JEE MAIN – 2021

16th Mar Shift – II

Section A

Multiple Choice Question:

1. The least value of |z| where z is complex number which satisfies the inequality $\exp\left(\frac{(|z|+3)(|z|-1)}{||z|+1|}\log_e 2\right) \geq \log_{\sqrt{2}}|5\sqrt{7}+9i|, i=\sqrt{-1}$, is equal to :

(1) 2
(2) 3
(3) 8
(4) $\sqrt{5}$

Ans. (2)

Sol. $2^{\frac{(|z|+3)(|z|-1)}{(|z|+1)}} \geq 2^3 \Rightarrow \frac{(|z|+3)(|z|-1)}{|z|+1} \geq 3$

$\Rightarrow |z|^2 + 2|z| - 3 \geq 3|z| + 3$

$\Rightarrow |z|^2 - |z| - 6 \geq 0$

$(|z|-3)(|z|+2) \geq 0$

$|z|_{min} = 3$

2. Let f : S $\rightarrow$ S where S = (0, ∞) be a twice differentiable function such that $f(x+1) = xf(x)$. If g : S $\rightarrow$ **R** be defined as $g(x) = \log_e f(x)$, then the value of $|g''(5) - g''(1)|$ is equal to :

(1) $\frac{197}{144}$
(2) $\frac{187}{144}$
(3) $\frac{205}{144}$
(4) 1

Ans. (3)

Sol. $f(x+1) = xf(x)$

$g(x+1) = \log_e(f(x+1))$

$g(x+1) = \log_e x + \log_e f(x)$

$g(x+1) - g(x) = \log_e x$

$g''(x+1) - g''(x) = -\frac{1}{x^2}$

$g''(2) - g''(1) = -1$

$g''(3) - g''(2) = -\frac{1}{4}$

$g''(4) - g''(3) = -\frac{1}{9}$

$g''(5) - g''(4) = -\frac{1}{16}$

$g''(5) - g''(1) = -\left[1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16}\right]$

$\therefore |g''(5) - g''(1)| = \left[\frac{144 + 36 + 16 + 9}{16 \times 9}\right] = \frac{205}{144}$

3. If y = y(x) is the solution of the differential equation $\frac{dy}{dx} + (\tan x)y = \sin x, 0 \le x \le \frac{\pi}{3}$, with y(0) = 0, then $y\left(\frac{\pi}{4}\right)$ equal to :

(1) $\log_e 2$

(2) $\frac{1}{2}\log_e 2$

(3) $\left(\frac{1}{2\sqrt{2}}\right)\log_e 2$

(4) $\frac{1}{4}\log_e 2$

Ans. **(3)**

Sol. I.F. = $e^{\int \tan x dx}$

$= e^{\ln \sec x}$

= sec x

Solution of the equation:

$y(\sec x) = \int (\sin x)(\sec x) dx$

$\Rightarrow \frac{y}{\cos x} = \ell n(\sec x) + c$

Put x = 0, c = 0

$\therefore$ y = cos x ℓn(sec x)

Put x = $\pi/4$

$y = \frac{1}{\sqrt{2}} \ln \sqrt{2} = \frac{1}{2\sqrt{2}} \ell n2$

$y = \frac{\ln 2}{2\sqrt{2}}$

4. If the foot of the perpendicular from point (4, 3, 8) on the line

$L_1 : \frac{x-a}{\ell} = \frac{y-2}{3} = \frac{z-b}{4}$, $\ell \neq 0$ is (3, 5, 7), then the shortest distance between the line

L_1 and line $L_2 : \frac{x-2}{3} = \frac{y-4}{4} = \frac{z-5}{5}$ is equal to :

(1) $\sqrt{\frac{2}{3}}$

(2) $\frac{1}{\sqrt{3}}$

(3) $\frac{1}{2}$

(4) $\frac{1}{\sqrt{6}}$

Ans. **(4)**

Sol. (3, 5, 7) lies on given line L_1

$$\frac{3-a}{\ell} = \frac{3}{3} = \frac{7-b}{4}$$

$$\frac{7-b}{4} = 1 \Rightarrow b = 3$$

$$\frac{3-a}{\ell} = 1 \Rightarrow 3 - a = \ell$$

A (4, 3, 8)

B (3, 5, 7)

DR's of AB = (1, −2, 1)

AB $\perp$ line L_1

$(1)(\ell) + (-2)(3) + 4(1) = 0$

$\Rightarrow \ell = 2$

a = 1

a = 1, b = 3, ℓ = 2

$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$$

$$\frac{x-2}{3} = \frac{y-4}{4} = \frac{z-5}{5}$$

$$\text{S.D.} = \frac{\begin{vmatrix} 1 & 2 & 2 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix}}{\left\| \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix} \right\|} = \frac{1}{\sqrt{6}}$$

5. If (x, y, z) be an arbitrary point lying on a plane P which passes through the points (42, 0, 0), (0, 42, 0) and (0, 0, 42), then the value of the expression

$$3+\frac{x-11}{(y-19)^2(z-12)^2}+\frac{y-19}{(x-11)^2(z-12)^2}+\frac{z-12}{(z-11)^2(y-19)^2}-\frac{x+y+z}{14(x-11)(y-19)(z-12)}$$

is equal to :

(1) 3

(2) 0

(3) 39

(4) –45

Ans. (1)

Sol. Equation of plane is x + y + z = 42

or, $(x-11)+(y-19)+(z-12)=0$

Now, $\frac{x-11}{(y-19)^2(z-12)^2}+\frac{y-19}{(x-11)^2(z-12)^2}+\frac{z-12}{(z-11)^2(y-19)^2}$

$=\frac{(x-11)^3+(y-19)^3+(z-12)^3}{(x-11)^2(y-19)^2(z-12)^2}$

$=\frac{3(x-11)(y-19)(z-12)}{(x-11)^2(y-19)^2(z-12)^2}$ [If $a+b+c=0$, then $a^3+b^3+c^3=3abc$]

$=\frac{3}{(x-11)(y-19)(z-12)}$

∴ The given expression is equal to

$3+\frac{3}{(x-11)(y-19)(z-12)}-\frac{3}{(x-11)(y-19)(z-12)}=3$

6. Consider the integral $I=\int_0^{10}\frac{[x]e^{[x]}}{e^{x-1}}dx$, where [x] denotes the greatest integer less than or equal to x. Then the value of I is equal to :

(1) 45 (e – 1)

(2) 45 (e + 1)

(3) 9(e – 1)

(4) 9(e + 1)

Ans. (1)

Sol. $I=\int_0^{10}[x]\cdot e^{[x]+1-x}dx$

$=\int_1^2 e^{2-x}dx+\int_2^3 2\cdot e^{3-x}dx+\int_3^4 3\cdot e^{4-x}dx+......+\int_9^{10}9e^{10-x}dx$

= – {(1 – e) + 2(1 – e) + 3(1 – e) +......+ 9(1 – e)}

= 45(e – 1)

7. Let A (–1, 1), B (3, 4) and C(2, 0) be given three points. A line y = mx, m > 0, intersects lines AC and BC at point P and Q respectively. Let A_1 and A_2 be the areas of ΔABC and ΔPQC respectively, such that $A_1 = 3A_2$, then the value of m is equal to :

(1) $\frac{4}{15}$

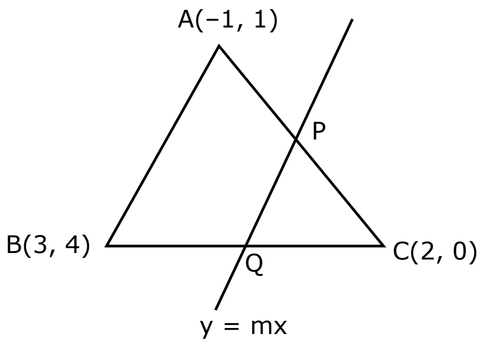
(2) 1

(3) 2

(4) 3

Ans. (2)

Sol.



$A_1 = \text{Area of } \Delta ABC = \frac{1}{2}\left\|\begin{vmatrix} -1 & 1 & 1 \\ 2 & 0 & 1 \\ 3 & 4 & 1 \end{vmatrix}\right\|$

$\Rightarrow A_1 = \frac{13}{2}$

Equation of line AC is $y - 1 = -\frac{1}{3}(x + 1)$

Solving it with line y = mx, we get $P\left(\frac{2}{3m+1}, \frac{2m}{3m+1}\right)$

Equation of line BC is $y - 0 = 4(x - 2)$

Solving it with line y = mx, we get $Q\left(\frac{-8}{m-4}, \frac{-8m}{m-4}\right)$

$A_2 = \text{Area of } \Delta PQC = \frac{1}{2}\left\|\begin{vmatrix} 2 & 0 & 1 \\ \frac{2}{3m+1} & \frac{2m}{3m+1} & 1 \\ \frac{-8}{m-4} & \frac{-8m}{m-4} & 1 \end{vmatrix}\right\| = \frac{A_1}{3} = \frac{13}{6}$

$\Rightarrow \frac{26m^2}{3m^2 - 11m - 4} = \pm\frac{13}{6}$

$\Rightarrow 12m^2 = \pm\left(3m^2 - 11m - 4\right)$

Taking +ve sign,

$9m^2 + 11m + 4 = 0$ (Rejected $\because$ m is imaginary)

Taking –ve sign,

$15m^2 - 11m - 4 = 0$

$m = 1, -\frac{4}{15}$

$\Rightarrow m = 1$ as $m > 0$

8. Let f be a real valued function, defined on **R** – {–1, 1} and given by

$f(x) = 3\log_e\left|\frac{x-1}{x+1}\right| - \frac{2}{x-1}$. Then in which of the following intervals, function $f(x)$ is increasing?

(1) $(-\infty,-1)\cup\left(\left[\frac{1}{2},\infty\right)-\{1\}\right)$

(2) $\left(-1,\frac{1}{2}\right]$

(3) $(-\infty, \infty) - \{-1, 1\}$

(4) $\left(-\infty,\frac{1}{2}\right]-\{-1\}$

Ans. **(1)**

Sol. $f'(x) = \left(\frac{x+1}{x-1}\times\frac{x+1-(x-1)}{(x+1)^2}\right)3+\frac{2}{(x-1)^2} = \frac{6}{(x-1)(x+1)}+\frac{2}{(x-1)^2}$

$= \frac{4(2x-1)}{(x+1)(x-1)^2}$

+   –   +

-1 $\frac{1}{2}$

$\therefore\ x \in (-\infty,-1)\cup\left[\frac{1}{2},\infty\right)-\{1\}$

9. Let the lengths of intercepts on x-axis and y-axis made by the circle $x^2 + y^2 + ax + 2ay + c = 0$, $(a<0)$ be $2\sqrt{2}$ and $2\sqrt{5}$, respectively. Then the shortest distance from origin to a tangent to this circle which is perpendicular to the line $x + 2y = 0$, is equal to :

(1) $\sqrt{10}$

(2) $\sqrt{6}$

(3) $\sqrt{11}$

(4) $\sqrt{7}$

Ans. **(2)**

Sol. $2\sqrt{a^2/4 - c} = 2\sqrt{2}$

$\sqrt{a^2-4c} = 2\sqrt{2}$

$a^2 - 4c = 8$ ... (1)

$2\sqrt{a^2-c} = 2\sqrt{5}$

$a^2 - c = 5$ ...(2)

(2) – (1), we get

$3c = -3 \Rightarrow c = -1$

$a^2 = 4 \Rightarrow a = -2$

$x^2 + y^2 - 2x - 4y - 1 = 0$

Equation of tangent $2x - y + \lambda = 0$

$\therefore p = r$

$\left|\frac{0-0+\lambda}{\sqrt{5}}\right| = \sqrt{6}$

$\Rightarrow \lambda = \pm\sqrt{30}$

$\therefore$ tangents are $2x - y \pm \sqrt{30} = 0$

Distance from origin $= \frac{\sqrt{30}}{\sqrt{5}} = \sqrt{6}$

10. Let A denote the event that a 6-digit integer formed by 0, 1, 2, 3, 4, 5, 6 without repetitions, be divisible by 3. Then probability of event A is equal to :

(1) $\frac{4}{9}$

(2) $\frac{9}{56}$

(3) $\frac{3}{7}$

(4) $\frac{11}{27}$

Ans. **(1)**

Sol. Total case = $6\lfloor 6$

Fav. case = (0, 1, 2, 3, 4, 5) + (0, 1, 2, 4, 5, 6) + (1, 2, 3, 4, 5, 6)

$= 5\lfloor 5 + 5\lfloor 5 + \lfloor 6$

= 1920

Probability = $\frac{1920}{6\lfloor 6} = \frac{4}{9}$

11. Let $\alpha \in \mathbf{R}$ be such that the function $f(x) = \begin{cases} \dfrac{\cos^{-1}\left(1-\{x\}^2\right)\sin^{-1}\left(1-\{x\}\right)}{\{x\}-\{x\}^3} & x \neq 0 \\ \alpha & x = 0 \end{cases}$ is continuous at x = 0, where {x} = x–[x], [x] is the greatest integer less than or equal to x. Then :

(1) $\alpha = \frac{\pi}{4}$

(2) No such α exists

(3) $\alpha = 0$

(4) $\alpha = \frac{\pi}{\sqrt{2}}$

Ans. **(2)**

Sol.

$$\text{RHL} = \lim_{x\to 0^+} \frac{\cos^{-1}\left(1-x^2\right)\sin^{-1}(1-x)}{x\left(1-x^2\right)} = \frac{\pi}{2}\lim_{x\to 0^+}\frac{\cos^{-1}\left(1-x^2\right)}{x}$$

$$= \frac{\pi}{2}\lim_{x\to 0^+}\frac{-1}{\sqrt{1-\left(1-x^2\right)^2}}(-2x) \qquad \text{(L' Hospital Rule)}$$

$$= \pi \lim_{x\to 0^+}\frac{x}{\sqrt{2x^2-x^4}} = \pi\lim_{x\to 0^+}\frac{1}{\sqrt{2-x^2}} = \frac{\pi}{\sqrt{2}}$$

$$\text{LHL} = \lim_{x\to 0^-}\frac{\cos^{-1}\left(1-(1+x)^2\right)\sin^{-1}(-x)}{(1+x)-(1+x)^3} = \frac{\pi}{2}\lim_{x\to 0^-}\frac{\sin^{-1}x}{(1+x)\left[(1+x)^2-1\right]} = \frac{\pi}{2}\lim_{x\to 0^-}\frac{\sin^{-1}x}{x^2+2x}$$

$$= \frac{\pi}{2}\left(\frac{1}{2}\right) = \frac{\pi}{4}$$

As LHL $\neq$ RHL, so f(x) is not continuous at x = 0

12. The maximum value of $f(x) = \begin{vmatrix} \sin^2 x & 1+\cos^2 x & \cos 2x \\ 1+\sin^2 x & \cos^2 x & \cos 2x \\ \sin^2 x & \cos^2 x & \sin 2x \end{vmatrix}$, $x \in \mathbf{R}$ is :

(1) $\sqrt{7}$

(2) $\sqrt{5}$

(3) 5

(4) $\frac{3}{4}$

Ans. **(2)**

Sol. $C_1 \to C_1 + C_2$

$$\begin{vmatrix} 2 & 1+\cos^2 x & \cos 2x \\ 2 & \cos^2 x & \cos 2x \\ 1 & \cos^2 x & \sin 2x \end{vmatrix}$$

$R_1 \to R_1 - R_2$

$$\begin{vmatrix} 0 & 1 & 0 \\ 2 & \cos^2 x & \cos 2x \\ 1 & \cos^2 x & \sin 2x \end{vmatrix}$$

$= (-1)[2\sin 2x - \cos 2x] = \cos 2x - 2\sin 2x$

Maximum value = $\sqrt{5}$

13. Consider a rectangle ABCD having 5, 7, 6, 9 points in the interior of the line segments AB, CD, BC, DA respectively. Let α be the number of triangles having these points from different sides as vertices and β be the number of quadrilaterals having these points from different sides as vertices. Then $(\beta - \alpha)$ is equal to:

(1) 1890

(2) 795

(3) 717

(4) 1173

Ans. **(3)**

Sol. $\alpha = {}^6C_1\,{}^7C_1\,{}^9C_1 + {}^5C_1\,{}^7C_1\,{}^9C_1 + {}^5C_1\,{}^6C_1\,{}^9C_1 + {}^5C_1\,{}^6C_1\,{}^7C_1 = 378 + 315 + 270 + 210 = 1173$

$\beta = {}^5C_1\,{}^6C_1\,{}^7C_1\,{}^9C_1 = 1890$

$\Rightarrow \beta - \alpha = 1890 - 1173 = 717$

14. Let C be the locus of the mirror image of a point on the parabola $y^2 = 4x$ with respect to the line $y = x$. Then the equation of tangent to C at P(2, 1) is :

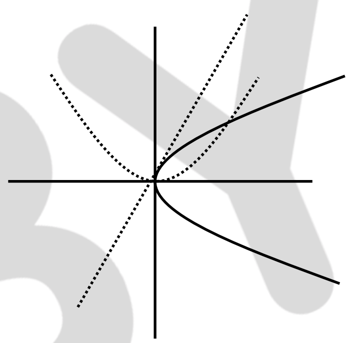
(1) $2x + y = 5$

(2) $x + 2y = 4$

(3) $x + 3y = 5$

(4) $x - y = 1$

Ans. **(4)**



Sol. Image of $y^2 = 4x$ w.r.t. $y = x$ is $x^2 = 4y$

Tangent from (2, 1)

$xx_1 = 2(y + y_1)$

$2x = 2(y + 1)$

$x = y + 1$

15. Given that the inverse trigonometric functions take principal values only. Then, the number of real values of x which satisfy $\sin^{-1}\left(\frac{3x}{5}\right) + \sin^{-1}\left(\frac{4x}{5}\right) = \sin^{-1} x$ is equal to :

(1) 1

(2) 2

(3) 3

(4) 0

Ans. **(3)**

Sol. Taking sine on both sides

$$\frac{3x}{5}\sqrt{1-\frac{16x^2}{25}}+\frac{4x}{5}\sqrt{1-\frac{9x^2}{25}}=x$$

$$\Rightarrow 3x\sqrt{25-16x^2}=25x-4x\sqrt{25-9x^2}$$

$$\Rightarrow x=0 \text{ or } 3\sqrt{25-16x^2}=25-4\sqrt{25-9x^2}$$

$$\Rightarrow 9\left(25-16x^2\right)=625-200\sqrt{25-9x^2}+16\left(25-9x^2\right)$$

$$\Rightarrow 200\sqrt{25-9x^2}=800$$

$$\Rightarrow \sqrt{25-9x^2}=4$$

$$\Rightarrow x^2=1$$

$$\Rightarrow x=\pm 1$$

$\therefore$ Total number of solutions = 3

16. Let C_1 be the curve obtained by solution of differential equation $2xy\frac{dy}{dx}=y^2-x^2, x>0$. Let the curve C_2 be the solution of $\frac{2xy}{x^2-y^2}=\frac{dy}{dx}$. If both the curves pass through (1, 1), then the area enclosed by the curves C_1 and C_2 is equal to :

(1) $\frac{\pi}{2}-1$

(2) $\frac{\pi}{4}+1$

(3) $\pi-1$

(4) $\pi+1$

Ans. **(1)**

Sol. $\frac{dy}{dx}=\frac{y^2-x^2}{2xy}$

Put y = vx

$$v+x\frac{dv}{dx}=\frac{v^2x^2-x^2}{2vx^2}=\frac{v^2-1}{2v}$$

$$x\frac{dv}{dx}=\frac{v^2-1-2v^2}{2v}=-\frac{(v^2+1)}{2v}$$

$$\Rightarrow \frac{2v}{v^2+1}dv=-\frac{dx}{x}$$

$$\ell n(v^2+1)=-\ell nx+\ell nc \Rightarrow v^2+1=\frac{c}{x}$$

$\Rightarrow \frac{y^2}{x^2}+1=\frac{c}{x} \Rightarrow x^2+y^2=cx$

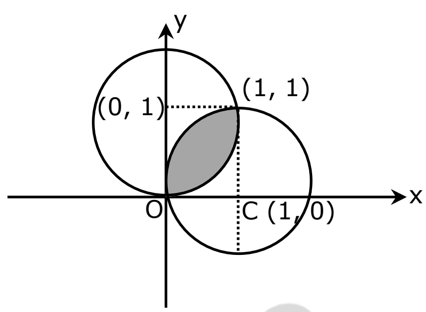
It passes through (1, 1)

$\therefore x^2 + y^2 - 2x=0$

Similarly, for second differential equation $\frac{dx}{dy}=\frac{x^2-y^2}{2xy}$

equation of curve is $x^2+y^2-2y=0$

Now required area is



$= \left(\frac{1}{4}\times\pi\times1^2-\frac{1}{2}\times1\times1\right)\times2$

$= \left(\frac{\pi}{2}-1\right)$ sq. units

17. Let $\vec{a}=\hat{i}+2\hat{j}-3\hat{k}$ and $\vec{b}=2\hat{i}-3\hat{j}+5\hat{k}$. If $\vec{r}\times\vec{a}=\vec{b}\times\vec{r}, \vec{r}\cdot\left(\alpha\hat{i}+2\hat{j}+\hat{k}\right)=3$ and $\vec{r}\cdot\left(2\hat{i}+5\hat{j}-\alpha\hat{k}\right)=-1, \alpha\in R$, then the value of $\alpha+|\vec{r}|^2$ is equal to :

(1) 11
(2) 15
(3) 9
(4) 13

Ans. **(2)**

Sol. $\vec{r}\times\vec{a}=-\vec{r}\times\vec{b}$

$\vec{r}\times\left(\vec{a}+\vec{b}\right)=0 \qquad \left(\vec{a}+\vec{b}=3\hat{i}-\hat{j}+2\hat{k}\right)$

$\vec{r} \,||\, \left(\vec{a}+\vec{b}\right)$

$\vec{r}=\lambda\left(\vec{a}+\vec{b}\right)$

$\because \vec{r}\cdot\left(2\hat{i}+5\hat{j}-\alpha\hat{k}\right)=-1$

$\lambda\left[3\hat{i}-\hat{j}+2\hat{k}\right]\cdot\left[2\hat{i}+5\hat{j}-\alpha\hat{k}\right]=-1$

$\Rightarrow \lambda(6 - 5 - 2\alpha) = -1$

$\lambda(1 - 2\alpha) = -1$ ...(1)

$\vec{r} \cdot \left(\alpha\hat{i} + 2\hat{j} + \hat{k}\right) = 3$

$\lambda\left(3\hat{i} - \hat{j} + 2\hat{k}\right) \cdot \left(\alpha\hat{i} + 2\hat{j} + \hat{k}\right) = 3$

$\Rightarrow \lambda[3\alpha - 2 + 2] = 3 \Rightarrow \lambda\alpha = 1$ ...(2)

Solving (1) & (2)

$\lambda\left[1 - \frac{2}{\lambda}\right] = -1$

$\lambda - 2 = -1 \Rightarrow \lambda = 1$ and $\alpha = 1$

$\vec{r} = 3\hat{i} - \hat{j} + 2\hat{k}$

$\alpha + |\vec{r}|^2 \Rightarrow 1 + 14 = 15$

18. Let $P(x) = x^2 + bx + c$ be a quadratic polynomial with real coefficients such that $\int_0^1 P(x)\,dx = 1$ and P(x) leaves remainder 5 when it is divided by (x – 2). Then the value of 9(b+c) is equal to :

(1) 7
(2) 11
(3) 15
(4) 9

Ans. **(1)**

Sol. $(x - 2)Q(x) + 5 = x^2 + bx + c$

Put x = 2

$5 = 2b+c+4$ ...(1)

$\int_0^1 \left(x^2 + bx + c\right) dx = 1$

$\Rightarrow \frac{1}{3} + \frac{b}{2} + c = 1$

$\frac{b}{2} + c = \frac{2}{3}$ ...(2)

Solving (1) & (2)

$b = \frac{2}{9}$

$c = \frac{5}{9}$

$9(b+c) = 7$

19. If the points of intersections of the ellipse $\frac{x^2}{16}+\frac{y^2}{b^2}=1$ and the circle $x^2 + y^2 = 4b$, $b > 4$ lie on the curve $y^2 = 3x^2$, then b is equal to :

(1) 5
(2) 6
(3) 12
(4) 10

Ans. **(3)**

Sol. $\frac{x^2}{16}+\frac{y^2}{b^2}=1$ ...(1)

$x^2+y^2=4b$ ...(2)

$y^2=3x^2$ ...(3)

From eqns. (2) and (3), $x^2 = b$ and $y^2 = 3b$

From eqn. (1), $\frac{b}{16}+\frac{3b}{b^2}=1$

$\Rightarrow b^2+48=16b$

$\Rightarrow b=12$

20. Let A = {2, 3, 4, 5, ……, 30} and '$\cong$' be an equivalence relation on A×A, defined by $(a,b)\cong(c,d)$, if and only if ad = bc. Then the number of ordered pairs which satisfy this equivalence relation with ordered pair (4, 3) is equal to :

(1) 7
(2) 5
(3) 6
(4) 8

Ans. **(1)**

Sol. ad = bc

(a, b) R (4, 3) $\Rightarrow$ 3a = 4b

$a = \frac{4}{3}b$

b must be multiple of 3

b can be 3, 6, 9, ……, 30

Also, a must be less than or equal to 30.

(a, b) = (4, 3), (8, 6), (12, 9) (16, 12), (20, 15), (24, 18), (28, 21)

$\Rightarrow$ 7 ordered pairs

Section B

Integer Type:

1. Let $\vec{c}$ be a vector perpendicular to the vectors $\vec{a} = \hat{i} + \hat{j} - \hat{k}$ and $\vec{b} = \hat{i} + 2\hat{j} + \hat{k}$. If $\vec{c} \cdot (\hat{\imath} + \hat{\jmath} + 3\hat{k}) = 8$, then the value of $\vec{c} \cdot (\vec{a} \times \vec{b})$ is equal to _______

Ans. **(28)**

Sol. $\vec{c} \cdot (\vec{a} \times \vec{b}) = [\vec{c}\,\vec{a}\,\vec{b}]$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & -1 \\ 1 & 2 & 1 \end{vmatrix} = (3, -2, 1)$$

$\vec{c} \perp \vec{a}, \vec{c} \perp \vec{b} \Rightarrow C \,||\, \vec{a} \times \vec{b}$

$\vec{c} = \lambda(\vec{a} \times \vec{b})$

$\Rightarrow \vec{c} = \lambda(3\hat{i} - 2\hat{j} + \hat{k})$

$\vec{c}(\hat{i} + \hat{j} + 3\hat{k}) = 8$

$\Rightarrow 3\lambda - 2\lambda + 3\lambda = 8$

$\Rightarrow 4\lambda = 8 \Rightarrow \lambda = 2$

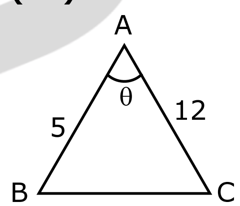
$\vec{c} = 6\hat{i} - 4\hat{j} + 2\hat{k}$

$$\vec{c} \cdot (\vec{a} \times \vec{b}) = [\vec{c}\,\vec{a}\,\vec{b}] = \begin{vmatrix} 6 & -4 & 2 \\ 1 & 1 & -1 \\ 1 & 2 & 1 \end{vmatrix}$$

$= 18 + 8 + 2 = 28$

2. In ΔABC, the lengths of sides AC and AB are 12 cm and 5 cm, respectively. If the area of ΔABC is 30 cm^2 and R and r are respectively the radii of circumcircle and incircle of ΔABC, then the value of 2R + r (in cm) is equal to _______

Ans. **(15)**

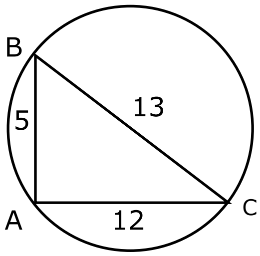


Sol.

Area $= \frac{1}{2}(5)(12)\sin\theta = 30$

$\sin\theta = 1 \Rightarrow \theta = \frac{\pi}{2}$

ABC is right angled at A.



$r = (s-a)\tan\frac{A}{2}$

$r = (s-a)$

$2R + r = s \quad (\because a = 2R)$

$2R + r = \frac{30}{2} = 15$

3. Consider the statistics of two sets of observations as follows:

	Size	Mean	Variance
Observation I	10	2	2
Observation II	n	3	1

If the variance of the combined set of these two observations is $\frac{17}{9}$, then the value of n is equal to _______

Ans. **(5)**

Sol. For group-1 : $\frac{\sum x_i}{10} = 2 \Rightarrow \sum x_i = 20$

$\frac{\sum x_i^2}{10} - (2)^2 = 2 \Rightarrow \sum x_i^2 = 60$

For group-2 : $\frac{\sum y_i}{n} = 3 \Rightarrow \sum y_i = 3n$

$\frac{\sum y_i^2}{n} - 3^2 = 1 \Rightarrow \sum y_i^2 = 10n$

Now, combined variance

$$\sigma^2 = \frac{\sum\left(x_i^2 + y_i^2\right)}{10+n} - \left(\frac{\sum(x_i + y_i)}{10+n}\right)^2$$

$$\Rightarrow \frac{17}{9} = \frac{60+10n}{10+n} - \frac{(20+3n)^2}{(10+n)^2}$$

$$\Rightarrow 17\left(n^2 + 20n + 100\right) = 9\left(n^2 + 40n + 200\right)$$

$$\Rightarrow 8n^2 - 20n - 100 = 0$$

$$\Rightarrow 2n^2 - 5n - 25 = 0 \Rightarrow n = 5$$

4. Let

$S_n(x) = \log_{a^{\frac{1}{2}}} x + \log_{a^{\frac{1}{3}}} x + \log_{a^{\frac{1}{6}}} x + \log_{a^{\frac{1}{11}}} x + \log_{a^{\frac{1}{18}}} x + \log_{a^{\frac{1}{27}}} x + \ldots..$ up to n-terms, where a >1. If $S_{24}(x)$ = 1093 and $S_{12}(2x)$ = 265, then value of a is equal to______

Ans. **(16)**

Sol. $S_n(x) = 2\log_a x + 3\log_a x + 6\log_a x + 11\log_a x + ...$

$S_n(x) = \log_a x(2 + 3 + 6 + 11 + ...)$

$S_r = 2 + 3 + 6 + 11$

General term, $T_r = r^2 - 2r + 3$

$S_n(x) = \sum_{r=1}^{n} \log_a x\,(r^2 - 2r + 3)$

$\Rightarrow S_{24}(x) = \log_a x \sum_{r=1}^{24}(r^2 - 2r + 3)$

$\Rightarrow 1093 = 4372 \log_a x$

$\Rightarrow \log_a x = \frac{1}{4}$

$\Rightarrow x = a^{1/4}$

$S_{12}(2x) = \log_a(2x)\sum_{r=1}^{12}(r^2 - 2r + 3)$

$\Rightarrow 265 = 530 \log_a(2x)$

$\Rightarrow \log_a(2x) = \frac{1}{2}$

$\Rightarrow 2x = a^{1/2}$

After solving (i) and (ii), we get

$a^{1/4} = 2$

$\Rightarrow a = 16$

5. Let n be a positive integer. Let $A = \sum_{k=0}^{n}(-1)^k\, {}^nC_k\left[\left(\frac{1}{2}\right)^k + \left(\frac{3}{4}\right)^k + \left(\frac{7}{8}\right)^k + \left(\frac{15}{16}\right)^k + \left(\frac{31}{32}\right)^k\right]$

If $63A = 1 - \frac{1}{2^{30}}$, then n is equal to _______

Ans. **(6)**

Sol. $A = \left(\frac{1}{2}\right)^n + \left(\frac{1}{4}\right)^n + \left(\frac{1}{8}\right)^n + \left(\frac{1}{16}\right)^n + \left(\frac{1}{32}\right)^n$

$= \frac{1}{2^n} + \frac{1}{2^{2n}} + \frac{1}{2^{3n}} + \frac{1}{2^{4n}} + \frac{1}{2^{5n}}$

$$= \frac{1}{2^n}\left[\frac{1 - \left(\frac{1}{2^n}\right)^5}{1 - \frac{1}{2^n}}\right]$$

$$A = \frac{2^{5n} - 1}{2^{5n}(2^n - 1)}$$

$$63A = \frac{63(2^{5n}-1)}{2^{5n}(2^n-1)}$$

$$= \frac{63}{2^n-1}\left(1-\frac{1}{2^{5n}}\right) = 1-\frac{1}{2^{30}}$$

$\Rightarrow n = 6$

6. Let $f : \mathbf{R} \to \mathbf{R}$ and $g : \mathbf{R} \to \mathbf{R}$ be defined as

$f(x) = \begin{cases} x+a, x<0 \\ |x-1|, x \geq 0 \end{cases}$ and $g(x) = \begin{cases} x+1, & x<0 \\ (x-1)^2+b, & x \geq 0 \end{cases}$

where a, b are non-negative real numbers. If (gof) (x) is continuous for all $x \in \mathbf{R}$, then a + b is equal to ______

Ans. **(1)**

Sol. $g[f(x)] = \begin{cases} f(x)+1 & f(x)<0 \\ (f(x)-1)^2+b & f(x) \geq 0 \end{cases}$

$$g[f(x)] = \begin{cases} x+a+1 & x+a<0 \ \& \ x<0 \\ |x-1|+1 & |x-1|<0 \ \& \ x \geq 0 \\ (x+a-1)^2+b & x+a \geq 0 \ \& \ x<0 \\ (|x-1|-1)^2+b & |x-1| \geq 0 \ \& \ x \geq 0 \end{cases}$$

$$g[f(x)] = \begin{cases} x+a+1 & x \in (-\infty,-a) \ \& \ x \in (-\infty,0) \\ |x-1|+1 & x \in \phi \\ (x+a-1)^2+b & x \in [-a,\infty) \ \& \ x \in (-\infty,0) \\ (|x-1|-1)^2+b & x \in R \ \& \ x \in [0,\infty) \end{cases}$$

$$g[f(x)] = \begin{cases} x+a+1 & x \in (-\infty,-a) \\ (x+a-1)^2+b & x \in [-a,0) \\ (|x-1|-1)^2+b & x \in [0,\infty) \end{cases}$$

Given g(f(x)) is continuous.

So, at x = –a & at x = 0

1 = b + 1 & $(a-1)^2 + b = b$

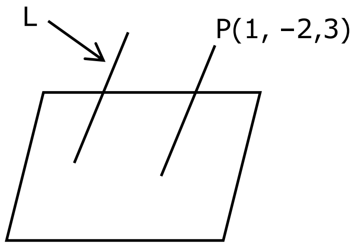
$\Rightarrow$ b = 0 & a = 1

$\Rightarrow$ a + b = 1

7. If the distance of the point (1, −2, 3) from the plane $x + 2y - 3z + 10 = 0$ measured parallel to the line, $\frac{x-1}{3} = \frac{2-y}{m} = \frac{z+3}{1}$ is $\sqrt{\frac{7}{2}}$, then the value of $|m|$ is equal to______

Ans. **(2)**

Sol.



$$\frac{x-1}{3} = \frac{y+2}{-m} = \frac{z-3}{1} = \lambda$$

Point $Q(3\lambda+1, -m\lambda-2, \lambda+3)$ lies on plane.

$$(3\lambda+1) + 2(-m\lambda-2) - 3(\lambda+3) + 10 = 0$$

$$\Rightarrow 3\lambda - 2m\lambda - 3\lambda + 1 - 4 - 9 + 10 = 0$$

$$\Rightarrow -2m\lambda = 2$$

$$m\lambda = -1 \Rightarrow \lambda = -\frac{1}{m}$$

$$Q\left[-\frac{3}{m} + 1, -1, -\frac{1}{m} + 3\right]$$

$$PQ = \sqrt{\frac{7}{2}}$$

$$\sqrt{\left(-\frac{3}{m}\right)^2 + 1 + \left(-\frac{1}{m}\right)^2} = \sqrt{\frac{7}{2}}$$

$$\Rightarrow \frac{10+m^2}{m^2} = \frac{7}{2}$$

$$\Rightarrow 20 + 2m^2 = 7m^2$$

$$m^2 = 4 \quad \Rightarrow |m| = 2$$

8. Let $\frac{1}{16}$, a and b be in G.P. and $\frac{1}{a}, \frac{1}{b}$, 6 be in A.P., where a, b, > 0. Then 72(a+b) is equal to _______

Ans. **(14)**

Sol. $a^2 = \frac{b}{16}$ and $\frac{2}{b} = \frac{1}{a} + 6$

Solving, we get $a = \frac{1}{12}$ or $a = -\frac{1}{4}$ [rejected]

If $a = \frac{1}{12} \Rightarrow b = \frac{1}{9}$

$$\therefore 72(a+b) = 72\left(\frac{1}{12} + \frac{1}{9}\right) = 14$$

9. Let $A = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ and $B = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ be two 2×1 matrices with real entries such that A = XB, where $X = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & -1 \\ 1 & k \end{bmatrix}$, and k ∈ **R**. If $a_1^2 + a_2^2 = \frac{2}{3}\left(b_1^2 + b_2^2\right)$ and $\left(k^2+1\right)b_2^2 \neq -2b_1b_2$, then the value of k is _______

Ans. **(1)**

Sol.

$XB = A$

$\Rightarrow \frac{1}{\sqrt{3}}\begin{bmatrix} b_1 - b_2 \\ b_1 + kb_2 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$

$b_1 - b_2 = \sqrt{3}a_1 \Rightarrow 3a_1^2 = b_1^2 + b_2^2 - 2b_1b_2$

$b_1 + kb_2 = \sqrt{3}a_2 \Rightarrow 3a_2^2 = b_1^2 + k^2b_2^2 + 2kb_1b_2$

Adding both equations, we get

$3(a_1^2 + a^2) = 2b_1^2 + (k^2+1)b_2^2 + 2b_1b_2(k-1)$

$\Rightarrow 2b_1^2 + 2b_2^2 = 2b_1^2 + (k^2+1)b_2^2 + 2b_1b_2(k-1)$

$\Rightarrow (k^2-1)b_2^2 + 2b_1b_2(k-1) = 0$

$\Rightarrow (k-1)\left((k+1)b_2^2 + 2b_1b_2\right) = 0$

$\Rightarrow k = 1$

10. For real numbers α, β, γ and δ, if

$$\int \frac{\left(x^2-1\right) + \tan^{-1}\left(\frac{x^2+1}{x}\right)}{\left(x^4+3x^2+1\right)\tan^{-1}\left(\frac{x^2+1}{x}\right)}dx$$

$$= \alpha \log_e\left(\tan^{-1}\left(\frac{x^2+1}{x}\right)\right) + \beta\tan^{-1}\left(\frac{\gamma\left(x^2-1\right)}{x}\right) + \delta\tan^{-1}\left(\frac{x^2+1}{x}\right) + C$$

where C is an arbitrary constant, then the value of 10(α+βγ+δ) is equal to______

Ans. **(6)**

Sol. $\int \frac{x^2-1}{\left(x^4+3x^2+1\right)\tan^{-1}\left(\frac{x^2+1}{x}\right)}dx + \int \frac{1}{x^4+3x^2+1}dx$

$$\underbrace{\int \frac{1-\frac{1}{x^2}}{\left[\left(x+\frac{1}{x}\right)^2+1\right]\tan^{-1}\left(x+\frac{1}{x}\right)}dx}_{I_1} + \underbrace{\int \frac{dx}{x^4+3x^2+1}}_{I_2}$$

$\tan^{-1}\left(x+\frac{1}{x}\right) = t$

$$I_1 = \int \frac{dt}{t}$$

$$I_1 = \ln(t) = \ln\left|\tan^{-1}\left(x + \frac{1}{x}\right)\right|$$

Now,

$$I_2 = \int \frac{dx}{x^4 + 3x^2 + 1}$$

$$= \frac{1}{2}\int \frac{(x^2+1)-(x^2-1)}{x^4+3x^2+1}dx$$

$$= \frac{1}{2}\left[\int \frac{1+\frac{1}{x^2}}{x^2+3+\frac{1}{x^2}}dx - \int \frac{\left(1-\frac{1}{x^2}\right)}{x^2+3+\frac{1}{x^2}}dx\right]$$

$$= \frac{1}{2}\left[\int \frac{1+\frac{1}{x^2}}{\left(x-\frac{1}{x}\right)^2+5} - \int \frac{\left(1-\frac{1}{x^2}\right)}{\left(x+\frac{1}{x}\right)^2+1}dx\right]$$

$\downarrow$ $x - \frac{1}{x} = u$ $\qquad$ $\downarrow$ $x + \frac{1}{x} = v$

$$= \frac{1}{2}\left[\int \frac{du}{u^2+(\sqrt{5})^2} - \int \frac{dv}{v^2+1}\right]$$

$$I_2 = \frac{1}{2\sqrt{5}}\tan^{-1}\left(\frac{x-\frac{1}{x}}{\sqrt{5}}\right) - \frac{1}{2}\tan^{-1}\left(x+\frac{1}{x}\right)$$

$$I = I_1 + I_2 = \ln\left|\tan^{-1}\left(x+\frac{1}{x}\right)\right| + \frac{1}{2\sqrt{5}}\ln\left(\frac{x^2-1}{\sqrt{5}x}\right) - \frac{1}{2}\tan^{-1}\left(\frac{x^2+1}{x}\right) + C$$

$$\therefore \alpha = 1,\ \beta = \frac{1}{2\sqrt{5}},\ \gamma = \frac{1}{\sqrt{5}},\ \delta = -\frac{1}{2}$$

$$10(\alpha+\beta\lambda+\delta) = 10\left[1+\frac{1}{10}-\frac{1}{2}\right]$$

$$= 10\left(\frac{1}{10}+\frac{1}{2}\right)$$

$$= 1 + 5 = 6$$