JEE Main 2021 24 Feb Shift 2 Maths Paper

1. Let $a, b \in \mathbf{R}$. If the mirror image of the point $P(a, 6, 9)$ with respect to the line $\frac{x-3}{7} = \frac{y-2}{5} = \frac{z-1}{-9}$ is $(20, b, -a-9)$, then $|a+b|$ is equal to :

(1) 86 (2) 88 (3)84 (4)90

Ans. **(2)**

Sol. P(a, 6, 9), Q (20, b, –a–9)

Mid point of $PQ = \left(\frac{a+20}{2}, \frac{b+6}{2}, -\frac{a}{2}\right)$ lie on the line.

$\frac{\frac{a+20}{2}-3}{7} = \frac{\frac{b+6}{2}-2}{5} = \frac{-\frac{a}{2}-1}{-9}$

$\Rightarrow \frac{a+20-6}{14} = \frac{b+6-4}{10} = \frac{-a-2}{-18}$

$\Rightarrow \frac{a+14}{14} = \frac{a+2}{18}$

$\Rightarrow 18a + 252 = 14a + 28$

$\Rightarrow 4a = -224$

$\boxed{a = -56}$

$\frac{b+2}{10} = \frac{a+2}{18}$

$\Rightarrow \frac{b+2}{10} = \frac{-54}{18}$

$\Rightarrow \frac{b+2}{10} = -3 \Rightarrow b = -32$

$|a + b| = |-56 - 32| = 88$

2. Let f be a twice differentiable function defined on $\mathbf{R}$ such that $f(0) = 1, f'(0) = 2$ and $f'(x) \neq 0$ for all $x \in \mathbf{R}$. If $\begin{vmatrix} f(x) & f'(x) \\ f'(x) & f''(x) \end{vmatrix} = 0$, for all $x \in \mathbf{R}$, then the value of $f(1)$ lies in the interval :

(1) (9, 12) (2) (6, 9) (3)(3, 6) (4)(0, 3)

Ans. **(2)**

Sol. Given $f(x)f''(x) - \left(f'(x)\right)^2 = 0$

Let $h(x) = \frac{f(x)}{f'(x)}$

Then $h'(x) = 0 \Rightarrow h(x) = k$

$\Rightarrow \frac{f(x)}{f'(x)} = k \quad \Rightarrow f(x) = kf'(x)$

$\Rightarrow f(0) = kf'(0)$

$\Rightarrow k = \frac{1}{2}$

Now, $f(x) = \frac{1}{2}f'(x)$

$\Rightarrow \int 2\,dx = \int \frac{f'(x)}{f(x)} dx$

$\Rightarrow 2x = \ln|f(x)| + C$

As $f(0) = 1 \Rightarrow C = 0$

$\Rightarrow 2x = \ln|f(x)|$

$\Rightarrow f(x) = \pm e^{2x}$

As $f(0) = 1 \Rightarrow f(x) = e^{2x}$

$\therefore f(1) = e^2 \approx 7.38$

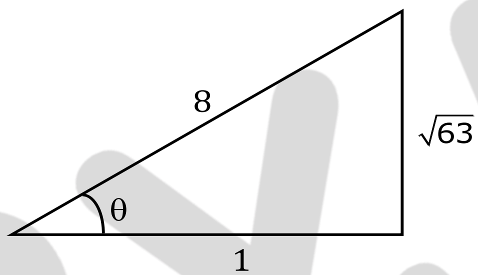
3. A possible value of $\tan\left(\frac{1}{4}\sin^{-1}\frac{\sqrt{63}}{8}\right)$ is :

(1) $\frac{1}{2\sqrt{2}}$ (2) $\frac{1}{\sqrt{7}}$ (3) $\sqrt{7}-1$ (4) $2\sqrt{2}-1$

Ans. **(2)**

Sol. $\tan\left(\frac{1}{4}\sin^{-1}\frac{\sqrt{63}}{8}\right)$

Let $\sin^{-1}\left(\frac{\sqrt{63}}{8}\right) = \theta$ $\sin\theta = \frac{\sqrt{63}}{8}$



$\cos\theta = \frac{1}{8}$

$2\cos^2\frac{\theta}{2} - 1 = \frac{1}{8}$

$\Rightarrow \cos^2\frac{\theta}{2} = \frac{9}{16}$

$\cos\frac{\theta}{2} = \frac{3}{4}$

$\Rightarrow \frac{1-\tan^2\frac{\theta}{4}}{1+\tan^2\frac{\theta}{4}} = \frac{3}{4}$

$\tan\frac{\theta}{4} = \frac{1}{\sqrt{7}}$

4. The probability that two randomly selected subsets of the set $\{1, 2, 3, 4, 5\}$ have exactly two elements in their intersection, is :

(1) $\frac{65}{2^7}$ (2) $\frac{135}{2^9}$ (3) $\frac{65}{2^8}$ (4) $\frac{35}{2^7}$

Ans. **(2)**

Sol. Let A and B be two subsets.

For each $x \in \{1, 2, 3, 4, 5\}$, there are four possibilities :

$x \in A \cap B,\ x \in A' \cap B,\ x \in A \cap B',\ x \in A' \cap B'$

So, the number of elements in sample space $= 4^5$

Required probability

$= \frac{{}^5C_2 \times 3^3}{4^5}$

$= \frac{10 \times 27}{2^{10}} = \frac{135}{2^9}$

5. The vector equation of the plane passing through the intersection of the planes $\vec{r} \cdot (\hat{i} + \hat{j} + \hat{k}) = 1$ and $\vec{r} \cdot (\hat{i} - 2\hat{j}) = -2$, and the point (1, 0, 2) is :

(1) $\vec{r} \cdot (\hat{\imath} - 7\hat{\jmath} + 3\hat{k}) = \frac{7}{3}$

(2) $\vec{r} \cdot (\hat{\imath} + 7\hat{\jmath} + 3\hat{k}) = 7$

(3) $\vec{r} \cdot (3\hat{\imath} + 7\hat{\jmath} + 3\hat{k}) = 7$

(4) $\vec{r} \cdot (\hat{\imath} + 7\hat{\jmath} + 3\hat{k}) = \frac{7}{3}$

Ans. **(2)**

Sol. Family of planes passing through intersection of planes is

$\{\vec{r} \cdot (\hat{\imath} + \hat{\jmath} + \hat{k}) - 1\} + \lambda\{\vec{r} \cdot (\hat{\imath} - 2\hat{\jmath}) + 2\} = 0$

The above curve passes through $\hat{\imath} + 2\hat{k}$,

$(3 - 1) + \lambda(1 + 2) = 0 \Rightarrow \lambda = -\frac{2}{3}$

Hence, equation of plane is

$3\{\vec{r} \cdot (\hat{\imath} + \hat{\jmath} + \hat{k}) - 1\} - 2\{\vec{r} \cdot (\hat{\imath} - 2\hat{\jmath}) + 2\} = 0$

$\Rightarrow \vec{r} \cdot (\hat{\imath} + 7\hat{\jmath} + 3\hat{k}) = 7$

TRICK: Only option (2) satisfies the point (1, 0, 2)

6. If P is a point on the parabola $y = x^2 + 4$ which is closest to the straight line $y = 4x - 1$, then the co-ordinates of P are :

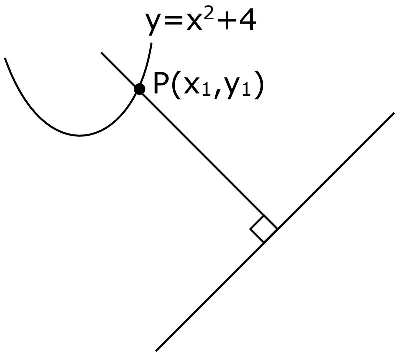
(1) (–2, 8) (2) (1, 5) (3) (3, 13) (4) (2, 8)

Ans. **(4)**

Sol. Tangent at P is parallel to the given line.

$\frac{dy}{dx}|_P = 4$

$\Rightarrow 2x_1 = 4$



$\Rightarrow x_1 = 2$

Required point is (2, 8)

7. Let a, b, c be in arithmetic progression. Let the centroid of the triangle with vertices $(a,c),(2,b)$ and (a,b) be $\left(\frac{10}{3},\frac{7}{3}\right)$. If α, β are the roots of the equation $ax^2 + bx + 1 = 0$, then the value of $\alpha^2 + \beta^2 - \alpha\beta$ is :

(1) $\frac{71}{256}$ (2) $-\frac{69}{256}$ (3) $\frac{69}{256}$ (4) $-\frac{71}{256}$

Ans. **(4)**

Sol. 2b = a + c

$\frac{2a+2}{3} = \frac{10}{3}$ and $\frac{2b+c}{3} = \frac{7}{3}$

$\Rightarrow a = 4$ $\left.\begin{array}{l} 2b + c = 7 \\ 2b - c = 4 \end{array}\right\}$, solving

$b = \frac{11}{4}$ and $c = \frac{3}{2}$

$\therefore$ Quadratic equation is $4x^2 + \frac{11}{4}x + 1 = 0$

$\therefore$ The value of $(\alpha + \beta)^2 - 3\alpha\beta = \frac{121}{256} - \frac{3}{4} = -\frac{71}{256}$

8. The value of the integral, $\int_1^3 \left[x^2 - 2x - 2\right] dx$, where [x] denotes the greatest integer less than or equal to x, is :

(1) –4 (2) –5 (3) $-\sqrt{2} - \sqrt{3} - 1$ (4) $-\sqrt{2} - \sqrt{3} + 1$

Ans. **(3)**

Sol. I = $\int_1^3 -3dx + \int_1^3 [(x-1)^2]dx$

Put $x - 1 = t$; $dx = dt$

I = (–6) + $\int_0^2 [t^2]dt$

I = –6 + $\int_0^1 0dt + \int_1^{\sqrt{2}} 1dt + \int_{\sqrt{2}}^{\sqrt{3}} 2dt + \int_{\sqrt{3}}^{2} 3dt$

$I = -6 + (\sqrt{2} - 1) + 2\sqrt{3} - 2\sqrt{2} + 6 - 3\sqrt{3}$

$I = -1 - \sqrt{2} - \sqrt{3}$

9. Let $f : \mathbf{R} \to \mathbf{R}$ be defined as

$$f(x) = \begin{cases} -55x, & \text{if } x < -5 \\ 2x^3 - 3x^2 - 120x, & \text{if } -5 \le x \le 4 \\ 2x^3 - 3x^2 - 36x - 336, & \text{if } x > 4 \end{cases}$$

Let $A = \{x \in \mathbf{R} : f \text{ is increasing}\}$. Then A is equal to :

(1) $(-5, -4) \cup (4, \infty)$

(2) $(-5, \infty)$

(3) $(-\infty, -5) \cup (4, \infty)$

(4) $(-\infty, -5) \cup (-4, \infty)$

Ans. **(1)**

Sol.

$$f'(x) = \begin{cases} -55 & ; \quad x < -5 \\ 6(x^2 - x - 20) & ; \quad -5 < x < 4 \\ 6(x^2 - x - 6) & ; \quad x > 4 \end{cases}$$

$$\Rightarrow f'(x) = \begin{cases} -55 & ; \quad x < -5 \\ 6(x-5)(x+4) & ; \quad -5 < x < 4 \\ 6(x-3)(x+2) & ; \quad x > 4 \end{cases}$$

Hence, $f(x)$ is monotonically increasing in $(-5, -4) \cup (4, \infty)$

10. If the curve $y = ax^2 + bx + c, x \in \mathbf{R}$ passes through the point (1, 2) and the tangent line to this curve at origin is $y = x$, then the possible values of a, b, c are :

(1) $a = 1, b = 1, c = 0$

(2) $a = -1, b = 1, c = 1$

(3) $a = 1, b = 0, c = 1$

(4) $a = \frac{1}{2}, b = \frac{1}{2}, c = 1$

Ans. **(1)**

Sol. $2 = a + b + c$

$\frac{dy}{dx} = 2ax + b, \left(\frac{dy}{dx}\right)_{(0,0)} = 1$

$\Rightarrow b = 1$ and $a + c = 1$

Since (0, 0) lies on curve,

$\therefore c = 0, a = 1$

TRICK : (0,0) lies on the curve. Only option (1) has $c = 0$

11. The negation of the statement $\sim p \wedge (p \vee q)$ is :

(1) $\sim p \wedge q$

(2) $p \wedge \sim q$

(3) $\sim p \vee q$

(4) $p \vee \sim q$

Ans. **(4)**

Sol. Negation of $\sim p \wedge (p \vee q)$ is

$\sim [\sim p \wedge (p \vee q)]$

$\equiv p \vee \sim (p \vee q)$

$\equiv p \vee (\sim p \wedge \sim q)$

$\equiv (p \vee \sim p) \wedge (p \vee \sim q)$

$\equiv T \wedge (p \vee \sim q)$, where T is tautology.

$\equiv p \vee \sim q$

12. For the system of linear equations :

$x-2y=1, x-y+kz=-2, ky+4z=6, k \in \mathbf{R}$

consider the following statements :

(A) The system has unique solution if $k \neq 2, k \neq -2$.

(B) The system has unique solution if $k=-2$.

(C) The system has unique solution if $k=2$.

(D) The system has no-solution if $k=2$.

(E) The system has infinite number of solutions if $k \neq -2$.

Which of the following statements are **correct** ?

(1) (B) and (E) only

(2) (C) and (D) only

(3) (A) and (D) only

(4) (A) and (E) only

Ans. **(3)**

Sol. $x-2y+0.z=1$

$x-y+kz=-2$

$0.x+ky+4z=6$

$$\Delta = \begin{vmatrix} 1 & -2 & 0 \\ 1 & -1 & k \\ 0 & k & 4 \end{vmatrix} = 4-k^2$$

For unique solution, $4-k^2 \neq 0$

$$\boxed{k \neq \pm 2}$$

For $k=2$,

$x-2y+0.z=1$

$x-y+2z=-2$

$0.x + 2y + 4z = 6$

$$\Delta_x = \begin{vmatrix} 1 & -2 & 0 \\ -2 & -1 & 2 \\ 6 & 2 & 4 \end{vmatrix} = (-8) + 2[-20]$$

$\Rightarrow \Delta_x = -48 \neq 0$

For $k = 2, \Delta_x \neq 0$

So, for $k = 2$, the system has no solution.

13. For which of the following curves, the line $x + \sqrt{3}y = 2\sqrt{3}$ is the tangent at the point $\left(\frac{3\sqrt{3}}{2}, \frac{1}{2}\right)$?

(1) $x^2 + 9y^2 = 9$

(2) $2x^2 - 18y^2 = 9$

(3) $y^2 = \frac{1}{6\sqrt{3}}x$

(4) $x^2 + y^2 = 7$

Ans. **(1)**

Sol. Tangent to $x^2 + 9y^2 = 9$ at point $\left(\frac{3\sqrt{3}}{2}, \frac{1}{2}\right)$ is $x\left(\frac{3\sqrt{3}}{2}\right) + 9y\left(\frac{1}{2}\right) = 9$

$\Rightarrow 3\sqrt{3}x + 9y = 18 \Rightarrow x + \sqrt{3}y = 2\sqrt{3}$

$\Rightarrow$ Option (1) is true.

14. The angle of elevation of a jet plane from a point A on the ground is 60°. After a flight of 20 seconds at the speed of 432 km/hour, the angle of elevation changes to 30°. If the jet plane is flying at a constant height, then its height is :

(1) $1200\sqrt{3}$ m

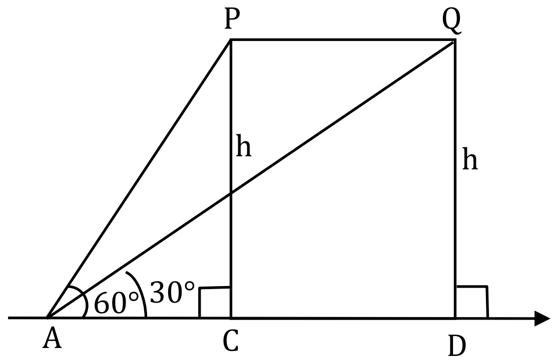
(2) $1800\sqrt{3}$ m

(3) $3600\sqrt{3}$ m

(4) $2400\sqrt{3}$ m

Ans. **(1)**

Sol.



v = 432 × $\frac{1000}{60\times60}$ m/sec = 120 m/sec

Distance PQ = v × 20 = 2400 m

In ΔPAC

$\tan 60° = \frac{h}{AC} \quad \Rightarrow AC = \frac{h}{\sqrt{3}}$

In ΔAQD

$\tan 30° = \frac{h}{AD} \quad \Rightarrow AD = \sqrt{3}h$

AD = AC + CD

$\Rightarrow \sqrt{3}h = \frac{h}{\sqrt{3}} + 2400 \ \Rightarrow \frac{2h}{\sqrt{3}} = 2400$

⇒ h = $1200\sqrt{3}$ m

15. For the statements p and q, consider the following compound statements :

(a) $(\sim q \wedge (p \rightarrow q)) \rightarrow \sim p$

(b) $\big((p \vee q)\big) \wedge \sim p) \rightarrow q$

Then which of the following statements is **correct** ?

(1) (a) is a tautology but not (b)

(2) (a) and (b) both are not tautologies.

(3) (a) and (b) both are tautologies.

(4) (b) is a tautology but not (a).

Ans. **(3)**

Sol. (a)

p	q	$\sim q$	$p \to q$	$\sim q \wedge (p \to q)$	$\sim p$	$(\sim q) \wedge (p \to q) \to \sim p$
T	T	F	T	F	F	T
T	F	T	F	F	F	T
F	T	F	T	F	T	T
F	F	T	T	T	T	T

(a) is tautology.

(b)

p	q	$p \vee q$	$\sim p$	$(p \vee q) \wedge \sim p$	$((p \vee q) \wedge \sim p) \to q$
T	T	T	F	F	T
T	F	T	F	F	T
F	T	T	T	T	T
F	F	F	T	F	T

(b) is tautology.

$\therefore$ (a) and (b) both are tautologies.

16. Let A and B be 3×3 real matrices such that A is symmetric matrix and B is skew-symmetric matrix. Then the system of linear equations $\left(A^2B^2 - B^2A^2\right)X = O$, where X is a 3×1 column matrix of unknown variables and O is a 3×1 null matrix, has :

(1) a unique solution
(2) exactly two solutions
(3) infinitely many solutions
(4) no solution

Ans. **(3)**

Sol. $A^T=A$, $B^T=-B$

Let $A^2B^2 - B^2A^2 = P$

$P^T = (A^2B^2 - B^2A^2)^T = (A^2B^2)^T - (B^2A^2)^T$

$= (B^2)^T (A^2)^T - (A^2)^T (B^2)^T$

$= B^2A^2 - A^2B^2$

$\Rightarrow$ P is a skew-symmetric matrix.

$$\begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore$ ay + bz = 0 ...(1)

−ax + cz = 0 ...(2)

−bx − cy =0 ...(3)

From equation (1), (2), (3)

$\Delta = 0$ and $\Delta_1 = \Delta_2 = \Delta_3 = 0$

$\therefore$ System of equations has infinite number of solutions.

17. If $n \geq 2$ is a positive integer, then the sum of the series

$^{n+1}C_2 + 2\left(^{2}C_2 + ^{3}C_2 + ^{4}C_2 + \ldots + ^{n}C_2\right)$ is :

(1) $\frac{n(n+1)^2(n+2)}{12}$ (2) $\frac{n(n-1)(2n+1)}{6}$

(3) $\frac{n(n+1)(2n+1)}{6}$ (4) $\frac{n(2n+1)(3n+1)}{6}$

Ans. **(3)**

Sol. $^{2}C_2 = ^{3}C_3$

Let S = $^{3}C_3 + ^{3}C_2 + \ldots\ldots + ^{n}C_2 = ^{n+1}C_3$ $(\because ^{n}C_r + ^{n}C_{r-1} = ^{n+1}C_r)$

$\therefore ^{n+1}C_2 + ^{n+1}C_3 + ^{n+1}C_3$

$= ^{n+2}C_3 + ^{n+1}C_3$

$= \frac{(n+2)!}{3!(n-1)!} + \frac{(n+1)!}{3!(n-2)!}$

$= \frac{(n+2)(n+1)n}{6} + \frac{(n+1)(n)(n-1)}{6} = \frac{n(n+1)(2n+1)}{6}$

TRICK : Put $n = 2$ and verify the options.

18. If a curve $y = f(x)$ passes through the point (1, 2) and satisfies $x\frac{dy}{dx} + y = bx^4$, then for what value of b, $\int_1^2 f(x)dx = \frac{62}{5}$?

(1) 5 (2) $\frac{62}{5}$ (3) $\frac{31}{5}$ (4) 10

Ans. **(4)**

Sol. $\frac{dy}{dx} + \frac{y}{x} = bx^3$

I.F. $= e^{\int \frac{dx}{x}} = x$

$\therefore yx = \int bx^4 dx = \frac{bx^5}{5} + c$

Above curve passes through (1,2).

$2 = \frac{b}{5} + c$

Also, $\int_1^2 \left(\frac{bx^4}{5} + \frac{c}{x}\right) dx = \frac{62}{5}$

$\Rightarrow \frac{b}{25} \times 32 + c \ln 2 - \frac{b}{25} = \frac{62}{5}$

$\Rightarrow c = 0$ and $b = 10$

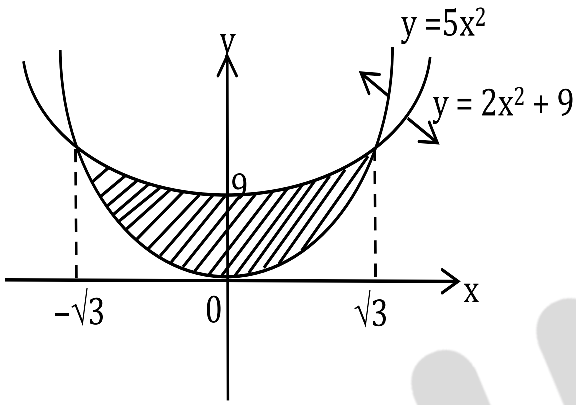
19. The area of the region : $R = \{(x, y) : 5x^2 \le y \le 2x^2 + 9\}$ is :

(1) $9\sqrt{3}$ square units (2) $12\sqrt{3}$ square units

(3) $11\sqrt{3}$ square units (4) $6\sqrt{3}$ square units

Ans. **(2)**

Sol.



Required area

$= 2\int_0^{\sqrt{3}}(2x^2 + 9 - 5x^2)dx$

$= 2\int_0^{\sqrt{3}}(9 - 3x^2)dx$

$= 2|9x - x^3|_0^{\sqrt{3}} = 12\sqrt{3}$

20. Let $f(x)$ be a differentiable function defined on $[0, 2]$ such that $f'(x) = f'(2 - x)$ for all $x \in (0, 2)$, $f(0) = 1$ and $f(2) = e^2$. Then the value of $\int_0^2 f(x)dx$ is :

(1) $1 + e^2$ (2) $1 - e^2$ (3) $2(1 - e^2)$ (4) $2(1 + e^2)$

Ans. **(1)**

Sol. $f'(x) = f'(2 - x)$

On integrating both sides, we get

$f(x) = -f(2 - x) + c$

Put $x = 0$

$f(0) + f(2) = c$

$\Rightarrow c = 1 + e^2$

$\Rightarrow f(x) + f(2 - x) = 1 + e^2$

$I = \int_0^2 f(x)dx = \int_0^1 \{f(x) + f(2-x)\}dx = 1 + e^2$

Section B

1. The number of the real roots of the equation $(x+1)^2 + |x-5| = \frac{27}{4}$ is ______.

Ans. **2**

Sol. For $x \geq 5$,

$(x+1)^2 + (x-5) = \frac{27}{4}$

$\Rightarrow x^2 + 3x - 4 = \frac{27}{4}$

$\Rightarrow x^2 + 3x - \frac{43}{4} = 0$

$\Rightarrow 4x^2 + 12x - 43 = 0$

$x = \frac{-12 \pm \sqrt{144+688}}{8}$

$x = \frac{-12 \pm \sqrt{832}}{8} = \frac{-12 \pm 28.8}{8}$

$= \frac{-3 \pm 7.2}{2}$

$= \frac{-3+7.2}{2}, \frac{-3-7.2}{2}$ (therefore, no solution)

For $x < 5$,

$(x+1)^2 - (x-5) = \frac{27}{4}$

$\Rightarrow x^2 + x + 6 - \frac{27}{4} = 0$

$\Rightarrow 4x^2 + 4x - 3 = 0$

$x = \frac{-4 \pm \sqrt{16+48}}{8}$

$x = \frac{-4 \pm 8}{8} \Rightarrow x = -\frac{12}{8}, \frac{4}{8}$

$\therefore$ 2 real roots.

2. The students $S_1, S_2, \ldots, S_{10}$ are to be divided into 3 groups A, B and C such that each group has at least one student and the group C has at most 3 students. Then the total number of possibilities of forming such groups is ______.

Ans. **31650**

Sol.

$C \to 1 \qquad 9\left[\begin{matrix} A \\ B \end{matrix}\right.$

$C \to 2 \qquad 8\left[\begin{matrix} A \\ B \end{matrix}\right.$

$C \to 3 \qquad 7\left[\begin{matrix} A \\ B \end{matrix}\right.$

Number of ways

$= {}^{10}C_1\,[2^9 - 2] + {}^{10}C_2\,[2^8 - 2] + {}^{10}C_3\,[2^7 - 2]$

$= 2^7[{}^{10}C_1 \times 4 + {}^{10}C_2 \times 2 + {}^{10}C_3] - 20 - 90 - 240$

$= 128\,[40 + 90 + 120] - 350$

$= (128 \times 250) - 350$

$= 10[3165] = 31650$

3. If $a + \alpha = 1, b + \beta = 2$ and $af(x) + \alpha f\left(\frac{1}{x}\right) = bx + \frac{\beta}{x}, x \neq 0$, then the value of the expression $\dfrac{f(x) + f\left(\frac{1}{x}\right)}{x + \frac{1}{x}}$ is ______.

Ans. **2**

Sol. $af(x) + \alpha f\left(\frac{1}{x}\right) = bx + \frac{\beta}{x}$ ...(i)

Replace x by $\frac{1}{x}$

$af\left(\frac{1}{x}\right) + \alpha f(x) = \frac{b}{x} + \beta x$ ...(ii)

(i) + (ii)

$(a + \alpha)\left[f(x) + f\left(\frac{1}{x}\right)\right] = \left(x + \frac{1}{x}\right)(b + \beta)$

$\Rightarrow \dfrac{f(x) + f\left(\frac{1}{x}\right)}{x + \frac{1}{x}} = \frac{2}{1} = 2$

4. If the variance of 10 natural numbers $1, 1, 1, \ldots, 1, k$ is less than 10, then the maximum possible value of k is ______.

Ans. **11**

Sol. $\sigma^2 = \frac{\Sigma x^2}{n} - \left(\frac{\Sigma x}{n}\right)^2$

$\Rightarrow \sigma^2 = \frac{(9+k^2)}{10} - \left(\frac{9+k}{10}\right)^2 < 10$

$\Rightarrow 10(9+k^2) - (81 + k^2 + 18k) < 1000$

$\Rightarrow 90 + 10k^2 - k^2 - 18k - 81 < 1000$

$\Rightarrow 9k^2 - 18k + 9 < 1000$

$\Rightarrow (k-1)^2 < \frac{1000}{9} \Rightarrow k - 1 < \frac{10\sqrt{10}}{3}$

$\Rightarrow k < \frac{10\sqrt{10}}{3} + 1$

Maximum possible integral value of k is 11.

5. Let λ be an integer. If the shortest distance between the lines $x - \lambda = 2y - 1 = -2z$ and $x = y + 2\lambda = z - \lambda$ is $\frac{\sqrt{7}}{2\sqrt{2}}$, then the value of $|\lambda|$ is ______.

Ans. **1**

Sol. $\frac{x-\lambda}{1} = \frac{y-\frac{1}{2}}{\frac{1}{2}} = \frac{z}{-\frac{1}{2}}$

$\frac{x-\lambda}{2} = \frac{y-\frac{1}{2}}{1} = \frac{z}{-1}$ ...(1) Point on line $= \left(\lambda, \frac{1}{2}, 0\right)$

$\frac{x}{1} = \frac{y+2\lambda}{1} = \frac{z-\lambda}{1}$ ...(2) Point on line $= (0, -2\lambda, \lambda)$

Distance between skew lines $= \frac{[\vec{a}_2 - \vec{a}_1 \ \ \vec{b}_1 \ \ \vec{b}_2]}{|\vec{b}_1 \times \vec{b}_2|}$

$$\frac{\begin{vmatrix} \lambda & \frac{1}{2}+2\lambda & -\lambda \\ 2 & 1 & -1 \\ 1 & 1 & 1 \end{vmatrix}}{\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & -1 \\ 1 & 1 & 1 \end{vmatrix}}$$

$= \frac{\left|-5\lambda - \frac{3}{2}\right|}{\sqrt{14}} = \frac{\sqrt{7}}{2\sqrt{2}}$ (Given)

$\Rightarrow |10\lambda + 3| = 7 \Rightarrow \lambda = -1$ as λ is an integer.

$\Rightarrow |\lambda| = 1$

6. Let $i=\sqrt{-1}$. If $\frac{(-1+i\sqrt{3})^{21}}{(1-i)^{24}}+\frac{(1+i\sqrt{3})^{21}}{(1+i)^{24}}=k$, and $n=[|k|]$ be the greatest integral part of $|k|$. Then $\sum_{j=0}^{n+5}(j+5)^2-\sum_{j=0}^{n+5}(j+5)$ is equal to _____.

Ans. 310

Sol. $\frac{\left(2e^{i\frac{2\pi}{3}}\right)^{21}}{\left(\sqrt{2}e^{-i\frac{\pi}{4}}\right)^{24}}+\frac{\left(2e^{i\frac{\pi}{3}}\right)^{21}}{\left(\sqrt{2}e^{i\frac{\pi}{4}}\right)^{24}}$

$=\frac{2^{21}.e^{i14\pi}}{2^{12}.e^{-i6\pi}}+\frac{2^{21}(e^{i7\pi})}{2^{12}(e^{i6\pi})}$

$=2^9e^{i(20\pi)}+2^9e^{i\pi}$

$=2^9+2^9(-1)=0=k$

$\therefore n=0$

$\sum_{j=0}^{5}(j+5)^2-\sum_{j=0}^{5}(j+5)$

$=[5^2+6^2+7^2+8^2+9^2+10^2]-[5+6+7+8+9+10]$

$=[(1^2+2^2+\ldots+10^2)-(1^2+2^2+3^2+4^2)]-[(1+2+3+\ldots+10)-(1+2+3+4)]$

$=(385-30)-[55-10]$

$=355-45=310$

7. Let a point P be such that its distance from the point (5, 0) is thrice the distance of P from the point $(-5,0)$. If the locus of the point P is a circle of radius r, then $4r^2$ is equal to _______.

Ans. 56.25

Sol. Let P be (h,k), A(5, 0) and B(-5, 0)

Given PA = 3PB

$\Rightarrow PA^2=9PB^2$

$\Rightarrow (h-5)^2+k^2=9[(h+5)^2+k^2]$

$\Rightarrow 8h^2+8k^2+100h+200=0$

$\therefore$ Locus of P is $x^2+y^2+\left(\frac{25}{2}\right)x+25=0$

Centre $\equiv\left(\frac{-25}{4},0\right)$

$\therefore\ r^2=\left(\frac{-25}{4}\right)^2-25$

$= \frac{625}{16} - 25$

$= \frac{225}{16}$

$\therefore 4r^2 = 4 \times \frac{225}{16} = \frac{225}{4} = 56.25$

8. For integers n and r, let $\binom{n}{r} = \begin{cases} {}^nC_r, & \text{if } n \geq r \geq 0 \\ 0, & \text{otherwise} \end{cases}$

The maximum value of k for which the sum

$\sum_{i=0}^{k} \binom{10}{i}\binom{15}{k-i} + \sum_{i=0}^{k+1} \binom{12}{i}\binom{13}{k+1-i}$ exists, is equal to_______.

Ans. *

Sol. BONUS

$(1+x)^{10} = {}^{10}C_0 + {}^{10}C_1x + {}^{10}C_2x^2 + \ldots\ldots + {}^{10}C_{10}x^{10}$

$(1+x)^{15} = {}^{15}C_0 + {}^{15}C_1x + \ldots.{}^{15}C_{k-1}x^{k-1} + {}^{15}C_kx^k + {}^{15}C_{k+1}x^{k+1} + \ldots\ldots{}^{15}C_{15}x^{15}$

$\sum_{i=0}^{k}(10C_i)(15C_{k-i}) = {}^{10}C_0.\ {}^{15}C_k + {}^{10}C_1.{}^{15}C_{k-1} + \ldots.. + {}^{10}C_k.{}^{15}C_0$

Coefficient of x_k in $(1+x)^{25}$

$= {}^{25}C_k$

$\sum_{i=0}^{k+1}(12C_i)(13C_{k+1-i}) = {}^{12}C_0.{}^{13}C_{k+1} + {}^{12}C_1.\ {}^{13}C_k + \ldots\ldots + {}^{12}C_{k+1}.{}^{13}C_0$

Coefficient of x^{k+1} in $(1+x)^{25}$

$= {}^{25}C_{k+1}$

${}^{25}C_k + {}^{25}C_{k+1} = {}^{26}C_{k+1}$

By the given definition of $\binom{n}{r}$, k can be as large as possible.

9. The sum of first four terms of a geometric progression (G.P.) is $\frac{65}{12}$ and the sum of their respective reciprocals is $\frac{65}{18}$. If the product of first three terms of the G.P. is 1, and the third term is α, then 2α is_________.

Ans. **3**

Sol. a, ar, ar^2, ar^3

$a + ar + ar^2 + ar^3 = \frac{65}{12}$ ....(1)

$\frac{1}{a} + \frac{1}{ar} + \frac{1}{ar^2} + \frac{1}{ar^3} = \frac{65}{18}$

$\Rightarrow \frac{1}{a}\left(\frac{r^3+r^2+r+1}{r^3}\right) = \frac{65}{18}$ ....(2)

$\frac{(1)}{(2)}$, we get

$a^2r^3 = \frac{18}{12} = \frac{3}{2}$

Also, $a^3r^3 = 1 \Rightarrow a\left(\frac{3}{2}\right) = 1 \Rightarrow a = \frac{2}{3}$

$\frac{4}{9}r^3 = \frac{3}{2} \Rightarrow r^3 = \frac{3^3}{2^3} \Rightarrow r = \frac{3}{2}$

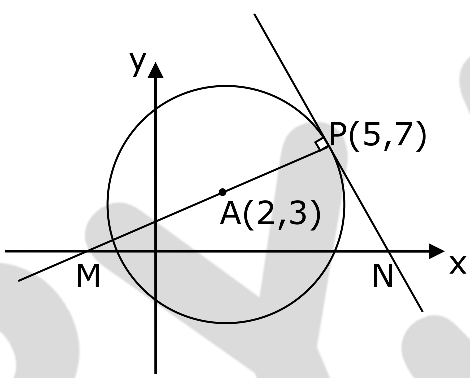
$\alpha = ar^2 = \frac{2}{3}.\left(\frac{3}{2}\right)^2 = \frac{3}{2}$

$\therefore 2\alpha = 3$

10. If the area of the triangle formed by the positive x-axis, the normal and the tangent to the circle $(x-2)^2+(y-3)^2=25$ at the point (5, 7) is A, then 24A is equal to ______.

Ans.

Sol.



Equation of normal at P is

$(y-7) = \left(\frac{7-3}{5-2}\right)(x-5)$

⇒ 3y – 21 = 4x – 20

⇒ 4x – 3y + 1 = 0

$\Rightarrow M$ is $\left(-\frac{1}{4}, 0\right)$

Equation of tangent at P is

$(y-7) = -\frac{3}{4}(x-5)$

⇒ 4y – 28 = –3x + 15

⇒ 3x + 4y = 43

$\Rightarrow N$ is $\left(\frac{43}{3}, 0\right)$

The question is **wrong**. The normal cuts at a point on the negative axis.