EXERCISE 4.10 PAGE NO: 4.66

1. Evaluate:
(i) Cot ($\sin^{-1}$ (3/4) + $\sec^{-1}$ (4/3))
(ii) Sin ($\tan^{-1}$ x + $\tan^{-1}$ 1/x) for x < 0
(iii) Sin ($\tan^{-1}$ x + $\tan^{-1}$ 1/x) for x > 0
(iv) Cot ($\tan^{-1}$ a + $\cot^{-1}$ a)
(v) Cos ($\sec^{-1}$ x + cosec^{-1} x), |x| ≥ 1

Solution:
(i) Given Cot ($\sin^{-1}$ (3/4) + $\sec^{-1}$ (4/3))

$$= \cot\left(\sin^{-1}\frac{3}{4} + \cos^{-1}\frac{3}{4}\right)$$

$$\left(\because \sec^{-1}x = \cos^{-1}\frac{1}{x}\right)$$

We have

$$\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$$

By substituting these values in given question, we get

$$= \cot\frac{\pi}{2}$$

= 0

(ii) Given Sin ($\tan^{-1}$ x + $\tan^{-1}$ 1/x) for x < 0

$$= \sin\left(\tan^{-1}x + (\cot^{-1}x - \pi)\right) \quad \left(\because \tan^{-1}\theta = \cot^{-1}\frac{1}{\theta} - \pi \qquad \text{for } x < 0\right)$$

$= \sin\left(\frac{\pi}{2} - \pi\right)\left(\because \tan^{-1}\theta + \cot^{-1}\theta = \frac{\pi}{2}\right)$

On simplifying, we get

$= \sin\left(-\frac{\pi}{2}\right)$

We know that sin (- θ) = – sin θ

$= -\sin\frac{\pi}{2} = -1$

(iii) Given Sin ($\tan^{-1}$ x + $\tan^{-1}$ 1/x) for x > 0

$= \sin\left(\tan^{-1}x + \cot^{-1}x\right)\left(\because \tan^{-1}\theta = \cot^{-1}\frac{1}{\theta} \quad \text{for } x > 0\right)$

Again we know that,

$\tan^{-1}\theta + \cot^{-1}\theta = \frac{\pi}{2}$

Now by substituting above identity in given question we get,

$= \sin\frac{\pi}{2}$

= 1

(iv) Given Cot ($\tan^{-1}$ a + $\cot^{-1}$ a)

We know that,

$\tan^{-1}\theta + \cot^{-1}\theta = \frac{\pi}{2}$

Now by substituting above identity in given question we get,

$= \cot\left(\frac{\pi}{2}\right)$

= 0

(v) Given Cos ($\sec^{-1} x + \text{cosec}^{-1} x$), $|x| \geq 1$

We know that

$$\sec^{-1}\theta = \cos^{-1}\frac{1}{\theta}$$

Again we have

$$\text{cosec}^{-1}\theta = \sin^{-1}\frac{1}{\theta}$$

By substituting these values in given question we get,

$= \cos\left(\cos^{-1}\frac{1}{x} + \sin^{-1}\frac{1}{x}\right)$

We know that from the identities,

$$\sin^{-1}\theta + \cos^{-1}\theta = \frac{\pi}{2}$$

Now by substituting we get,

$= \cos\frac{\pi}{2}$

= 0

2. If $\cos^{-1} x + \cos^{-1} y = \pi/4$, find the value of $\sin^{-1} x + \sin^{-1} y$.

Solution:

Given $\cos^{-1} x + \cos^{-1} y = \pi/4$

We know that

$$\sin^{-1}\theta + \cos^{-1}\theta = \frac{\pi}{2}$$

Now substituting above identity in given question we get,

$$\Rightarrow \left(\frac{\pi}{2} - \sin^{-1}x\right) + \left(\frac{\pi}{2} - \sin^{-1}y\right) = \frac{\pi}{4}$$

Adding and simplifying we get,

$$\Rightarrow \pi - \left(\sin^{-1}x + \sin^{-1}y\right) = \frac{\pi}{4}$$

On rearranging,

$$\Rightarrow \sin^{-1}x + \sin^{-1}y = \pi - \frac{\pi}{4}$$

$$\Rightarrow \sin^{-1}x + \sin^{-1}y = \frac{3\pi}{4}$$

3. If $\sin^{-1}x + \sin^{-1}y = \pi/3$ and $\cos^{-1}x - \cos^{-1}y = \pi/6$, find the values of x and y.

Solution:

Given $\sin^{-1}x + \sin^{-1}y = \pi/3$ ……. Equation (i)

And $\cos^{-1}x - \cos^{-1}y = \pi/6$ ……… Equation (ii)

Subtracting Equation (ii) from Equation (i), we get

$$\Rightarrow (\sin^{-1}x - \cos^{-1}x) + (\sin^{-1}y + \cos^{-1}y) = \frac{\pi}{3} - \frac{\pi}{6}$$

We know that,

$$\sin^{-1}\theta + \cos^{-1}\theta = \frac{\pi}{2},$$

By substituting above identity, we get

$$\Rightarrow (\sin^{-1}x - \cos^{-1}x) + \left(\frac{\pi}{2}\right) = \frac{\pi}{6}$$

Replacing $\sin^{-1}x$ by $\pi/2 - \cos^{-1}x$ and rearranging we get,

$$\Rightarrow \left(\frac{\pi}{2} - \cos^{-1}x\right) - \cos^{-1}x = -\frac{\pi}{3}$$

Now by adding,

$$\Rightarrow 2\cos^{-1}x = \frac{5\pi}{6}$$

$$\Rightarrow \cos^{-1}x = \frac{5\pi}{12}$$

$$\Rightarrow x = \cos\left(\frac{5\pi}{12}\right)$$

$$\Rightarrow x = \cos\left(\frac{\pi}{4} + \frac{\pi}{6}\right)$$

We know that Cos (A + B) = Cos A. Cos B – Sin A. Sin B, substituting this we get,

$$\Rightarrow x = \cos\frac{\pi}{4}.\cos\frac{\pi}{6} - \sin\frac{\pi}{4}.\sin\frac{\pi}{6}$$

$$\Rightarrow x = \frac{1}{\sqrt{2}}.\frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}}.\frac{1}{2}$$

$\Rightarrow x = \frac{\sqrt{3}-1}{2\sqrt{2}}$

Now, putting the value of $\cos^{-1} x$ in equation (ii)

$\Rightarrow \frac{5\pi}{12} - \cos^{-1} y = \frac{\pi}{6}$

$\Rightarrow \cos^{-1} y = \frac{\pi}{4}$

$\Rightarrow y = \frac{1}{\sqrt{2}}$

$\Rightarrow x = \frac{\sqrt{3}-1}{2\sqrt{2}}$ And $y = \frac{1}{\sqrt{2}}$

4. If cot ($\cos^{-1} 3/5 + \sin^{-1} x$) = 0, find the value of x.

Solution:

Given cot ($\cos^{-1} 3/5 + \sin^{-1} x$) = 0

On rearranging we get,

($\cos^{-1} 3/5 + \sin^{-1} x$) = $\cot^{-1}$ (0)

($\text{Cos}^{-1} 3/5 + \sin^{-1} x$) = π/2

We know that $\cos^{-1} x + \sin^{-1} x = \pi/2$

Then $\sin^{-1} x = \pi/2 - \cos^{-1} x$

Substituting the above in ($\cos^{-1} 3/5 + \sin^{-1} x$) = π/2 we get,

($\text{Cos}^{-1} 3/5 + \pi/2 - \cos^{-1} x$) = π/2

Now on rearranging we get,

($\text{Cos}^{-1} 3/5 - \cos^{-1} x$) = π/2 - π/2

($\text{Cos}^{-1} 3/5 - \cos^{-1} x$) = 0

Therefore $\text{Cos}^{-1} 3/5 = \cos^{-1} x$

On comparing the above equation we get,

x = 3/5

5. If $(\sin^{-1} x)^2 + (\cos^{-1} x)^2 = 17 \pi^2/36$, find x.

Solution:

Given $(\sin^{-1} x)^2 + (\cos^{-1} x)^2 = 17 \pi^2/36$

We know that $\cos^{-1} x + \sin^{-1} x = \pi/2$

Then $\cos^{-1} x = \pi/2 - \sin^{-1} x$

Substituting this in $(\sin^{-1} x)^2 + (\cos^{-1} x)^2 = 17 \pi^2/36$ we get

$(\sin^{-1} x)^2 + (\pi/2 - \sin^{-1} x)^2 = 17 \pi^2/36$

Let $y = \sin^{-1} x$

$y^2 + ((\pi/2) – y)^2 = 17 \pi^2/36$

$y^2 + \pi^2/4 – y^2 – 2y ((\pi/2) – y) = 17 \pi^2/36$

$\pi^2/4 – \pi y + 2 y^2 = 17 \pi^2/36$

On rearranging and simplifying, we get

$2y^2 - \pi y + 2/9 \pi^2 = 0$

$18y^2 - 9 \pi y + 2 \pi^2 = 0$

$18y^2 - 12 \pi y + 3 \pi y + 2 \pi^2 = 0$

$6y (3y - 2\pi) + \pi (3y - 2\pi) = 0$

Now, $(3y - 2\pi) = 0$ and $(6y + \pi) = 0$

Therefore $y = 2\pi/3$ and $y = - \pi/6$

Now substituting $y = - \pi/6$ in $y = \sin^{-1} x$ we get

$\sin^{-1} x = - \pi/6$

$x = \sin (- \pi/6)$

$x = -1/2$

Now substituting $y = -2\pi/3$ in $y = \sin^{-1} x$ we get

$x = \sin (2\pi/3)$

$x = \sqrt{3}/2$

Now substituting $x = \sqrt{3}/2$ in $(\sin^{-1} x)^2 + (\cos^{-1} x)^2 = 17 \pi^2/36$ we get,

$= \pi/3 + \pi/6$

$= \pi/2$ which is not equal to $17 \pi^2/36$

So we have to neglect this root.

Now substituting $x = -1/2$ in $(\sin^{-1} x)^2 + (\cos^{-1} x)^2 = 17 \pi^2/36$ we get,

$= \pi^2/36 + 4 \pi^2/9$

$= 17 \pi^2/36$

Hence $x = -1/2$.