

EXERCISE 4.14
PAGE NO: 4.115
1. Evaluate the following:

(i) $\tan \{2 \tan^{-1} (1/5) - \pi/4\}$

(ii) $\tan \{1/2 \sin^{-1} (3/4)\}$

(iii) $\sin \{1/2 \cos^{-1} (4/5)\}$

(iv) $\sin (2 \tan^{-1} 2/3) + \cos (\tan^{-1} \sqrt{3})$

Solution:

(i) Given $\tan \{2 \tan^{-1} (1/5) - \pi/4\}$

We know that,

$$2 \tan^{-1}(x) = \tan^{-1}\left(\frac{2x}{1-x^2}\right), \text{ if } |x| < 1$$

 And $\frac{\pi}{4}$ can be written as $\tan^{-1}(1)$

Now substituting these values we get,

$$= \tan \left\{ \tan^{-1} \left(\frac{2 \times \frac{1}{5}}{1 - \frac{1}{25}} \right) - \tan^{-1} 1 \right\}$$

$$= \tan \left\{ \tan^{-1} \left(\frac{5}{12} \right) - \tan^{-1} 1 \right\}$$

Again we know that,

$$\tan^{-1} x - \tan^{-1} y = \tan^{-1} \frac{x-y}{1+xy}$$

Now substituting this formula, we get

$$= \tan \left\{ \tan^{-1} \left(\frac{\frac{5}{12} - 1}{1 + \frac{5}{12}} \right) \right\}$$

$$= \tan \left\{ \tan^{-1} \left(\frac{-7}{17} \right) \right\}$$

$$= -\frac{7}{17}$$

(ii) Given $\tan \left\{ \frac{1}{2} \sin^{-1} \left(\frac{3}{4} \right) \right\}$

$$\text{Let } \frac{1}{2} \sin^{-1} \frac{3}{4} = t$$

Therefore,

$$\Rightarrow \sin^{-1} \frac{3}{4} = 2t$$

$$\Rightarrow \sin 2t = \frac{3}{4}$$

Now, by Pythagoras theorem, we have

$$\Rightarrow \sin 2t = \frac{3}{4} = \frac{\text{perpendicular}}{\text{hypotenuse}}$$

$$\Rightarrow \cos 2t = \frac{\sqrt{4^2 - 3^2}}{4} = \frac{\text{Base}}{\text{hypotenuse}}$$

$$\Rightarrow \cos 2t = \frac{\sqrt{7}}{4}$$

By considering, given question

$$\tan \left\{ \frac{1}{2} \sin^{-1} \left(\frac{3}{4} \right) \right\}$$

$$= \tan(t)$$

We know that,

$$\tan(x) = \sqrt{\frac{1 - \cos 2x}{1 + \cos 2x}}$$

$$= \sqrt{\frac{1 - \cos 2t}{1 + \cos 2t}}$$

$$= \sqrt{\frac{1 - \frac{\sqrt{7}}{4}}{1 + \frac{\sqrt{7}}{4}}}$$

$$= \sqrt{\frac{4 - \sqrt{7}}{4 + \sqrt{7}}}$$

Now by rationalizing the denominator, we get

$$= \sqrt{\frac{(4 - \sqrt{7})(4 - \sqrt{7})}{(4 + \sqrt{7})(4 - \sqrt{7})}}$$

$$= \sqrt{\frac{(4 - \sqrt{7})^2}{9}}$$

$$= \frac{4 - \sqrt{7}}{3}$$

Hence

$$\tan \left\{ \frac{1}{2} \sin^{-1} \frac{3}{4} \right\} = \frac{4 - \sqrt{7}}{3}$$

(iii) Given $\sin \left\{ \frac{1}{2} \cos^{-1} \left(\frac{4}{5} \right) \right\}$

We know that

$$\cos^{-1} x = 2 \sin^{-1} \left(\pm \sqrt{\frac{1-x}{2}} \right)$$

Now by substituting this formula we get,

$$\sin\left(\frac{1}{2}2 \sin^{-1}\left(\pm\sqrt{\frac{1-\frac{4}{5}}{2}}\right)\right)$$

$$= \sin\left(\sin^{-1}\left(\pm\sqrt{\frac{1}{2 \times 5}}\right)\right)$$

$$= \sin\left(\sin^{-1}\left(\pm\frac{1}{\sqrt{10}}\right)\right)$$

As we know that

$$\sin(\sin^{-1}x) = x \text{ as } x \in [-1, 1]$$

$$= \pm\frac{1}{\sqrt{10}}$$

Hence, $\sin\left(\frac{1}{2}\cos^{-1}\frac{4}{5}\right) = \pm\frac{1}{\sqrt{10}}$

(iv) Given $\sin(2 \tan^{-1} 2/3) + \cos(\tan^{-1} \sqrt{3})$

We know that

$$\sin^{-1}\left(\frac{2x}{1+x^2}\right) = 2 \tan^{-1}(x);$$

$$\cos^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right) = \tan^{-1}(x);$$

Now by substituting these formulae we get,

$$= \sin\left(\sin^{-1}\left(\frac{2 \times \frac{2}{3}}{1+\frac{4}{9}}\right)\right) + \cos\left(\cos^{-1}\left(\frac{1}{\sqrt{1+3}}\right)\right)$$

$$= \sin\left(\sin^{-1}\left(\frac{12}{13}\right)\right) + \cos\left(\cos^{-1}\left(\frac{1}{2}\right)\right)$$

$$= \frac{12}{13} + \frac{1}{2}$$

$$= \frac{37}{26}$$

Hence,

$$\sin\left(2 \tan^{-1}\left(\frac{2}{3}\right)\right) + \cos\left(\tan^{-1}\sqrt{3}\right) = \frac{37}{26}$$

2. Prove the following results:

- (i) $2 \sin^{-1}(3/5) = \tan^{-1}(24/7)$
- (ii) $\tan^{-1} \frac{1}{4} + \tan^{-1}(2/9) = \frac{1}{2} \cos^{-1}(3/5) = \frac{1}{2} \sin^{-1}(4/5)$
- (iii) $\tan^{-1}(2/3) = \frac{1}{2} \tan^{-1}(12/5)$
- (iv) $\tan^{-1}(1/7) + 2 \tan^{-1}(1/3) = \pi/4$
- (v) $\sin^{-1}(4/5) + 2 \tan^{-1}(1/3) = \pi/2$
- (vi) $2 \sin^{-1}(3/5) - \tan^{-1}(17/31) = \pi/4$
- (vii) $2 \tan^{-1}(1/5) + \tan^{-1}(1/8) = \tan^{-1}(4/7)$
- (viii) $2 \tan^{-1}(3/4) - \tan^{-1}(17/31) = \pi/4$
- (ix) $2 \tan^{-1}(1/2) + \tan^{-1}(1/7) = \tan^{-1}(31/17)$
- (x) $4 \tan^{-1}(1/5) - \tan^{-1}(1/239) = \pi/4$

Solution:

(i) Given $2 \sin^{-1}(3/5) = \tan^{-1}(24/7)$

Consider LHS

$$2 \sin^{-1} \frac{3}{5}$$

We know that

$$\sin^{-1}(x) = \tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$$

Now by substituting the above formula we get,

$$\begin{aligned}
 & 2 \times \tan^{-1}\left(\frac{\frac{3}{5}}{\sqrt{1-\frac{9}{25}}}\right) \\
 = & \\
 & 2 \times \tan^{-1}\left(\frac{\frac{3}{4}}{\frac{3}{5}}\right) \\
 = & \\
 & 2 \times \tan^{-1}\left(\frac{3}{4}\right)
 \end{aligned}$$


Again we know that

$$2 \tan^{-1}(x) = \tan^{-1}\left(\frac{2x}{1-x^2}\right), \text{ if } |x| < 1$$

Therefore,

$$\begin{aligned}
 & \tan^{-1}\left(\frac{2 \times \frac{3}{4}}{1-\frac{9}{16}}\right) \\
 = & \\
 & \tan^{-1}\left(\frac{\frac{3}{2}}{\frac{7}{16}}\right) \\
 = & \\
 & \tan^{-1}\left(\frac{24}{7}\right) \\
 = & \text{RHS}
 \end{aligned}$$

$$\text{So, } 2 \sin^{-1}\frac{3}{5} = \tan^{-1}\left(\frac{24}{7}\right)$$

Hence, proved. 

(ii) Given $\tan^{-1} \frac{1}{4} + \tan^{-1} \left(\frac{2}{9}\right) = \frac{1}{2} \cos^{-1} \left(\frac{3}{5}\right) = \frac{1}{2} \sin^{-1} \left(\frac{4}{5}\right)$

Consider LHS

$$= \tan^{-1}\left(\frac{1}{4}\right) + \tan^{-1}\left(\frac{2}{9}\right)$$

We know that

$$\tan^{-1} x + \tan^{-1} y = \tan^{-1} \frac{x+y}{1-xy}$$

Now by substituting this formula, we get

$$= \tan^{-1} \left(\frac{\frac{1}{4} + \frac{2}{9}}{1 - \frac{1}{4} \times \frac{2}{9}} \right)$$

$$= \tan^{-1} \left(\frac{\frac{9+8}{36}}{\frac{36-2}{36}} \right)$$

$$= \tan^{-1} \left(\frac{17}{34} \right)$$

$$= \tan^{-1} \left(\frac{1}{2} \right)$$

Multiplying and dividing by 2

$$= \frac{1}{2} \left\{ 2 \tan^{-1} \left(\frac{1}{2} \right) \right\}$$

Again we know that

$$2 \tan^{-1} x = \cos^{-1} \left(\frac{1-x^2}{1+x^2} \right)$$

$$= \frac{1}{2} \cos^{-1} \left(\frac{1-\frac{1}{4}}{1+\frac{1}{4}} \right)$$

$$= \frac{1}{2} \cos^{-1} \left(\frac{\frac{3}{4}}{\frac{5}{4}} \right)$$

$$= \frac{1}{2} \cos^{-1} \left(\frac{3}{5} \right)$$

= RHS

$$\text{So, } \tan^{-1} \left(\frac{1}{4} \right) + \tan^{-1} \left(\frac{2}{9} \right) = \frac{1}{2} \cos^{-1} \left(\frac{3}{5} \right)$$

Now,

$$= \frac{1}{2} \cos^{-1} \left(\frac{3}{5} \right)$$

We know that,

$$\cos^{-1} x = \sin^{-1} \sqrt{1 - x^2}$$

By substituting this, we get

$$= \frac{1}{2} \sin^{-1} \sqrt{1 - \frac{9}{25}}$$

$$= \frac{1}{2} \sin^{-1} \sqrt{\frac{16}{25}}$$

$$= \frac{1}{2} \sin^{-1} \frac{4}{5}$$

= RHS

$$\text{So, } \tan^{-1} \left(\frac{1}{4} \right) + \tan^{-1} \left(\frac{2}{9} \right) = \frac{1}{2} \cos^{-1} \left(\frac{3}{5} \right) = \frac{1}{2} \sin^{-1} \frac{4}{5}$$

Hence, proved.

(iii) Given $\tan^{-1} (2/3) = \frac{1}{2} \tan^{-1} (12/5)$

Consider LHS

$$= \tan^{-1} \left(\frac{2}{3} \right)$$

Now, Multiplying and dividing by 2, we get

$$= \frac{1}{2} \left\{ 2 \tan^{-1} \left(\frac{2}{3} \right) \right\}$$

We know that

$$2\tan^{-1} x = \tan^{-1} \left(\frac{2x}{1-x^2} \right)$$

By substituting the above formula we get

$$= \frac{1}{2} \tan^{-1} \left(\frac{2 \times \frac{2}{3}}{1 - \frac{4}{9}} \right)$$

$$= \frac{1}{2} \tan^{-1} \left(\frac{4}{5} \right)$$

$$= \frac{1}{2} \tan^{-1} \left(\frac{12}{5} \right)$$

= RHS

$$\text{So, } \tan^{-1} \left(\frac{2}{3} \right) = \frac{1}{2} \tan^{-1} \left(\frac{12}{5} \right)$$

Hence, proved.

(iv) Given $\tan^{-1} (1/7) + 2 \tan^{-1} (1/3) = \pi/4$

Consider LHS

$$= \tan^{-1} \left(\frac{1}{7} \right) + 2 \tan^{-1} \left(\frac{1}{3} \right)$$

We know that,

$$2\tan^{-1} x = \tan^{-1} \left(\frac{2x}{1-x^2} \right)$$

By substituting the above formula we get,

$$= \tan^{-1} \left(\frac{1}{7} \right) + \tan^{-1} \left(\frac{2 \times \frac{1}{3}}{1 - \frac{1}{9}} \right)$$

$$= \tan^{-1}\left(\frac{1}{7}\right) + \tan^{-1}\left(\frac{\frac{2}{3}}{\frac{2}{9}}\right)$$

$$= \tan^{-1}\left(\frac{1}{7}\right) + \tan^{-1}\left(\frac{3}{4}\right)$$

Again we know that

$$\tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy}$$

$$= \tan^{-1}\left(\frac{\frac{1}{7} + \frac{3}{4}}{1 - \frac{1}{7} \times \frac{3}{4}}\right)$$

$$= \tan^{-1}\left(\frac{\frac{25}{28}}{\frac{28}{28}}\right)$$

$$= \tan^{-1}(1)$$

$$= \frac{\pi}{4}$$

$$= \text{RHS}$$

$$\text{So, } \tan^{-1}\left(\frac{1}{7}\right) + 2\tan^{-1}\left(\frac{1}{3}\right) = \frac{\pi}{4}$$

Hence, proved.

(v) Given $\sin^{-1}(4/5) + 2 \tan^{-1}(1/3) = \pi/2$

Consider LHS

$$= \sin^{-1}\left(\frac{4}{5}\right) + 2\tan^{-1}\left(\frac{1}{3}\right)$$

We know that,

$$\sin^{-1}(x) = \tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$$

And, $2\tan^{-1} x = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$

Now by substituting the formula we get,

$$= \tan^{-1}\left(\frac{\frac{4}{5}}{\sqrt{1-\frac{16}{25}}}\right) + \tan^{-1}\left(\frac{\frac{2 \times \frac{1}{3}}{1-\frac{1}{9}}}\right)$$

$$= \tan^{-1}\left(\frac{\frac{4}{5}}{\sqrt{\frac{9}{25}}}\right) + \tan^{-1}\left(\frac{\frac{2}{9}}{\frac{8}{9}}\right)$$

$$= \tan^{-1}\left(\frac{4}{3}\right) + \tan^{-1}\left(\frac{3}{4}\right)$$

We know that,

$$\tan^{-1} x + \tan^{-1} y = \tan^{-1} \frac{x+y}{1-xy}$$

$$= \tan^{-1}\left(\frac{\frac{4}{3} + \frac{3}{4}}{1 - \frac{4}{3} \times \frac{3}{4}}\right)$$

$$= \tan^{-1}\left(\frac{\frac{25}{12}}{0}\right)$$

$$= \tan^{-1}(\infty)$$

$$= \frac{\pi}{2}$$

$$= \text{RHS}$$



$$\text{So, } \sin^{-1}\left(\frac{4}{5}\right) + 2\tan^{-1}\left(\frac{1}{3}\right) = \frac{\pi}{2}$$

Hence Proved

$$(vi) \text{ Given } 2 \sin^{-1}\left(\frac{3}{5}\right) - \tan^{-1}\left(\frac{17}{31}\right) = \pi/4$$

Consider LHS

$$= 2\sin^{-1}\left(\frac{3}{5}\right) - \tan^{-1}\left(\frac{17}{31}\right)$$

We know that

$$\sin^{-1}(x) = \tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$$

According to the formula we have,

$$= 2\tan^{-1}\left(\frac{\frac{3}{5}}{\sqrt{1-\frac{9}{25}}}\right) - \tan^{-1}\left(\frac{17}{31}\right)$$

$$= 2\tan^{-1}\left(\frac{\frac{3}{5}}{\sqrt{\frac{16}{25}}}\right) - \tan^{-1}\left(\frac{17}{31}\right)$$

$$= 2 \tan^{-1}\left(\frac{3}{4}\right) - \tan^{-1}\left(\frac{17}{31}\right)$$

Again we know that,

$$2\tan^{-1} x = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$$

By substituting this formula, we get

$$= \tan^{-1}\left(\frac{2 \times \frac{3}{4}}{1-\frac{9}{16}}\right) - \tan^{-1}\left(\frac{17}{31}\right)$$



$$= \tan^{-1}\left(\frac{3}{7}\right) - \tan^{-1}\left(\frac{17}{31}\right)$$

$$= \tan^{-1}\left(\frac{24}{7}\right) - \tan^{-1}\left(\frac{17}{31}\right)$$

Again we have,

$$\tan^{-1} x - \tan^{-1} y = \tan^{-1} \frac{x - y}{1 + xy}$$

$$= \tan^{-1} \left(\frac{\frac{24}{7} - \frac{17}{31}}{1 + \frac{24}{7} \times \frac{17}{31}} \right)$$

$$= \tan^{-1} \left(\frac{\frac{744 - 119}{217}}{\frac{217 + 408}{217}} \right)$$

$$= \tan^{-1} \left(\frac{625}{625} \right)$$

$$= \tan^{-1}(1)$$

$$= \frac{\pi}{4} = \text{RHS}$$

$$\text{So, } 2\sin^{-1}\left(\frac{3}{5}\right) - \tan^{-1}\left(\frac{17}{31}\right) = \frac{\pi}{4}$$

Hence the proof.

(vii) Given $2 \tan^{-1} (1/5) + \tan^{-1} (1/8) = \tan^{-1} (4/7)$

Consider LHS

$$= 2 \tan^{-1}\left(\frac{1}{5}\right) + \tan^{-1}\left(\frac{1}{8}\right)$$

We know that

$$2\tan^{-1} x = \tan^{-1} \left(\frac{2x}{1-x^2} \right)$$

Now by substituting the formula we get,

$$= \tan^{-1} \left(\frac{2 \times \frac{1}{5}}{1 - \frac{1}{25}} \right) + \tan^{-1} \left(\frac{1}{8} \right)$$

$$= \tan^{-1} \left(\frac{\frac{2}{5}}{\frac{24}{25}} \right) + \tan^{-1} \left(\frac{1}{8} \right)$$

$$= \tan^{-1} \left(\frac{5}{12} \right) + \tan^{-1} \left(\frac{1}{8} \right)$$

Again from the formula we have,

$$\tan^{-1} x + \tan^{-1} y = \tan^{-1} \frac{x+y}{1-xy}$$

$$= \tan^{-1} \left(\frac{\frac{5}{12} + \frac{1}{8}}{1 - \frac{5}{12} \times \frac{1}{8}} \right)$$

$$= \tan^{-1} \left(\frac{\frac{10+3}{24}}{\frac{96-5}{96}} \right)$$

$$= \tan^{-1} \left(\frac{13}{24} \times \frac{96}{91} \right)$$

$$= \tan^{-1} \left(\frac{4}{7} \right)$$

= RHS

$$\text{So, } 2 \tan^{-1} \left(\frac{1}{5} \right) + \tan^{-1} \left(\frac{1}{8} \right) = \tan^{-1} \left(\frac{4}{7} \right)$$

Hence, proved.

(viii) Given $2 \tan^{-1} (3/4) - \tan^{-1} (17/31) = \pi/4$

Consider LHS

$$= 2 \tan^{-1} \left(\frac{3}{4} \right) - \tan^{-1} \left(\frac{17}{31} \right)$$

We know that,

$$2 \tan^{-1} x = \tan^{-1} \left(\frac{2x}{1-x^2} \right)$$

Now by substituting the formula we get,

$$\begin{aligned} &= \tan^{-1} \left(\frac{2 \times \frac{3}{4}}{1 - \frac{9}{16}} \right) - \tan^{-1} \left(\frac{17}{31} \right) \\ &= \tan^{-1} \left(\frac{3}{2} \times \frac{16}{7} \right) - \tan^{-1} \left(\frac{17}{31} \right) \\ &= \tan^{-1} \left(\frac{24}{7} \right) - \tan^{-1} \left(\frac{17}{31} \right) \end{aligned}$$

We know that,

$$\tan^{-1} x - \tan^{-1} y = \tan^{-1} \frac{x-y}{1+xy}$$

Again by substituting the formula we get,

$$\begin{aligned} &= \tan^{-1} \left(\frac{\frac{24}{7} - \frac{17}{31}}{1 + \frac{24}{7} \times \frac{17}{31}} \right) \\ &= \tan^{-1} \left(\frac{\frac{744-119}{217}}{\frac{217+408}{217}} \right) \\ &= \tan^{-1} \left(\frac{625}{625} \right) \\ &= \tan^{-1}(1) \\ &= \frac{\pi}{4} \end{aligned}$$

= RHS

$$\text{So, } 2\tan^{-1}\left(\frac{3}{4}\right) - \tan^{-1}\left(\frac{17}{31}\right) = \frac{\pi}{4}$$

Hence, proved.

(ix) Given $2 \tan^{-1} (1/2) + \tan^{-1} (1/7) = \tan^{-1} (31/17)$

Consider LHS

$$= 2 \tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{7}\right)$$

We know that,

$$2\tan^{-1} x = \tan^{-1} \left(\frac{2x}{1-x^2} \right)$$

Now by substituting the formula we get,

$$= \tan^{-1} \left(\frac{2 \times \frac{1}{2}}{1 - \frac{1}{4}} \right) + \tan^{-1} \left(\frac{1}{7} \right)$$

$$= \tan^{-1} \left(\frac{2}{\frac{3}{4}} \right) + \tan^{-1} \left(\frac{1}{7} \right)$$

$$= \tan^{-1} \left(\frac{4}{3} \right) + \tan^{-1} \left(\frac{1}{7} \right)$$

Again by using the formula, we can write as

$$\tan^{-1} x + \tan^{-1} y = \tan^{-1} \frac{x+y}{1-xy}$$

$$= \tan^{-1} \left(\frac{\frac{4}{3} + \frac{1}{7}}{1 - \frac{4}{7} \times \frac{1}{3}} \right)$$

$$= \tan^{-1} \left(\frac{\frac{31}{21}}{\frac{21}{21}} \right)$$

$$= \tan^{-1} \left(\frac{31}{17} \right)$$

= RHS

$$\text{So, } 2 \tan^{-1} \left(\frac{1}{2} \right) + \tan^{-1} \left(\frac{1}{7} \right) = \tan^{-1} \left(\frac{31}{17} \right)$$

Hence, proved.

(x) Given $4 \tan^{-1}(1/5) - \tan^{-1}(1/239) = \pi/4$

Consider LHS

$$= 4 \tan^{-1} \left(\frac{1}{5} \right) - \tan^{-1} \left(\frac{1}{239} \right)$$

We know that,

$$4 \tan^{-1} x = \tan^{-1} \left(\frac{4x - 4x^3}{1 - 6x^2 + x^4} \right)$$

Now by substituting the formula, we get

$$= \tan^{-1} \left(\frac{4 \times \frac{1}{5} - 4 \left(\frac{1}{5} \right)^3}{1 - 6 \left(\frac{1}{5} \right)^2 + \left(\frac{1}{5} \right)^4} \right) - \tan^{-1} \left(\frac{1}{239} \right)$$

$$= \tan^{-1} \left(\frac{120}{119} \right) - \tan^{-1} \left(\frac{1}{239} \right)$$

Again we know that,

$$\tan^{-1} x - \tan^{-1} y = \tan^{-1} \frac{x - y}{1 + xy}$$

$$= \tan^{-1} \left(\frac{\frac{120}{119} - \frac{1}{239}}{1 - \frac{120}{119} \times \frac{1}{239}} \right)$$

$$= \tan^{-1} \left(\frac{120 \times 239 - 119}{119 \times 239 + 120} \right)$$

$$= \tan^{-1} \left(\frac{28561}{28561} \right)$$

$$= \tan^{-1}(1)$$

$$= \frac{\pi}{4}$$

$$= \text{RHS}$$

So,

$$4 \tan^{-1} \left(\frac{1}{5} \right) - \tan^{-1} \left(\frac{1}{239} \right) = \frac{\pi}{4}$$

Hence, proved.

3. If $\sin^{-1} \left(\frac{2a}{1+a^2} \right) - \cos^{-1} \left(\frac{1-b^2}{1+b^2} \right) = \tan^{-1} \left(\frac{2x}{1-x^2} \right)$, then prove that $x = \frac{a-b}{1+ab}$

Solution:

$$\text{Given } \sin^{-1} \left(\frac{2a}{1+a^2} \right) - \cos^{-1} \left(\frac{1-b^2}{1+b^2} \right) = \tan^{-1} \left(\frac{2x}{1-x^2} \right)$$

Consider,

$$\Rightarrow \sin^{-1} \left(\frac{2a}{1+a^2} \right) - \cos^{-1} \left(\frac{1-b^2}{1+b^2} \right) = \tan^{-1} \left(\frac{2x}{1-x^2} \right)$$

We know that,

$$2 \tan^{-1} x = \sin^{-1} \left(\frac{2x}{1+x^2} \right)$$

$$2 \tan^{-1} x = \cos^{-1} \left(\frac{1-x^2}{1+x^2} \right)$$

$$2 \tan^{-1} x = \tan^{-1} \left(\frac{2x}{1-x^2} \right)$$

Now by applying these formulae in given equation we get,

$$\Rightarrow 2\tan^{-1}(a) - 2\tan^{-1}(b) = 2\tan^{-1}(x)$$

$$\Rightarrow 2(\tan^{-1}(a) - \tan^{-1}(b)) = 2\tan^{-1}(x)$$

$$\Rightarrow \tan^{-1}(a) - \tan^{-1}(b) = \tan^{-1}(x)$$

Again we know that,

$$\tan^{-1} x - \tan^{-1} y = \tan^{-1} \frac{x - y}{1 + xy}$$

Now by substituting this in above equation we get,

$$\Rightarrow \tan^{-1}\left(\frac{a-b}{1+ab}\right) = \tan^{-1}(x)$$

On comparing we get,

$$\Rightarrow x = \frac{a-b}{1+ab}$$

Hence, proved.

4. Prove that:

(i) $\tan^{-1}\{(1 - x^2)/ 2x\} + \cot^{-1}\{(1 - x^2)/ 2x\} = \pi/2$

(ii) $\sin \{\tan^{-1} (1 - x^2)/ 2x\} + \cos^{-1} (1 - x^2)/ (1 + x^2)\} = 1$

Solution:

(i) Given $\tan^{-1}\{(1 - x^2)/ 2x\} + \cot^{-1}\{(1 - x^2)/ 2x\} = \pi/2$

Consider LHS

$$= \tan^{-1} \frac{1-x^2}{2x} + \cot^{-1} \frac{1-x^2}{2x}$$

We know that,

$$\cot^{-1} x = \tan^{-1} \left(\frac{1}{x}\right)$$

Now by applying the above formula we get,

$$= \tan^{-1} \left(\frac{1-x^2}{2x} \right) + \tan^{-1} \left(\frac{2x}{1-x^2} \right)$$

Again we know

$$\tan^{-1} x + \tan^{-1} y = \tan^{-1} \frac{x+y}{1-xy}$$

By substituting this we get,

$$= \tan^{-1} \left(\frac{\left(\frac{1-x^2}{2x} \right) + \left(\frac{2x}{1-x^2} \right)}{1 - \left(\frac{1-x^2}{2x} \right) \times \left(\frac{2x}{1-x^2} \right)} \right)$$

$$= \tan^{-1} \left(\frac{\frac{1+x^4-2x^2+4x^2}{2x(1-x^2)}}{\frac{2x(1-x^2)-2x(1-x^2)}{2x(1-x^2)}} \right)$$

$$= \tan^{-1} \left(\frac{1+x^4+2x^2}{0} \right)$$

$$= \tan^{-1}(\infty)$$

$$= \frac{\pi}{2} = \text{RHS}$$

$$\tan^{-1} \frac{1-x^2}{2x} + \cot^{-1} \frac{1-x^2}{2x} = \frac{\pi}{2}$$

Hence, proved.

(ii) Given $\sin \left\{ \tan^{-1} \left(\frac{1-x^2}{2x} \right) + \cos^{-1} \left(\frac{1-x^2}{1+x^2} \right) \right\}$

Consider LHS

$$= \sin \left(\tan^{-1} \frac{1-x^2}{2x} + \cos^{-1} \frac{1-x^2}{1+x^2} \right)$$

We know that,

$$2\tan^{-1} x = \cos^{-1} \left(\frac{1-x^2}{1+x^2} \right)$$

Now by applying the formula in above question we get,

$$= \sin \left(\tan^{-1} \frac{1-x^2}{2x} + 2\tan^{-1} x \right)$$

Again, we have

$$2\tan^{-1} x = \tan^{-1} \left(\frac{2x}{1-x^2} \right)$$

Now by substituting the formula we get,

$$= \sin \left(\tan^{-1} \frac{1-x^2}{2x} + \tan^{-1} \left(\frac{2x}{1-x^2} \right) \right)$$

Again we know that,

$$\tan^{-1} x + \tan^{-1} y = \tan^{-1} \frac{x+y}{1-xy}$$

Now by applying the formula,

$$= \sin \left(\tan^{-1} \left(\frac{\frac{1-x^2}{2x} + \left(\frac{2x}{1-x^2} \right)}{1 - \frac{1-x^2}{2x} \times \left(\frac{2x}{1-x^2} \right)} \right) \right)$$

$$= \sin \left(\tan^{-1} \left(\frac{\frac{1+x^4-2x^2+4x^2}{2x(1-x^2)}}{\frac{2x(1-x^2)-2x(1-x^2)}{2x(1-x^2)}} \right) \right)$$

$$= \sin \left(\tan^{-1} \left(\frac{1+x^4-2x^2+4x^2}{0} \right) \right)$$

$$= \sin(\tan^{-1}(\infty))$$

$$= \sin\left(\frac{\pi}{2}\right)$$

$$= 1$$

$$= \text{RHS}$$

So,

$$\sin^{-1}\left(\tan^{-1}\frac{1-x^2}{2x} + \cos^{-1}\frac{1-x^2}{1+x^2}\right) = 1$$

Hence, proved.

5. If $\sin^{-1}\left(\frac{2a}{1+a^2}\right) + \sin^{-1}\left(\frac{2b}{1+b^2}\right) = 2 \tan^{-1} x$, prove that $x = \frac{a+b}{1-ab}$

Solution:

Given $\sin^{-1}\left(\frac{2a}{1+a^2}\right) + \sin^{-1}\left(\frac{2b}{1+b^2}\right) = 2 \tan^{-1} x$

Consider

$$\sin^{-1}\left(\frac{2a}{1+a^2}\right) + \sin^{-1}\left(\frac{2b}{1+b^2}\right) = 2 \tan^{-1}(x)$$

We know that,

$$2 \tan^{-1} x = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$$

Now by applying the above formula we get,

$$\Rightarrow 2 \tan^{-1}(a) + 2 \tan^{-1}(b) = 2 \tan^{-1}(x)$$

$$\Rightarrow 2(\tan^{-1}(a) + \tan^{-1}(b)) = 2 \tan^{-1}(x)$$

$$\Rightarrow \tan^{-1}(a) + \tan^{-1}(b) = \tan^{-1}(x)$$

Again we have,

$$\tan^{-1} x + \tan^{-1} y = \tan^{-1} \frac{x + y}{1 - xy}$$

Now by substituting, we get

$$\Rightarrow \tan^{-1} \left(\frac{a+b}{1-ab} \right) = \tan^{-1}(x)$$

On comparing we get,

$$\Rightarrow x = \frac{a+b}{1-ab}$$

Hence, proved.

