqrt{x^2 + x + 1}\right| + c$$

Thus,

$$\int (x + 2)\sqrt{x^2 + x + 1}dx = \frac{1}{3}(x^2 + x + 1)^{\frac{3}{2}} + \frac{3}{8}(2x + 1)\sqrt{x^2 + x + 1} + \frac{9}{16}\ln\left|x + \frac{1}{2} + \sqrt{x^2 + x + 1}\right| + c$$

EXERCISE 19.30

Evaluate the following integrals:

1. $\int \frac{2x+1}{(x+1)(x-2)}\,dx$

Solution:

Here the denominator is already factored.

So let

$$\frac{2x+1}{(x+1)(x-2)} = \frac{A}{x+1} + \frac{B}{x-2} \ldots\ldots \text{(i)}$$

$$\Rightarrow \frac{2x+1}{(x+1)(x-2)} = \frac{A(x-2)+B(x+1)}{(x+1)(x-2)}$$

$\Rightarrow 2x + 1 = A(x – 2) + B(x + 1)$...... (ii)

We need to solve for A and B. One way to do this is to pick values for x which will cancel each variable.

Put x = 2 in the above equation, we get

$\Rightarrow 2(2) + 1 = A(2 – 2) + B(2 + 1)$

$\Rightarrow 3B = 5$

$$\Rightarrow B = \frac{5}{3}$$

Now put x = – 1 in equation (ii), we get

$\Rightarrow 2(– 1) + 1 = A((– 1) – 2) + B((– 1) + 1)$

$\Rightarrow – 3A = – 1$

$$\Rightarrow A = \frac{1}{3}$$

We put the values of A and B values back into our partial fractions in equation

(i) Now replace this as the integrand. We get

$$\int \left[\frac{A}{x+1} + \frac{B}{x-2}\right] dx$$

$$\Rightarrow \int \left[\frac{\frac{1}{3}}{x+1} + \frac{\frac{5}{3}}{x-2}\right] dx$$

Split up the integral,

$$\Rightarrow \frac{1}{3}\int \left[\frac{1}{x+1}\right] dx + \frac{5}{3}\int \left[\frac{1}{x-2}\right] dx$$

Let substitute u = x + 1 ⇒ du = dx and z = x – 2 ⇒ dz = dx, so the above equation becomes,

$$\Rightarrow \frac{1}{3}\int \left[\frac{1}{u}\right] du + \frac{5}{3}\int \left[\frac{1}{z}\right] dz$$

On integrating we get

$$\Rightarrow \frac{1}{3}\log|u| + \frac{5}{3}\log|z| + C$$

Substituting back, we get

$$\Rightarrow \frac{1}{3}\log|x+1| + \frac{5}{3}\log|x-2| + C$$

The absolute value signs account for the domain of the natural log function (x>0).

Hence,

$$\int \frac{2x+1}{(x+1)(x-2)} dx = \frac{1}{3}\log|x+1| + \frac{5}{3}\log|x-2| + C$$

2. $\int \frac{1}{x(x-2)(x-4)} dx$

Solution:

In the given equation the denominator is already factored.

So let

$$\frac{1}{x(x-2)(x-4)} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x-4} \ldots\ldots (i)$$

$$\Rightarrow \frac{1}{x(x-2)(x-4)} = \frac{A(x-2)(x-4) + Bx(x-4) + Cx(x-2)}{x(x-2)(x-4)}$$

⇒ 1 = A(x – 2) (x – 4) + Bx (x – 4) + C x (x – 2)...... (ii)

We need to solve for A, B and C. One way to do this is to pick values for x which will cancel each variable.

Put x = 0 in the above equation, we get

⇒ 1 = A (0 – 2) (0 – 4) + B (0) (0 – 4) + C (0) (0 – 2)

⇒ 1 = 8A + 0 + 0

$$\Rightarrow A = \frac{1}{8}$$

Now put x = 2 in equation (ii), we get

⇒ 1 = A (2 – 2) (2 – 4) + B (2) (2 – 4) + C (2) (2 – 2)

⇒ 1 = 0 – 4B + 0

$$\Rightarrow B = -\frac{1}{4}$$

Now put x = 4 in equation (ii), we get

⇒ 1 = A (4 – 2) (4 – 4) + B (4) (4 – 4) + C (4) (4 – 2)

⇒ 1 = 0 + 0 + 8C

$$\Rightarrow C = \frac{1}{8}$$

We put the values of A, B, and C values back into our partial fractions in equation (i) and replace this as the integrand. We get

$$\int \left[\frac{A}{x} + \frac{B}{x-2} + \frac{C}{x-4}\right] dx$$

$$\Rightarrow \int \left[\frac{\frac{1}{8}}{x} + \frac{-\frac{1}{4}}{x-2} + \frac{\frac{1}{8}}{x-4}\right] dx$$

Split up the integral,

$$\Rightarrow \frac{1}{8}\int \left[\frac{1}{x}\right] dx - \frac{1}{4}\int \left[\frac{1}{x-2}\right] dx + \frac{1}{8}\int \left[\frac{1}{x-4}\right] dx$$

Let substitute u = x – 4 ⇒ du = dx and z = x – 2 ⇒ dz = dx, so the above equation becomes,

$$\Rightarrow \frac{1}{8}\int \left[\frac{1}{x}\right] dx - \frac{1}{4}\int \left[\frac{1}{z}\right] dz + \frac{1}{8}\int \left[\frac{1}{u}\right] du$$

On integrating we get

$$\Rightarrow \frac{1}{8}\log|x| - \frac{1}{4}\log|z| + \frac{1}{8}\log|u| + C$$

Substituting back, we get

$$\Rightarrow \frac{1}{8}\log|x| - \frac{1}{4}\log|x-2| + \frac{1}{8}\log|x-4| + C$$

We will take $\frac{1}{8}$ common, we get

$$\Rightarrow \frac{1}{8}[\log|x| - 2\log|x-2| + \log|x-4| + C]$$

Applying the logarithm rule we can rewrite the above equation as

$$\Rightarrow \frac{1}{8}\left[\log\left|\frac{x}{(x-2)^2}\right| + \log|x-4| + C\right]$$

$$\Rightarrow \frac{1}{8}\left[\log\left|\frac{x(x-4)}{(x-2)^2}\right|\right] + C$$

The absolute value signs account for the domain of the natural log function (x>0).

Hence,

$$\int \frac{1}{x(x-2)(x-4)} dx = \frac{1}{8}\left[\log\left|\frac{x(x-4)}{(x-2)^2}\right|\right] + C$$

3. $\int \frac{x^2+x-1}{x^2+x-6} dx$

Solution:

First we have to simplify numerator, we get

$$\frac{x^2 + x - 1}{x^2 + x - 6}$$

$$= \frac{x^2 + x - 6 + 5}{x^2 + x - 6}$$

$$= \frac{x^2 + x - 6}{x^2 + x - 6} + \frac{5}{x^2 + x - 6}$$

$$= 1 + \frac{5}{x^2 + x - 6}$$

Now we will factorize denominator by splitting the middle term, we get

$$1 + \frac{5}{x^2 + x - 6}$$

The above equation can be written as

$$= 1 + \frac{5}{x^2 + 3x - 2x - 6}$$

By taking factors common

$$= 1 + \frac{5}{x(x + 3) - 2(x + 3)}$$

$$= 1 + \frac{5}{(x + 3)(x - 2)}$$

Now the denominator is factorized, so let separate the fraction through partial fraction, hence let

$$\frac{5}{(x + 3)(x - 2)} = \frac{A}{x + 3} + \frac{B}{x - 2} \dots\dots (i)$$

$$\Rightarrow \frac{5}{(x + 3)(x - 2)} = \frac{A(x - 2) + B(x + 3)}{(x + 3)(x - 2)}$$

$\Rightarrow 5 = A(x - 2) + B(x + 3) \dots\dots (ii)$

We need to solve for A and B. One way to do this is to pick values for x which will cancel each variable.

Put x = 2 in the above equation, we get

$\Rightarrow 5 = A(2 - 2) + B(2 + 3)$

$\Rightarrow 5 = 0 + 5B$

$\Rightarrow B = 1$

Now put x = – 3 in equation (ii), we get

$\Rightarrow 5 = A((-3) - 2) + B((-3) + 3)$

$\Rightarrow 5 = -5A$

$\Rightarrow A = -1$

We put the values of A and B values back into our partial fractions in equation (i) and replace this as the integrand. We get

$$\int \left[1 + \frac{A}{x + 3} + \frac{B}{x - 2}\right] dx$$

$$\Rightarrow \int \left[1 + \frac{-1}{x + 3} + \frac{1}{x - 2}\right] dx$$

Split up the integral,

$$\Rightarrow \int 1dx - \int \left[\frac{1}{x+3}\right]dx + \int \left[\frac{1}{x-2}\right]dx$$

Let substitute u = x + 3 ⇒ du = dx and z = x – 2 ⇒ dz = dx, so the above equation becomes,

$$\Rightarrow \int 1dx - \int \left[\frac{1}{u}\right]du + \int \left[\frac{1}{z}\right]dz$$

On integrating we get

$\Rightarrow x - \log |u| + \log |z| + C$

Substituting back, we get

$\Rightarrow x - \log |x + 3| + \log |x - 2| + C$

Applying the logarithm rule, we can rewrite the above equation as

$$\Rightarrow x + \log\left|\frac{x-2}{x+3}\right| + C$$

The absolute value signs account for the domain of the natural log function (x>0).

Hence,

$$\int \frac{x^2+x-1}{x^2+x-6}dx = x + \log\left|\frac{x-2}{x+3}\right| + C$$

4. $\int \frac{3+4x-x^2}{(x+2)(x-1)}dx$

Solution:

First we simplify numerator, we get

$$\frac{3+4x-x^2}{(x+2)(x-1)}$$

$$= \frac{-(x^2-4x-3)}{x^2+x-2}$$

$$= \frac{-(x^2 + x - 5x - 2 - 1)}{x^2 + x - 2}$$

$$= \frac{-(x^2 + x - 2)}{x^2 + x - 2} + \frac{5x + 1}{x^2 + x - 2}$$

$$= -1 + \frac{5x + 1}{(x + 2)(x - 1)}$$

Now the denominator is factorized, so let separate the fraction through partial fraction, hence let

$$\frac{5x + 1}{(x + 2)(x - 1)} = \frac{A}{x + 2} + \frac{B}{x - 1} \ldots\ldots (i)$$

$$\Rightarrow \frac{5x + 1}{(x + 2)(x - 1)} = \frac{A(x - 1) + B(x + 2)}{(x + 2)(x - 1)}$$

$\Rightarrow 5x + 1 = A(x - 1) + B(x + 2)$…… (ii)

We need to solve for A and B. One way to do this is to pick values for x which will cancel each variable.

Put x = 1 in the above equation, we get

$\Rightarrow 5(1) + 1 = A\ 1 - 1) + B\ (1 + 2)$

$\Rightarrow 6 = 0 + 3B$

$\Rightarrow B = 2$

Now put x = – 2 in equation (ii), we get

$\Rightarrow 5(-2) + 1 = A\ ((-2) - 1) + B\ ((-2) + 2)$

$\Rightarrow -9 = -3A + 0$

$\Rightarrow A = 3$

We put the values of A and B values back into our partial fractions in equation (i) and replace this as the integrand. We get

$$\int \left[-1 + \frac{5x + 1}{(x + 2)(x - 1)}\right] dx$$

$$\Rightarrow \int \left[-1 + \frac{A}{x + 2} + \frac{B}{x - 1}\right] dx$$

$$\Rightarrow \int \left[-1 + \frac{3}{x + 2} + \frac{2}{x - 1}\right] dx$$

Split up the integral,

$$\Rightarrow -\int 1dx + 3\int \left[\frac{1}{x + 2}\right] dx + 2\int \left[\frac{1}{x - 1}\right] dx$$

Let substitute u = x + 2 ⇒ du = dx and z = x – 1 ⇒ dz = dx, so the above equation becomes,

$$\Rightarrow -\int 1dx + 3\int \left[\frac{1}{u}\right] du + 2\int \left[\frac{1}{z}\right] dz$$

On integrating we get

$$\Rightarrow -x + 3\log|u| + 2\log|z| + C$$

Substituting back, we get

$$\Rightarrow -x + 3\log|x + 2| + 2\log|x - 1| + C$$

The absolute value signs account for the domain of the natural log function (x>0).

Hence,

$$\int \frac{3 + 4x - x^2}{(x + 2)(x - 1)} dx = -x + 3\log|x + 2| + 2\log|x - 1| + C$$

5. $\int \frac{x^2 + 1}{x^2 - 1} dx$

Solution:
First we simplify numerator, we get

$$\frac{x^2 + 1}{x^2 - 1}$$

$$= \frac{x^2 - 1 + 2}{x^2 - 1}$$

$$= \frac{x^2 - 1}{x^2 - 1} + \frac{2}{x^2 - 1}$$

$$= 1 + \frac{2}{(x-1)(x+1)}$$

Now the denominator is factorized, so let separate the fraction through partial fraction, hence let

$$\frac{2}{(x+1)(x-1)} = \frac{A}{x+1} + \frac{B}{x-1} \ldots\ldots (i)$$

$$\Rightarrow \frac{2}{(x+1)(x-1)} = \frac{A(x-1) + B(x+1)}{(x+1)(x-1)}$$

$$\Rightarrow 2 = A(x-1) + B(x+1) \ldots\ldots (ii)$$

We need to solve for A and B. One way to do this is to pick values for x which will cancel each variable.

Put x = 1 in the above equation, we get

$\Rightarrow 2 = A(1-1) + B(1+1)$

$\Rightarrow 2 = 0 + 2B$

$\Rightarrow B = 1$

Now put x = – 1 in equation (ii), we get

$\Rightarrow 2 = A((-1) - 1) + B((-1) + 1)$

$\Rightarrow 2 = -2A + 0$

$\Rightarrow A = -1$

We put the values of A and B values back into our partial fractions in equation (i) and replace this as the integrand. We get

$$\int \left[1 + \frac{2}{(x-1)(x+1)}\right] dx$$

$$\Rightarrow \int \left[1 + \frac{A}{x + 1} + \frac{B}{x - 1}\right] dx$$

$$\Rightarrow \int \left[1 + \frac{-1}{x + 1} + \frac{1}{x - 1}\right] dx$$

Split up the integral,

$$\Rightarrow \int 1dx - \int \left[\frac{1}{x + 1}\right] dx + \int \left[\frac{1}{x - 1}\right] dx$$

Let substitute u = x + 1 ⇒ du = dx and z = x − 1 ⇒ dz = dx, so the above equation becomes,

$$\Rightarrow \int 1dx - \int \left[\frac{1}{u}\right] du + \int \left[\frac{1}{z}\right] dz$$

On integrating we get

$$\Rightarrow x - \log|u| + \log|z| + C$$

Substituting back, we get

$$\Rightarrow x - \log|x + 1| + \log|x - 1| + C$$

Applying the logarithm rule we get

$$\Rightarrow x + \log\left|\frac{x - 1}{x + 1}\right| + C$$

The absolute value signs account for the domain of the natural log function (x>0).

Hence,

$$\int \frac{x^2 + 1}{x^2 - 1} dx = x + \log\left|\frac{x - 1}{x + 1}\right| + C$$

EXERCISE 19.31 PAGE NO: 19.190

Evaluate the following integrals:

1. $\int \frac{x^2+1}{x^4+x^2+1} dx$

Solution:

The given equation can be written as,

$$\int \frac{1+\frac{1}{x^2}}{x^2+1+\frac{1}{x^2}} dx$$

$$= \int \frac{1+\frac{1}{x^2}}{\left(x-\frac{1}{x}\right)^2+3} dx$$

Let $x-\frac{1}{x}=t$

Then, $\left(1+\frac{1}{x^2}\right) dx = dt$

$$= \int \frac{1}{t^2+3} dt$$

Using standard identity we get

$$= \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{t}{\sqrt{3}}\right)+c$$

Substituting t as $x-\frac{1}{x}$, we get

$$= \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{\left(x-\frac{1}{x}\right)}{\sqrt{3}}\right)+c$$

$$= \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{x^2-1}{\sqrt{3}x}\right)+c$$

2. $\int \sqrt{\cot\theta}\, d\theta$

Solution:

Let cot θ = x^2

$$-\mathrm{cosec}^2\theta d\theta = 2x dx$$

$$d\theta = -\frac{2x}{1+\cot^2\theta}dx$$

$$d\theta = -\frac{2x}{1+x^4}dx$$

$$\int -\frac{2x^2}{1+x^4}dx$$

Re-writing the given equation as

$$-\int \frac{1+\frac{1}{x^2}+1-\frac{1}{x^2}}{\frac{1}{x^2}+x^2}dx$$

$$-\int \frac{1+\frac{1}{x^2}}{\left(x-\frac{1}{x}\right)^2+2}dx - \int \frac{1-\frac{1}{x^2}}{\left(x+\frac{1}{x}\right)^2-2}dx$$

Let $x-\frac{1}{x}=t$ and $x+\frac{1}{x}=z$

So $\left(1+\frac{1}{x^2}\right)dx = dt$ and $\left(1-\frac{1}{x^2}\right)dx = dz$

$$-\int \frac{dt}{(t)^2+2} - \int \frac{dz}{(z)^2-2}$$

Using identity $\int \frac{1}{x^2+a^2}dx = 1/a \tan^{-1}(x)$ and $\int \frac{dz}{(z)^2-1} = \frac{1}{2}\log\left|\frac{z-1}{z+1}\right| + c$

$$-\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{t}{\sqrt{2}}\right) - \frac{1}{2\sqrt{2}}\log\left|\frac{z-\sqrt{2}}{z+\sqrt{2}}\right| + c$$

Now, substituting t as x – 1/x and z as x + 1/x we have

$$= -\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x^2-1}{\sqrt{2}x}\right) - \frac{1}{2\sqrt{2}}\log\left|\frac{x^2+1-\sqrt{2}x}{x^2+1+\sqrt{2}x}\right| + c$$

Lastly, substituting x^2 as $\cot\theta$ we get

$$I = -\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\cot\theta-1}{\sqrt{2\cot\theta}}\right) - \frac{1}{2\sqrt{2}}\log\left|\frac{\cot\theta+1-\sqrt{2\cot\theta}}{\cot\theta+1-\sqrt{2\cot\theta}}\right| + c$$

3. $\int \frac{x^2+9}{x^4+81}dx$

Solution:

$$\int \frac{1+\frac{9}{x^2}}{x^2+\frac{81}{x^2}}dx$$

$$= \int \frac{1+\frac{9}{x^2}}{\left(x-\frac{9}{x}\right)^2+18}dx \quad \text{(By completing the square)}$$

Let $x - \frac{9}{x} = t$

$$\left(1+\frac{9}{x^2}\right)dx = dt$$

$$= \int \frac{dt}{t^2+18}$$

Using identity $\int \frac{1}{x^2+1}dx = \tan^{-1}(x)$

$$= \frac{1}{3\sqrt{2}}\tan^{-1}\left(\frac{t}{3\sqrt{2}}\right) + c$$

Substituting t as $x - \frac{1}{x}$

$$= \frac{1}{3\sqrt{2}}\tan^{-1}\left(\frac{x-\frac{1}{x}}{3\sqrt{2}}\right) + c$$

4. $\int \frac{1}{x^4 + x^2 + 1} dx$

Solution:

$$\int \frac{\frac{1}{x^2}}{x^2 + 1 + \frac{1}{x^2}} dx$$

$$= \frac{1}{2} \int \frac{1 + \frac{1}{x^2} + \frac{1}{x^2} - 1}{x^2 + 1 + \frac{1}{x^2}} dx$$ (Manipulating the numerator by multiplying and diving by 2)

$$= \frac{1}{2}\left[\int \frac{1 + \frac{1}{x^2}}{x^2 + 1 + \frac{1}{x^2}} dx + \int \frac{-1 + \frac{1}{x^2}}{x^2 + 1 + \frac{1}{x^2}} dx\right]$$

$$= \frac{1}{2}\left[\int \frac{1 + \frac{1}{x^2}}{\left(x - \frac{1}{x}\right)^2 + 3} dx + \int \frac{-1 + \frac{1}{x^2}}{\left(x + \frac{1}{x}\right)^2 - 1} dx\right]$$

Let $x - \frac{1}{x} = t$ and $x + \frac{1}{x} = z$

Then, $\left(1 + \frac{1}{x^2}\right) dx = dt$ and $\left(1 - \frac{1}{x^2}\right) dx = dz$

$$= \frac{1}{2}\left[\int \frac{dt}{(t)^2 + 3} - \int \frac{dz}{(z)^2 - 1}\right]$$

Using identity $\int \frac{1}{x^2+1} dx = \tan^{-1}(x)$ and $\int \frac{dz}{(z)^2-1} = \frac{1}{2}\log\left|\frac{z-1}{z+1}\right| + c$

$$= \frac{1}{2}\left[\frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{t}{\sqrt{3}}\right) - \frac{1}{2}\log\left|\frac{z-1}{z+1}\right|\right]$$

Substituting t as $x - \frac{1}{x}$ and z as $x + \frac{1}{x}$

We get,

$$= \frac{1}{2}\left[\frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{x - \frac{1}{x}}{\sqrt{3}}\right) - \frac{1}{2}\log\left|\frac{x + \frac{1}{x} - 1}{x + \frac{1}{x} + 1}\right|\right] + c$$

$$I = \frac{1}{2\sqrt{3}} tan^{-1}\left(\frac{x^2 - 1}{\sqrt{3}x}\right) - \frac{1}{4} log\left|\frac{x^2 + 1 - x}{x^2 + 1 + x}\right| + c$$

5. $\int \frac{x^2 - 3x + 1}{x^4 + x^2 + 1} dx$

Solution:

The given equation can be written as

$$\int \frac{1 - \frac{3}{x} + \frac{1}{x^2}}{x^2 + 1 + \frac{1}{x^2}} dx$$

$$\int \frac{1 + \frac{1}{x^2}}{\left(x - \frac{1}{x}\right)^2 + 3} dx - \int \frac{3x}{x^4 + x^2 + 1} dx$$

Substituting t as $x - \frac{1}{x}$ and z as x^2

$\left(1 + \frac{1}{x^2}\right) dx = dt$ And $2xdx = dz$

$$\int \frac{dt}{(t)^2 + 3} - \frac{3}{2}\int \frac{dz}{z^2 + z + 1}$$

$$\int \frac{dt}{(t)^2 + 3} - \frac{3}{2}\int \frac{dz}{\left(z + \frac{1}{2}\right)^2 + \frac{3}{4}}$$

Using identity $\int \frac{1}{x^2+1} dx = \tan^{-1}(x)$

$$\frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{t}{\sqrt{3}}\right) - \sqrt{3} \tan^{-1}\left(\frac{2z + 1}{\sqrt{3}}\right) + c$$

Substituting t as $x - \frac{1}{x}$ and z as x^2

$$= \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{x - \frac{1}{x}}{\sqrt{3}}\right) - \sqrt{3} \tan^{-1}\left(\frac{2x^2 + 1}{\sqrt{3}}\right) + c$$

Hence,

$$I = \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{x^2 - 1}{\sqrt{3}x}\right) - \sqrt{3} \tan^{-1}\left(\frac{2x^2 + 1}{\sqrt{3}}\right) + c$$

EXERCISE 19.32 PAGE NO: 19.196

Evaluate the following integrals:

1. $\int \frac{1}{(x-1)\sqrt{x+2}} dx$

Solution:

Assume $x + 2 = t^2$

$dx = 2tdt$

Now,

$$\int \frac{2dt}{(t^2-3)}$$

Using identity $\int \frac{dz}{(z)^2-1} = \frac{1}{2}\log\left|\frac{z-1}{z+1}\right| + c$

$$= \frac{1}{\sqrt{3}}\log\left|\frac{t-\sqrt{3}}{t+\sqrt{3}}\right| + c$$

$$= \frac{1}{\sqrt{3}}\log\left|\frac{\sqrt{(x+2)}-\sqrt{3}}{\sqrt{x+2}+\sqrt{3}}\right| + c$$

2. $\int \frac{1}{(x-1)\sqrt{2x+3}} dx$

Solution:

Assume $2x + 3 = t^2$

$dx = t\,dt$

$$\int \frac{dt}{\frac{t^2-3}{2}-1}$$

$$\int \frac{2dt}{(t^2-5)}$$

Using identity $\int \frac{dz}{(z)^2-1} = \frac{1}{2}\log\left|\frac{z-1}{z+1}\right| + c$

$$\frac{1}{\sqrt{5}}\log\left|\frac{t-\sqrt{5}}{t+\sqrt{5}}\right|+c$$

$$\frac{1}{\sqrt{5}}\log\left|\frac{\sqrt{(2x+3)}\ -\sqrt{5}}{\sqrt{2x+3}+\sqrt{5}}\right|+c$$

3. $\int \frac{x+1}{(x-1)\sqrt{x+2}}dx$

Solution:

The given equation can be written as

$$\int \frac{(x-1)+2}{(x-1)\sqrt{x+2}}dx$$

Now splitting the integral in two parts

$$\int \frac{(x-1)}{(x-1)\sqrt{x+2}}dx+\int \frac{2}{(x-1)\sqrt{x+2}}dx$$

For the first part using identity $\int x^n dx=\frac{x^{n+1}}{n+1}$

$$2\sqrt{x+2}$$

For the second part

Assume $x+2=t^2$

$dx = 2\,t\,dt$

$$\int \frac{4dt}{(t^2-3)}$$

Using identity $\int \frac{dz}{(z)^2-1}=\frac{1}{2}\log\left|\frac{z-1}{z+1}\right|+c$

$$\frac{2}{\sqrt{3}}\log\left|\frac{t-\sqrt{3}}{t+\sqrt{3}}\right|+c$$

$$\frac{2}{\sqrt{3}}\log\left|\frac{\sqrt{(x+2)}\ -\sqrt{3}}{\sqrt{x+2}+\sqrt{3}}\right|+c$$

Hence integral is

$$2\sqrt{x+2}+\frac{2}{\sqrt{3}}\log\left|\frac{\sqrt{(x+2)}\ -\sqrt{3}}{\sqrt{x+2}+\sqrt{3}}\right|+c$$

4. $\int \frac{x^2}{(x-1)\sqrt{x+2}}dx$

Solution:

The given equation can be written as

$$\int \frac{(x^2-1)+1}{(x-1)\sqrt{x+2}}dx$$

$$\int \frac{(x^2-1)}{(x-1)\sqrt{x+2}}dx+\int \frac{1}{(x-1)\sqrt{x+2}}dx$$

$$\int \frac{(x+1)}{\sqrt{x+2}}dx+\int \frac{1}{(x-1)\sqrt{x+2}}dx=\int\frac{(x+2)-1}{\sqrt{x+2}}dx+\int \frac{1}{(x-1)\sqrt{x+2}}dx$$

$$\int \frac{(1)}{\sqrt{x+2}}dx+\int \sqrt{x+2}\,dx+\int \frac{1}{(x-1)\sqrt{x+2}}dx$$

For the first- and second-part using identity $\int x^n dx=\frac{x^{n+1}}{n+1}$

$$\frac{2}{3}(x+2)^{\frac{3}{2}}-2\sqrt{x+2}$$

For the second part

Assume $x+2=t^2$

$dx = 2\,t\,dt$

So,

$$\int \frac{2\,dt}{(t^2-3)}$$

Using identity $\int \frac{dz}{(z)^2-1} = \frac{1}{2}\log\left|\frac{z-1}{z+1}\right| + c$

$$= 2.\frac{1}{2\sqrt{3}}\log\left|\frac{t-\sqrt{3}}{t+\sqrt{3}}\right| + c$$

$$= \frac{1}{\sqrt{3}}\log\left|\frac{\sqrt{(x+2)}-\sqrt{3}}{\sqrt{x+2}+\sqrt{3}}\right| + c \quad (\text{Using } t^2 = x+2)$$

Hence integral is

$$I = \frac{2}{3}(x+2)^{\frac{3}{2}} - 2\sqrt{x+2} + \frac{1}{\sqrt{3}}\log\left|\frac{\sqrt{(x+2)}-\sqrt{3}}{\sqrt{x+2}+\sqrt{3}}\right| + c$$

5. $\int \frac{x}{(x-3)\sqrt{x+1}}dx$

Solution:

The given equation can be written as

$$\int \frac{(x-3)+3}{(x-3)\sqrt{x+1}}dx$$

$$\int \frac{(x-3)}{(x-3)\sqrt{x+1}}dx + \int \frac{3}{(x-3)\sqrt{x+1}}dx$$

For the first part using identity $\int x^n dx = \frac{x^{n+1}}{n+1}$

$$2\sqrt{x+1} + c$$

For the second part $\int \frac{3}{(x-3)\sqrt{x+1}}dx$,

Assume $x + 1 = t^2$

$dx = 2t\ dt$

$$\int \frac{3.2dt}{(t^2-4)}$$

Using identity $\int \frac{dz}{(z)^2-1} = \frac{1}{2}\log\left|\frac{z-1}{z+1}\right| + c$

$$=3 \times 2 \times \frac{1}{2(2)} \log\left|\frac{t-2}{t+2}\right| + c$$

$$=\frac{3}{2}\log\left|\frac{\sqrt{(x+1)}-2}{\sqrt{x+1}+2}\right| + c$$

Hence integral is

$$= 2\sqrt{x+1} + \frac{3}{2}\log\left|\frac{\sqrt{(x+1)}-2}{\sqrt{x+1}+2}\right| + c$$