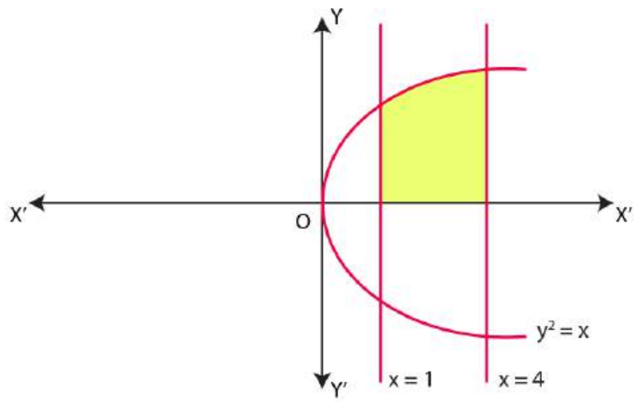
Exercise 8.1 Page No: 365

1. Find the area of the region bounded by the curve $y^2 = x$ and the lines x=1, x = 4 and the x- axis in the first quadrant.

Solution: Equation of the curve (rightward parabola) is $y^2 = x$.



$y = \sqrt{x}$ ……….(1)

Required area is shaded region:

$= \left|\int_1^4 y\, dx\right| = \left|\int_1^4 \sqrt{x}\, dx\right|$ [From equation (1)]

$= \left|\int_1^4 x^{\frac{1}{2}}\, dx\right|$

$= \left|\dfrac{\left(x^{\frac{3}{2}}\right)_1^4}{\frac{3}{2}}\right|$

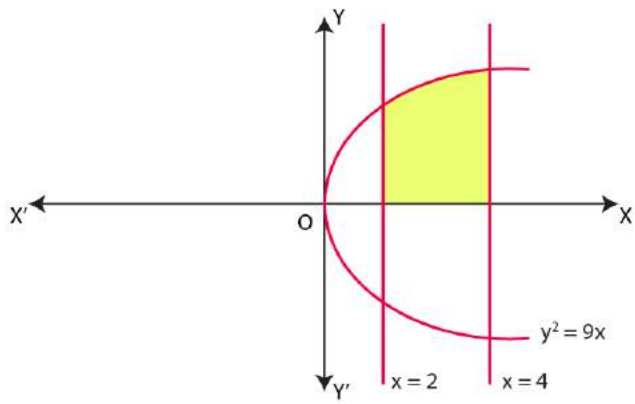
$= \left|\dfrac{2}{3}\left(4^{\frac{3}{2}} - 1^{\frac{3}{2}}\right)\right|$

$= \left|\frac{2}{3}\left(4^{\frac{1}{2}\times 3} - 1^{\frac{1}{2}\times 3}\right)\right| = \left|\frac{2}{3}(8-1)\right| = \frac{2}{3}\times 7 = \frac{14}{3}$ sq. units

2. Find the area of the region bounded by $y^2 = 9x, x = 2, x = 4$ and the x-axis in the first quadrant.

Solution: Equation of the curve (rightward parabola) is $y^2 = 9x$.

$y = 3\sqrt{x}$ ……….(1)



Required area is shaded region, which is bounded by curve $y^2 = 9x$, and vertical lines x=2, x=4 and x-axis in first quadrant.

$= \left|\int_2^4 y\,dx\right| = \left|\int_1^4 3\sqrt{x}\,dx\right|$ [From equation (1)]

$= \left|3\int_2^4 x^{\frac{1}{2}}\,dx\right| = \left|3\frac{\left(x^{\frac{3}{2}}\right)_2^4}{\frac{3}{2}}\right|$

$= \left|3.\frac{2}{3}\left(4^{\frac{3}{2}} - 2^{\frac{3}{2}}\right)\right| = \left|3.\frac{2}{3}\left(4^{\frac{1}{2}\times 3} - 2^{\frac{1}{2}\times 3}\right)\right|$

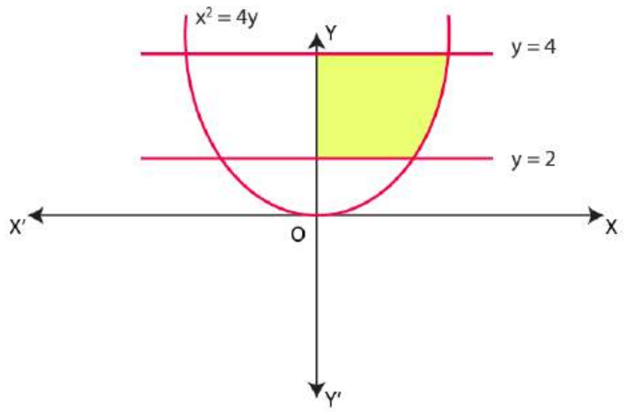
$= \left|2\left(8-2\sqrt{2}\right)\right| = \left(16-4\sqrt{2}\right)$ sq. units

3. Find the area of the region bounded by $x^2 = 4y, y = 2, y = 4$ and the y- axis in the first quadrant.

Solution: Equation of curve (parabola) is $x^2 = 4y.$

or $x = 2\sqrt{y}$ ……….(1)

Required region is shaded, that is area bounded by curve $x^2 = 4y,$ and Horizontal lines $y = 2, y = 4$ and y-axis in first quadrant.

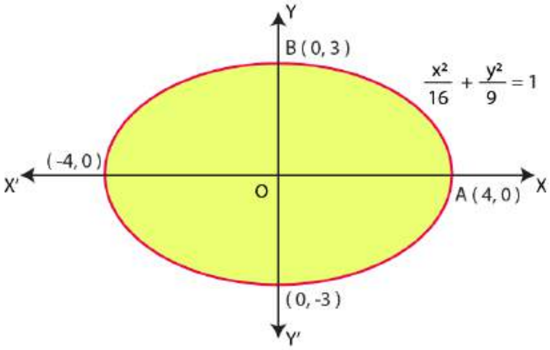


$= \left|\int_2^4 x\, dy\right| = \left|\int_2^4 2\sqrt{y}\, dy\right| = \left|2\int_2^4 y^{\frac{1}{2}}\, dy\right|$

$= \left|2\frac{\left(y^{\frac{3}{2}}\right)_2^4}{\frac{3}{2}}\right| = \frac{4}{3}\left(4^{\frac{3}{2}} - 2^{\frac{3}{2}}\right) = \left(\frac{32-8\sqrt{2}}{3}\right)$ sq. units

4. Find the area of the region bounded by the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1.$

Solution: Equation of ellipse is $\frac{x^2}{16}+\frac{y^2}{9}=1$ ……….(1)



Here $a^2(=16) > b^2(=9)$

From equation (1), $\frac{y^2}{9} = 1 - \frac{x^2}{16} = \frac{16-x^2}{16}$

$\Rightarrow y^2 = \frac{9}{16}(16-x^2)$

$\Rightarrow y^2 = \frac{3}{4}(16-x^2)$ ……….(2)

for arc of ellipse in first quadrant.

Ellipse (1) is symmetrical about x-axis and about y-axis (if we change y to –y or x to –x, equation remain same).

Intersections of ellipse (1) with x-axis (y=0)
Put y=0 in equation (1), we have

$\frac{x^2}{16} = 1 \Rightarrow x^2 = 16 \Rightarrow x = \pm 4$

Therefore, Intersections of ellipse (1) with x-axis are (0, 4) and (0, -4).

Now again,
Intersections of ellipse (1) with y-axis (x=0)

Putting x=0 in equation (1), $\frac{y^2}{9}=1 \Rightarrow y^2=9 \Rightarrow y=\pm 3$

Therefore, Intersections of ellipse (1) with y-axis are (0, 3) and (0,-3).

Now ,

Area of region bounded by ellipse (1) = Total shaded area = 4 x Area OAB of ellipse in first quadrant

= $4\left|\int_0^4 y\ dx\right|$ [$\because$ At end B of arc AB of ellipse; $x=0$ and at end A of arc AB ; $x=4$]

= $4\left|\int_0^4 \frac{3}{4}\sqrt{16-x^2}\ dx\right|$ = $4\left|\int_0^4 \frac{3}{4}\sqrt{4^2-x^2}\ dx\right|$

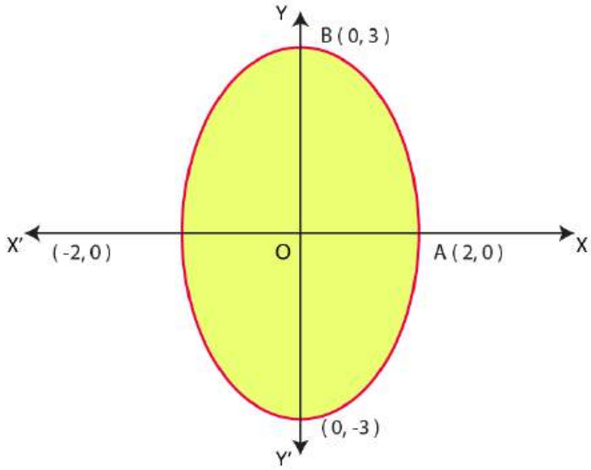
= $3\left[\frac{x}{2}\sqrt{4^2-x^2}+\frac{4^2}{2}\sin^{-1}\frac{x}{4}\right]_0^4$ $\left[\because \int\sqrt{a^2-x^2}\ dx=\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]$

= $3\left[\frac{4}{2}\sqrt{16-16}+8\sin^{-1}1-\left(0+8\sin^{-1}0\right)\right]$ = $3\left[0+\frac{8\pi}{2}\right]$

= $3(4\pi)=12\pi$ sq. units

5. Find the area of the region bounded by the ellipse $\frac{x^2}{4}+\frac{y^2}{9}=1$.

Solution: Equation of ellipse is $\frac{x^2}{4}+\frac{y^2}{9}=1$



Here $a^2(=4) < b^2(=9)$

From equation (1), $\frac{y^2}{9} = 1 - \frac{x^2}{4} = \frac{4 - x^2}{4}$

$\Rightarrow y^2 = \frac{9}{4}(4 - x^2)$

$\Rightarrow y^2 = \frac{3}{2}(4 - x^2)$ ……….(2)

For an arc of ellipse in first quadrant.

Ellipse (1) is symmetrical about x-axis and y-axis.

Intersections of ellipse (1) with x-axis (y=0)

Put y=0 in equation (1), $\frac{x^2}{4} = 1$

$\Rightarrow x^2 = 4$

$\Rightarrow x = \pm 2$

Therefore, Intersections of ellipse (1) with x-axis are (0, 2) and (0,-2).

Intersections of ellipse (1) with y-axis (x=0)

Put $x=0$ in equation (1), $\frac{y^2}{9}=1$

$\Rightarrow y^2=9 \Rightarrow y=\pm 3$

Therefore, Intersections of ellipse (1) with y-axis are (0, 3) and (0,-3).

Now,

Area of region bounded by ellipse (1) = Total shaded area = 4 x Area OAB of ellipse in first quadrant

= $4\left|\int_0^2 y\,dx\right|$ [$\because$ At end B of arc AB of ellipse; $x=0$ and at end A of arc AB ; $x=2$]

= $4\left|\int_0^2 \frac{3}{2}\sqrt{4-x^2}\,dx\right|$ = $4\left|\int_0^4 \frac{3}{2}\sqrt{2^2-x^2}\,dx\right|$

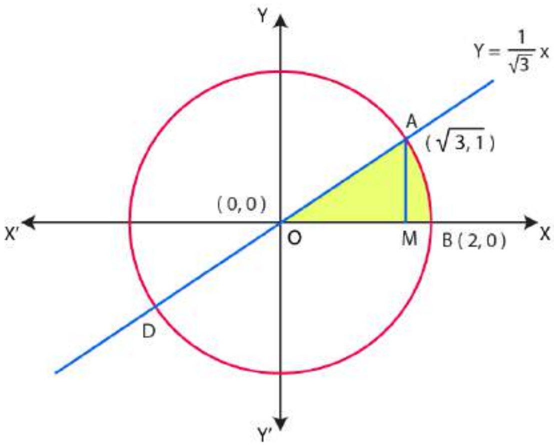
= $6\left[\frac{x}{2}\sqrt{2^2-x^2}+\frac{2^2}{2}\sin^{-1}\frac{x}{2}\right]_0^4$ $\left[\because \int\sqrt{a^2-x^2}\,dx=\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]$

= $6\left[\frac{2}{2}\sqrt{4-4}+2\sin^{-1}1-\left(0+2\sin^{-1}0\right)\right]$

= $6\left[0+2.\frac{\pi}{2}-0\right]=6\pi$ sq. units

6. Find the area of the region in the first quadrant enclosed by x-axis, line $x=\sqrt{3}y$ and the circle $x^2+y^2=4.$

Solution:
Step 1: To draw the graphs and shade the region whose area we are to find.



Equation of the circle is $x^2+y^2=2^2$ ……….(1)

We know that equation (1) represents a circle whose centre is (0, 0) and radius is 2.

Equation of the given line is $x=\sqrt{3}y$

$\Rightarrow y=\frac{1}{\sqrt{3}}x$ ……….(2)

We know that equation (2) being of the form $y=mx$ where $m=\frac{1}{\sqrt{3}}=\tan 30^\circ=\tan\theta$

$\Rightarrow \theta=30^\circ$ represents a straight line passing through the origin and making angle of 30° with x-axis.

Step 2: To find values of x and y.

Put $y=\frac{x}{\sqrt{3}}$ from equation (2) in equation (1),

$x^2+\frac{x^2}{3}=4$ $\Rightarrow 3x^2+x^2=12$ $\Rightarrow 4x^2=12$

$\Rightarrow x^2=3$ $\Rightarrow x=\pm 3$

Putting $x=\pm 3$ in $y=\frac{x}{\sqrt{3}}$, $y=1$ and $y=-1$

Therefore, the two points of intersections of circle (1) and line (2) are A$\left(\sqrt{3},1\right)$ and D$\left(-\sqrt{3},-1\right)$.

Step 3: Now shaded area OAM between segment OA of line (2) and x-axis

= $\left|\int_0^{\sqrt{3}} y\,dx\right|$ $\left[\because \text{ At O, } x=0 \text{ and at A, } x=\sqrt{3}\right]$

= $\left|\int_0^{\sqrt{3}} \frac{1}{\sqrt{3}}x\,dx\right|$ = $\frac{1}{\sqrt{3}}\left(\frac{x^2}{2}\right)_0^{\sqrt{3}}$ = $\frac{1}{\sqrt{3}}\left(\frac{3}{2}-0\right)$ = $\frac{3}{2\sqrt{3}}$ = $\frac{\sqrt{3}}{2}$ sq. units……….(3)

Step IV. Now shaded area AMB between are AB of circle and $x-$ axis.

= = $\left|\int_{\sqrt{3}}^{2} y\,dx\right|$ $\left[\because \text{ At O, } x=\sqrt{3} \text{ and at A, } x=2\right]$

= $\left|\int_{\sqrt{3}}^{2} \sqrt{2^2-x^2}\,dx\right|$ From equation (2),

= $\left(\frac{x}{2}\sqrt{2^2-x^2}+\frac{2^2}{2}\sin^{-1}\frac{x}{2}\right)_{\sqrt{3}}^{2}$ = $\left[\frac{2}{2}\sqrt{4-4}+2\sin^{-1}1-\left(\frac{\sqrt{3}}{2}\sqrt{4-3}+2\sin^{-1}\frac{\sqrt{3}}{2}\right)\right]$

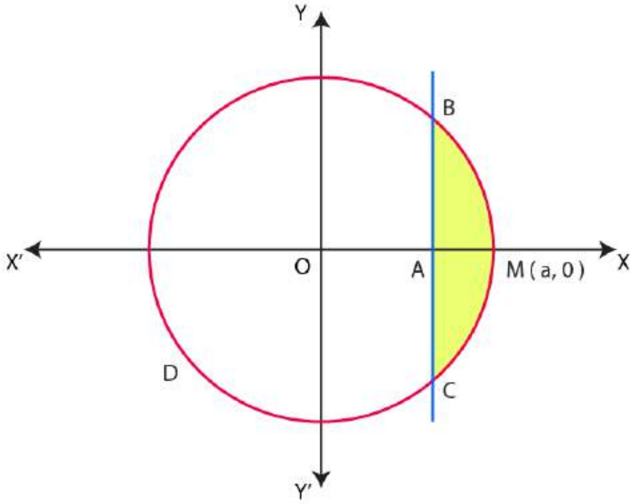
= $0+2.\frac{\pi}{2}-\frac{\sqrt{3}}{2}-2.\frac{\pi}{3}$ = $\pi-\frac{\sqrt{3}}{2}-\frac{2\pi}{3}$ = $\left(\frac{\pi}{3}-\frac{\sqrt{3}}{2}\right)$ sq. units……….(iv)

Step V. Required shaded area OAB = Area of OAM + Area of AMB

= $\frac{\sqrt{3}}{2}+\left(\frac{\pi}{3}-\frac{\sqrt{3}}{2}\right)=\frac{\pi}{3}$ sq. units

7. Find the area of the smaller part of the circle $x^2+y^2=a^2$ cut off by the line $x=\frac{a}{\sqrt{2}}$.

Solution: Equation of the circle is $x^2+y^2=a^2$ ……….(1)



$\therefore\ y^2 = a^2 - x^2$

$\Rightarrow\ y = \sqrt{a^2 - x^2}$ ……….(2)

Here,

Area of smaller part of the circle $x^2 + y^2 = a^2$ cut off by the line $x = \frac{a}{\sqrt{2}}$

= Area of ABMC = 2 x Area of ABM

$= 2\left|\int_{\frac{a}{\sqrt{2}}}^{a} y\, dx\right| = 2\left|\int_{\frac{a}{\sqrt{2}}}^{a} \sqrt{a^2 - x^2}\, dx\right|$ [From equation (2)]

$= 2\left[\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_{\frac{a}{\sqrt{2}}}^{a}$

$= 2\left[\frac{a}{2}\sqrt{a^2 - a^2} + \frac{a^2}{2}\sin^{-1}1 - \left(\frac{\frac{a^2}{\sqrt{2}}}{2}\sqrt{a^2 - \frac{a^2}{2}}\sin^{-1}\frac{\frac{a^2}{\sqrt{2}}}{2}\right)\right]$

$= 2\left[0+\frac{a^2}{2}.\frac{\pi}{2}-\frac{a}{2\sqrt{2}}\sqrt{\frac{a^2}{2}}-\frac{a^2}{2}\sin^{-1}\frac{1}{\sqrt{2}}\right]$

$= 2\left[\frac{\pi a^2}{4}-\frac{a}{2\sqrt{2}}\frac{a}{\sqrt{2}}-\frac{a^2}{2}\frac{\pi}{4}\right]$

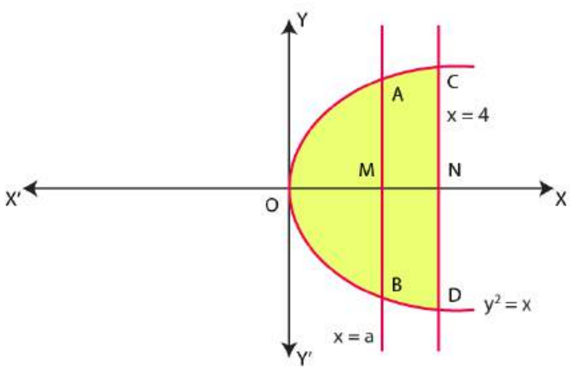
$= 2\left[\frac{\pi a^2}{4}-\frac{\pi a^2}{8}-\frac{a^2}{4}\right]$

$= 2a^2\left[\frac{2\pi-\pi-2}{8}\right]$

$= \frac{a^2}{4}(\pi-2)=\frac{a^2}{4}\left(\frac{\pi}{2}-1\right)$ sq. units

8. The area between $x=y^2$ and x=4 is divided into two equal parts by the line x=a find the value of a.

Solution: Equation of the curve (parabola) is $x=y^2$ ……(1)



$\Rightarrow y=\sqrt{x}$

Now area bounded by parabola (1) and vertical line x=4 is divided into two equal parts by the vertical line x=a.

Area OAMB = Area AMBDNC

$\Rightarrow 2\left|\int_0^a y\,dx\right| = 2\left|\int_a^4 y\,dx\right|$

$$\Rightarrow 2\left|\int_0^a x^{\frac{1}{2}}\,dx\right| = 2\left|\int_a^4 x^{\frac{1}{2}}\,dx\right|$$

$$\Rightarrow \frac{\left(x^{\frac{3}{2}}\right)_0^a}{\frac{3}{2}} = \frac{\left(x^{\frac{3}{2}}\right)_a^4}{\frac{3}{2}}$$

$$\Rightarrow \frac{2}{3}\left[a^{\frac{3}{2}} - 0\right] = \frac{2}{3}\left[4^{\frac{3}{2}} - a^{\frac{3}{2}}\right]$$

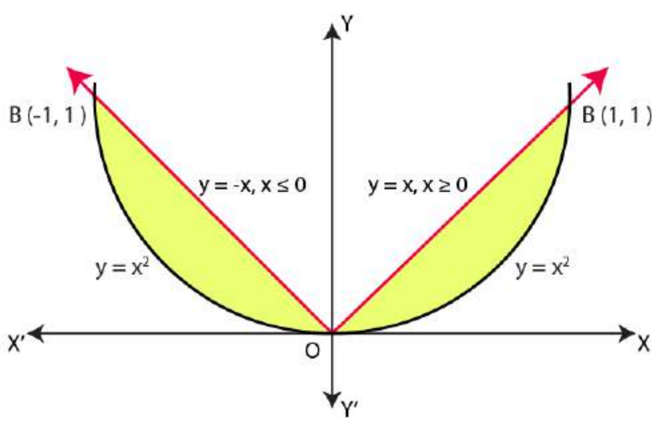
$$\Rightarrow a^{\frac{3}{2}} = 8 - a^{\frac{3}{2}}$$

$$\Rightarrow 2a^{\frac{3}{2}} = 8 \Rightarrow a^{\frac{3}{2}} = 4$$

$$\Rightarrow a = 4^{\frac{2}{3}}$$

9. Find the area of the region bounded by the parabola $y = x^2$ and $y = |x|$.

Solution: The required area is the area included between the parabola $y = x^2$ and the modulus function $y = |x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x \leq 0 \end{cases}$



To find: Area between the parabola $y = x^2$ and the ray $y = x$ for $x \geq 0$

Here, Limits of integration $\Rightarrow$ $y = x$

$\Rightarrow$ $x^2 = x$ $\Rightarrow x^2 - x = 0$

$\Rightarrow$ $x(x-1) = 0$ $\Rightarrow x = 0, x = 1$

Now, for $y = |x|$, table of values,

$y = x$ if $x \geq 0$

x	0	1	2
y	0	1	2

$y = -x$ if $x \leq 0$

x	0	-1	-2
y	0	1	2

Now, Area between parabola $y = x^2$ and x-axis between limits x=0 and x=1

$= \int_0^1 y\, dx = \int_0^1 x^2\, dx = \left(\frac{x^3}{3}\right)_0^1 = \frac{1}{3}$ ………..(1)

And Area of ray y=x and x-axis,

$= \int_0^1 y\, dx = \int_0^1 x\, dx = \left(\frac{x^2}{2}\right)_0^1 = \frac{1}{2}$ ………..(2)

So, Required shaded area in first quadrant

= Area between ray y=x for $x \geq 0$ and x-axis – Area between parabola $y = x^2$ and x-axis in first quadrant

= Area given by equation (2) – Area given by equation (1)

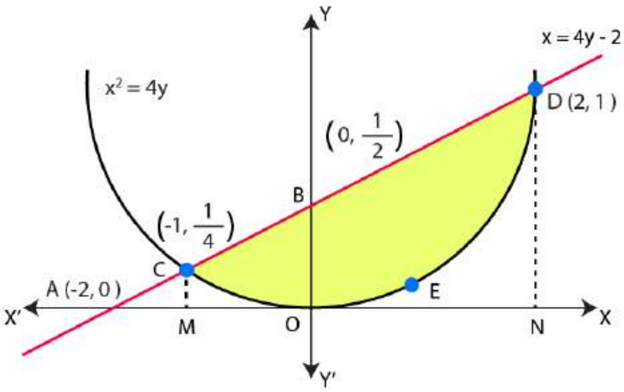
$= \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$ sq. units

Therefore, the required area = 2 x (1/6) = 1/3

10. Find the area bounded by the curve $x = 4y$ and the line $x = 4y - 2$.

Solution:

Step I. Graphs and region of Integration



Equation of the given curve is

$x^2 = 4y$ ....(1)

Equation of the given line is

$x = 4y - 2$ ....(2)

$\Rightarrow y = \frac{x+2}{4}$

x	0	1	-2
y	0	1/2	0

Step 2: Putting $y = \frac{x^2}{4}$ from equation (1) in equation (2),

$x = 4.\frac{x^2}{4} - 2 \Rightarrow x = x^2 - 2 \Rightarrow -x^2 + x + 2 = 0$

$\Rightarrow x^2 - x - 2 = 0 \Rightarrow$

$x^2 - 2x + x - 2 = 0 \Rightarrow x(x-2) + (x-2) = 0$

$\Rightarrow (x-2)(x+1) = 0 \Rightarrow x = 2$ or $x = -1$

For x=2,from equation (1), $y = \frac{x^2}{4} = \frac{4}{4} = 1$

So point is (2, 1)

For x=-1 from equation (1), $y = \frac{x^2}{4} = \frac{1}{4}$

So point is $\left(-1, \frac{1}{4}\right)$

Therefore, the two points of intersection of parabola (1) and line (2) are C $\left(-1, \frac{1}{4}\right)$ and D (2, 1).

Step 3. Area CMOEDN between parabola (1) and x-axis

$= \left|\int_{-1}^{2} y\, dx\right| = \left|\int_{-1}^{2} \frac{x^2}{4}\, dx\right|$

$= \left|\frac{\left(x^3\right)_{-1}^{2}}{12}\right| = \left|\frac{1}{12}\left\{2^3 - (-1)^3\right\}\right|$

$= \frac{1}{12}(8+1) = \frac{9}{12} = \frac{3}{4}$ sq. units……..(3)

Step 4. Area of trapezium CMND between line (2) and x-axis

$= \left|\int_{-1}^{2} y\, dx\right| = \left|\int_{-1}^{2} \frac{x+2}{4}\, dx\right|$

$= \left|\frac{1}{4}\int_{-1}^{2} (x+2)\, dx\right| = \frac{1}{4}\left|\left(\frac{x^2}{2} + 2x\right)_{-1}^{2}\right|$

$= \frac{1}{4}\left|\left(\frac{4}{2} + 4\right) - \left(\frac{1}{2} - 2\right)\right| = \frac{1}{4}\left|\left(2 + 4 - \frac{1}{2} + 2\right)\right|$

= $\frac{1}{4}\left|8-\frac{1}{2}\right|$ = $\frac{1}{4}\times\frac{15}{2}=\frac{15}{8}$ sq. units……….(4)

Therefore,

Required shaded area = Area given by equation (4) – Area given by equation (3)

= $\frac{15}{8}-\frac{3}{4}=\frac{15-6}{8}=\frac{9}{8}$ sq. units

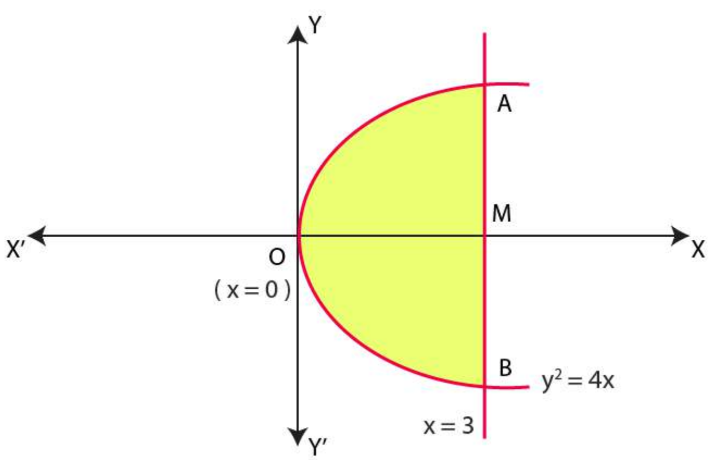
11. Find the area of the region bounded by the curve $y^2 = 4x$ and the line x=3.

Solution: Equation of the (parabola) curve is

$y^2 = 4x$ ………(1)

$\Rightarrow$ $y = 4x = 2x^{\frac{1}{2}}$ ………(2)

Here required shaded area OAMB = 2 x Area OAM



$= 2\left|\int_0^3 y\,dx\right| = 2\left|\int_0^3 2x^{\frac{1}{2}}\,dx\right| = 4\left|\frac{\left(x^{\frac{3}{2}}\right)_0^3}{\frac{3}{2}}\right|$

$= 4.\frac{2}{3}\left[3^{\frac{3}{2}} - 0\right] = \frac{8}{3}.3\sqrt{3} = 8\sqrt{3}$ sq. units

12. Choose the correct answer:

Area lying in the first quadrant and bounded by the circle $x^2 + y^2 = 4$ and the lines x = 0 and x = 2 is

(A) π **(B)** $\frac{\pi}{2}$ **(C)** $\frac{\pi}{3}$ **(D)** $\frac{\pi}{4}$

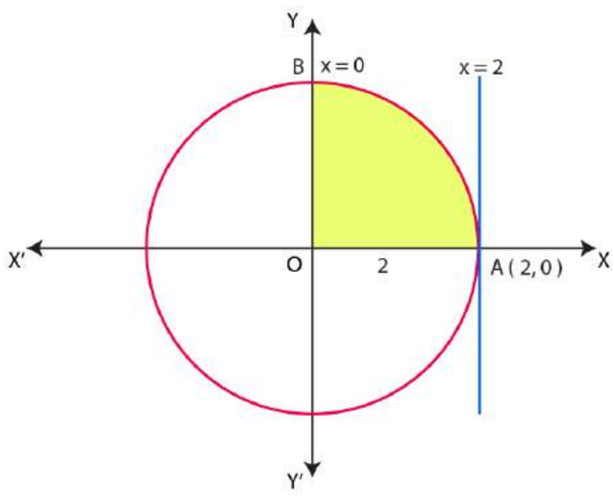
Solution:

Option (A) is correct.

Explanation:

Equation of the circle is $x^2 + y^2 = 2^2$ ……….(1)

$\Rightarrow y = \sqrt{2^2 - x^2}$ ……….(2)



Required area = $\left|\int_0^2 y\,dx\right| = \left|\int_0^2 \sqrt{2^2 - x^2}\,dx\right|$

$= \left|\left(\frac{x}{2}\sqrt{2^2 - x^2} + \frac{2^2}{2}\sin^{-1}\frac{x}{2}\right)_0^2\right|$

$= \frac{2}{2}\sqrt{4-4} + 2\sin^{-1}1 - \left(0 + 2\sin^{-1}0\right)$

$= 0 + 2.\frac{\pi}{2} - 0 - 0$ = π sq. units

13. Choose the correct answer:

Area of the region bounded by the curve y^2 = 4x, y- axis and the line $y = 3$ is:

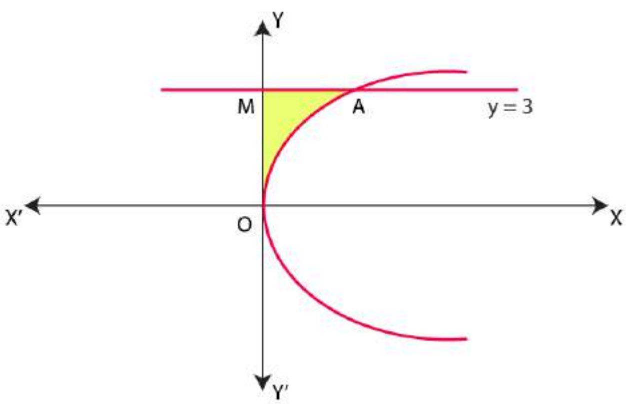
(A) 2 (B) 9/4 (C) 9/3 (D) 9/2

Solution:

Option (B) is correct.

Explanation:

Equation of the curve (parabola) is $y^2 = 4x$



Required area = Area OAM = $\left|\int_0^3 x\,dy\right|$ = $\left|\int_0^2 \frac{y^2}{4}\,dy\right|$

= $\frac{1}{4}\left|\left(\frac{y^3}{3}\right)_0^2\right|$ = $\frac{1}{4}\left|\frac{27}{3}-0\right| = \frac{9}{4}$ sq. units