JEE MAIN 2021

17th March Shift II
SECTION – A

1. If the Boolean expression $(p \wedge q) \circledast (p \otimes q)$ is a tautology, then $\circledast$ and $\otimes$ are respectively given by :

(1) $\wedge , \rightarrow$
(2) $\rightarrow , \rightarrow$
(3) $\vee , \rightarrow$
(4) $\wedge , \vee$

Ans. (2)

Sol. $(p \wedge q) \rightarrow (p \rightarrow q)$

$(p \wedge q) \rightarrow (\sim p \vee q)$

$(\sim p \vee \sim q) \vee (\sim p \vee q)$

$\sim p \vee (\sim q \vee q) \Rightarrow$ Tautology

$\Rightarrow \circledast \Rightarrow \rightarrow$

$\otimes \Rightarrow \rightarrow$

2. Let the tangent to the circle $x^2 + y^2 = 25$ at the point R(3,4) meet x-axis and y-axis at points P and Q, respectively. If r is the radius of the circle passing through the origin O and having centre at the incentre of the triangle OPQ, then r^2 is equal to :

(1) $\frac{625}{72}$
(2) $\frac{585}{66}$
(3) $\frac{125}{72}$
(4) $\frac{529}{64}$

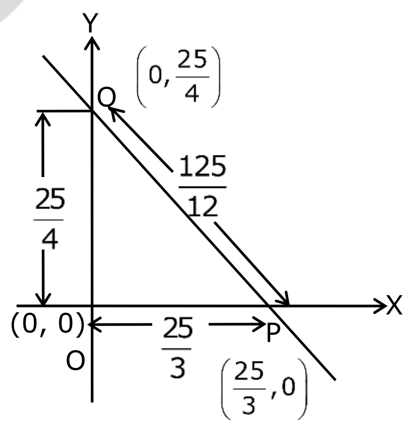
Ans. (1)

Sol. Given equation of circle

$x^2 + y^2 = 25$

$\therefore$ Tangent equation at (3, 4)

T : 3x + 4y = 25



Incentre of ΔOPQ.

$$I = \left(\frac{\frac{25}{4} \times \frac{25}{3}}{\frac{25}{3} + \frac{25}{4} + \frac{125}{12}}, \frac{\frac{25}{3} \times \frac{25}{4}}{\frac{25}{3} + \frac{25}{4} + \frac{125}{12}} \right).$$

$$\therefore I = \left(\frac{625}{75 + 100 + 125}, \frac{625}{75 + 100 + 125} \right) = \left(\frac{25}{12}, \frac{25}{12} \right)$$

$\because$ Distance from origin to incentre is r.

$$\therefore r^2 = \left(\frac{25}{12}\right)^2 + \left(\frac{25}{12}\right)^2 = \frac{625}{72}$$

Therefore, the correct answer is (1)

3. Let a computer program generate only the digits 0 and 1 to form a string of binary numbers with probability of occurrence of 0 at even places be $\frac{1}{2}$ and probability of occurrence of 0 at the odd place be $\frac{1}{3}$. Then the probability that '10' is followed by '01' is equal to :

(1) $\frac{1}{6}$

(2) $\frac{1}{18}$

(3) $\frac{1}{9}$

(4) $\frac{1}{3}$

Ans. (3)

Sol. P(0 at even place) = $\frac{1}{2}$, P(0 at odd place) = $\frac{1}{3}$

P(1 at even place) = $\frac{1}{2}$, P(1 at odd place) = $\frac{2}{3}$

P(10 is followed by 01)

$$= \left(\frac{2}{3} \times \frac{1}{2} \times \frac{1}{3} \times \frac{1}{2}\right) + \left(\frac{1}{2} \times \frac{1}{3} \times \frac{1}{2} \times \frac{2}{3}\right)$$

$$= \frac{1}{18} + \frac{1}{18}$$

$$= \frac{1}{9}$$

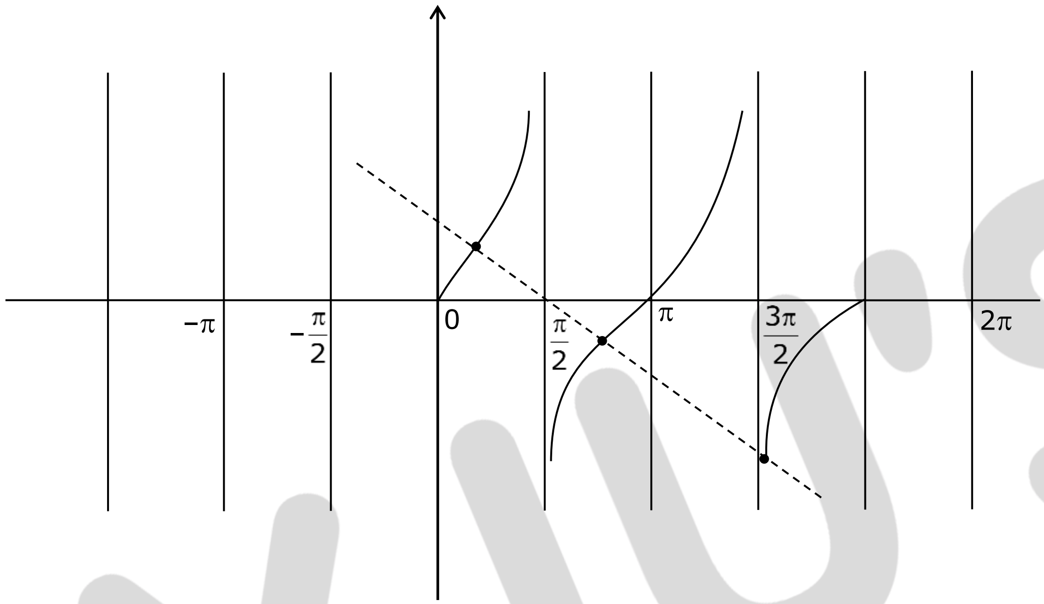
4. The number of solutions of the equation x + 2tanx = $\frac{\pi}{2}$ in the interval [0, 2π] is :

(1) 5 (2) 2

(3) 4 (4) 3

Ans. **(4)**

Sol.



$x + 2\tan x = \frac{\pi}{2}$ in $[0, 2\pi]$

$2\tan x = \frac{\pi}{2} - x$

$2\tan x = \frac{\pi}{2} - x$

$\tan x = \frac{\pi}{4} - \frac{x}{2}$

$y = \tan x$ and $y = \frac{-x}{2} + \frac{\pi}{4}$

3 intersection points

$\therefore$ 3 solutions

option (4)

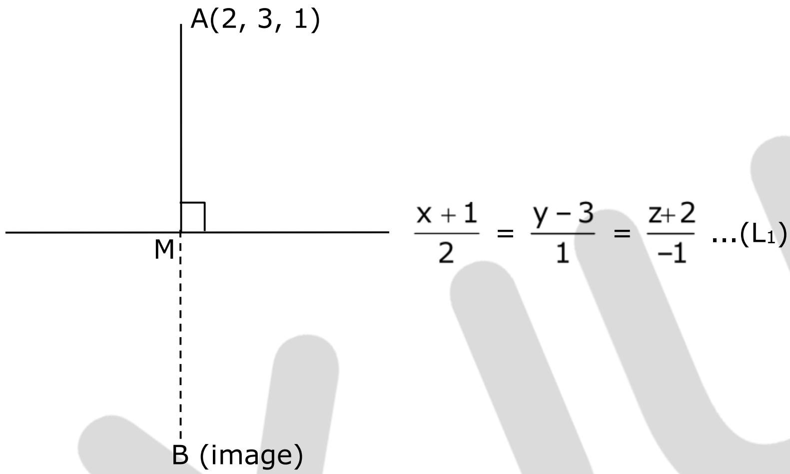
5. If the equation of plane passing through the mirror image of a point (2, 3, 1) with respect to line $\frac{x+1}{2}=\frac{y-3}{1}=\frac{z+2}{-1}$ and containing the line $\frac{x-2}{3}=\frac{1-y}{2}=\frac{z+1}{1}$ is $\alpha x+\beta y+\gamma z=24$, then $\alpha+\beta+\gamma$ is equal to :

(1) 21 (2) 19

(3) 18 (4) 20

Ans. (2)

Sol.



Let point M is $(2\lambda - 1, \lambda + 3, - \lambda - 2)$

D.R.'s of AM line are $2\lambda - 1 - 2,\quad \lambda + 3 - 3,\quad -\lambda - 2 - 1$

$2\lambda - 3,\quad \lambda,\quad -\lambda - 3$

AM $\perp$ line L_1

$\therefore 2(2\lambda - 3) + 1(\lambda) - 1(-\lambda - 3) = 0$

$6\lambda = 3,\ \lambda = \frac{1}{2}\ \therefore M \equiv \left(0, \frac{7}{2}, \frac{-5}{2}\right)$

M is mid-point of A & B

$M = \frac{A+B}{2}$

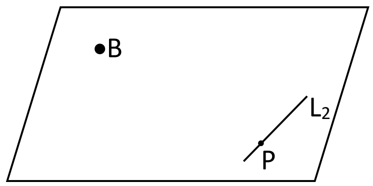
$B = 2M - A$

$B \equiv (-2, 4, -6)$

Now we have to find equation of plane passing through B (–2, 4, –6) & also containing the line

$$\frac{x-2}{3} = \frac{1-y}{2} = \frac{z+1}{1} \qquad \ldots(1)$$

$$\frac{x-2}{3} = \frac{y-1}{-2} = \frac{z+1}{1}$$



Point P on line is (2, 1, –1)

$\vec{b}_2$ of line L_2 is 3, –2, 1

$\vec{n} \parallel (\vec{b}_2 \times \vec{PB})$

$\vec{b}_2 = 3\hat{i} - 2\hat{j} + \hat{k}$

$\vec{PB} = -4\hat{i} + 3\hat{j} - 5\hat{k}$

$\vec{n} = 7\hat{i} + 11\hat{j} + \hat{k}$

$\therefore$ equation of plane is $\vec{r} \cdot \vec{n} = \vec{a} \cdot \vec{n}$

$\vec{r} \cdot (7\hat{i} + 11\hat{j} + \hat{k}) = (-2\hat{i} + 4\hat{j} - 6\hat{k}) \cdot (7\hat{i} + 11\hat{j} + \hat{k})$

$7x + 11y + z = -14 + 44 - 6$

$7x + 11y + z = 24$

$\therefore \alpha = 7$

$\beta = 11$

$\gamma = 1$

$\therefore \alpha + \beta + \gamma = 19$

option (2)

6. Consider the function f : R $\to$ R defined by $f(x)$ = $\begin{cases} \left(2-\sin\left(\frac{1}{x}\right)\right)|x|, & x \neq 0 \\ 0, & x = 0 \end{cases}$. Then f is

(1) monotonic on (0, ∞) only

(2) Not monotonic on (−∞, 0) and (0, ∞)

(3) monotonic on (−∞, 0) only

(4) monotonic on (−∞, 0) $\cup$ (0, ∞)

Ans. **(2)**

Sol.

$$f(x) = \begin{cases} -\left(2-\sin\frac{1}{x}\right)x, & x < 0 \\ 0, & x = 0 \\ \left(2-\sin\frac{1}{x}\right)x, & x > 0 \end{cases}$$

$$f'(x) = \begin{cases} -x\left(-\cos\frac{1}{x}\right)\left(-\frac{1}{x^2}\right)-\left(2-\sin\frac{1}{x}\right), x < 0 \\ x\left(-\cos\frac{1}{x}\right)\left(-\frac{1}{x^2}\right)+\left(2-\sin\frac{1}{x}\right), x > 0 \end{cases}$$

$$\begin{cases} -\frac{1}{x}\cos\frac{1}{x}+\sin\frac{1}{x}-2, & x < 0 \\ \frac{1}{x}\cos\frac{1}{x}-\sin\frac{1}{x}+2, & x > 0 \end{cases}$$

Then f is not monotonic on (−∞, 0) and (0, ∞).

7. Let O be the origin. Let $\overrightarrow{OP} = x\hat{i} + y\hat{j} - \hat{k}$ and $\overrightarrow{OQ} = -\hat{i} + 2\hat{j} + 3x\hat{k}$, x, y $\in$ R, x > 0, be such that $|\overrightarrow{PQ}| = \sqrt{20}$ and the vector $\overrightarrow{OP}$ is perpendicular to $\overrightarrow{OQ}$. If $\overrightarrow{OR} = 3\hat{i} + z\hat{j} - 7\hat{k}$, z $\in$ R, is coplanar with $\overrightarrow{OP}$ and $\overrightarrow{OQ}$, then the value of $x^2+y^2+z^2$ is equal to :

(1) 2 (2) 9

(3) 1 (4) 7

Ans. **(2)**

Sol. $\overrightarrow{OP} = x\hat{i} + y\hat{j} - \hat{k}$ $\overrightarrow{OP} \perp \overrightarrow{OQ}$

$\overrightarrow{OQ} = -\hat{i} + 2\hat{j} + 3x\hat{k}$

$$\overrightarrow{PQ} = (-1-x)\hat{i} + (2-y)\hat{j} + (3x+1)\hat{k}$$

$$|\overrightarrow{PQ}| = \sqrt{(-1-x)^2 + (2-y)^2 + (3x+1)^2}$$

$$\therefore \overrightarrow{OP}.\ \overrightarrow{OQ} = 0$$

$-x + 2y - 3x = 0$

$4x = 2y$

$$\sqrt{20}=\sqrt{(-1-x)^2+(2-y)^2+(3x+1)^2}$$

$$20=1+x^2+2x+4+y^2-4y+9x^2+1+6x$$

$$20=10x^2+y^2+8x+6-4y$$

$$20=10x^2+4x^2+8x+6-8x$$

$14 = 14x^2 \Rightarrow \boxed{x^2=1}$

$\therefore\ y^2=4x^2 \Rightarrow \boxed{y^2=4}$

$x = 1$ as $x > 0$ and $y = 2$

$$\therefore \begin{vmatrix} x & y & -1 \\ -1 & 2 & 3x \\ 3 & z & -7 \end{vmatrix}=0$$

$$\begin{vmatrix} 1 & 2 & -1 \\ -1 & 2 & 3 \\ 3 & z & -7 \end{vmatrix}$$

$1(-14 - 3z) - 2(7 - 9) -1(-z - 6)$

$-14 - 3z + 4 + z + 6 = 0$

$2z = -4$ $\boxed{z=-2}$

$x^2 + y^2 + z^2 = 9$

8. Let L be a tangent line to the parabola $y^2 = 4x - 20$ at (6, 2). If L is also a tangent to the ellipse $\frac{x^2}{2}+\frac{y^2}{b}=1$, then the value of b is equal to :

(1) 20 (2) 14

(3) 16 (4) 11

Ans. **(2)**

Sol. Parabola $y^2 = 4x - 20$

Tangent at P(6, 2) will be

$2y = 4\left(\frac{x+6}{2}\right) - 20$

$2y = 2x + 12 - 20$

$2y = 2x - 8$

$y = x - 4$

$x - y - 4 = 0$ ....(1)

This is also tangent to ellipse $\frac{x^2}{2}+\frac{y^2}{b}=1$

Apply $c^2 = a^2m^2 + b^2$

$(-4)^2 = (2)(1) + b$

$b = 14$

Option (2)

9. Let f : R $\rightarrow$ R be defined as $f(x) = e^{-x}\sin x$. If F : [0,1] R$\rightarrow$ is a differentiable function such that $F(x) = \int_0^x f(t)dt$, then the value of $\int_0^1 \left(F'(x)+f\left(x\right)\right)e^x dx$ lies in the interval

(1) $\left[\frac{330}{360},\frac{331}{360}\right]$ (2) $\left[\frac{327}{360},\frac{329}{360}\right]$

(3) $\left[\frac{331}{360},\frac{334}{360}\right]$ (4) $\left[\frac{335}{360},\frac{336}{360}\right]$

Ans. (1)

Sol. F'(x) = f(x) by Leibnitz theorem

$$\int_0^1 (F'(x)+f(x))e^x\,dx = \int_0^1 2f(x)e^x\,dx$$

$$I = \int_0^1 2\sin x\,dx$$

$I = 2(1 - \cos 1)$

$$=\left\{1-\left(1-\frac{1^2}{2!}+\frac{1^4}{4!}-\frac{1^6}{6!}+\ldots\right)\right\}$$

$$=2\left\{1-\left(1-\frac{1}{2}+\frac{1}{24}\right)\right\} < 2(1-\cos 1) < 2\left\{1-\left(1-\frac{1}{2}+\frac{1}{24}-\frac{1}{720}\right)\right\}$$

$$\frac{330}{360} < 2(1-\cos 1) < \frac{331}{360}$$

$$\frac{330}{360} < I < \frac{331}{360}$$

(1) is correct

10. If x, y, z are in arithmetic progression with common difference d, $x \neq 3d$, and the determinant of the matrix $\begin{bmatrix} 3 & 4\sqrt{2} & x \\ 4 & 5\sqrt{2} & y \\ 5 & k & z \end{bmatrix}$ is zero, then the value of k^2 is :

(1) 6
(2) 36
(3) 72
(4) 12

Ans. **(3)**

Sol. $\begin{vmatrix} 3 & 4\sqrt{2} & x \\ 4 & 5\sqrt{2} & y \\ 5 & k & z \end{vmatrix} = 0$

$R_1 \to R_1 + R_3 - 2R_2$

$\begin{vmatrix} 0 & 4\sqrt{2} - k - 10\sqrt{2} & 0 \\ 4 & 5\sqrt{2} & y \\ 5 & k & z \end{vmatrix} = 0 \qquad \{\because\ 2y = x + z\}$

$\Rightarrow (k - 6\sqrt{2})(4z - 5y) = 0$

$k = 6\sqrt{2}$ or $4z = 5y$ (Not possible $\because$ x, y, z in A.P.)

So, $k^2 = 72$

$\therefore$ Option (3)

11. If the integral $\int_0^{10} \frac{[\sin 2\pi x]}{e^{x-[x]}} dx = \alpha e^{-1} + \beta e^{-\frac{1}{2}} + \gamma$, where α, β, γ are integers and [x] denotes the greatest integer less than or equal to x, then the value of $\alpha + \beta + \gamma$ is equal to :

(1) 20
(2) 0
(3) 25
(4) 10

Ans. **(2)**

Sol. Given integral

$\int_0^{10} \frac{[\sin 2\pi x]}{e^{x-[x]}} dx = 10\int_0^{1} \frac{[\sin 2\pi x]}{e^{\{x\}}} dx$ (using property of definite in.)

$= 10\left[\int_0^{1/2} 0.dx + \int_{1/2}^{1} \frac{-1}{e^x} dx\right]$

$= -10\left[\frac{e^{-x}}{-1}\right]_{1/2}^{1} = 10\left[e^{-1} - e^{-1/2}\right]$

$= 10e^{-1} - 10e^{-1/2}$

comparing with the given relation,

$\alpha = 10$, $\beta = -10$, $\gamma = 0$

$\alpha + \beta + \gamma = 0.$

therefore, the correct answer is (2).

12. Let y = y(x) be the solution of the differential equation

cosx(3sinx + cosx + 3) dy = (1+y sinx(3sinx + cosx + 3))dx, $0 \le x \le \frac{\pi}{2}$, y(0) = 0.

Then, $y\left(\frac{\pi}{3}\right)$ is equal to :

(1) $2\log_e\left(\frac{2\sqrt{3}+10}{11}\right)$

(2) $2\log_e\left(\frac{\sqrt{3}+7}{2}\right)$

(3) $2\log_e\left(\frac{3\sqrt{3}-8}{4}\right)$

(4) $2\log_e\left(\frac{2\sqrt{3}+9}{6}\right)$

Ans. **(1)**

Sol. cosx (3 sin x + cos x + 3) dy = (1 + y sin x (3 sin x + cos x + 3)) dx ...(1)

(3 sin x + cos x + 3) (cos x dy − y sin x dx) = dx

$$\int d(y.\cos x) = \int \frac{dx}{3\sin x + \cos x + 3}$$

$$y \cos x = \int \frac{1}{3\left(\frac{2+\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right) + \left(\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right) + 3}$$

$$y \cos x = \int \frac{\sec^2\frac{x}{2}}{6\tan\frac{x}{2} + 1 - \tan^2\frac{x}{2} + 3 + 3\tan^2\frac{x}{2}}$$

$$y \cos x = \int \frac{\sec^2\frac{x}{2}}{2\tan^2\frac{x}{2} + 6\tan\frac{x}{2} + 4} = \int \frac{\frac{1}{2}\sec^2\frac{x}{2}dx}{\tan^2\frac{x}{2} + 3\tan\frac{x}{2} + 2}$$

$$y \cos x = \ln \left|\frac{\tan\frac{x}{2}+1}{\tan\frac{x}{2}+2}\right| + c$$

Put x =0 & y = 0

$$C = -\ln\left(\frac{1}{2}\right) = \ln(2)$$

$$y\left(\frac{\pi}{3}\right) = 2\ln\left|\frac{1+\sqrt{3}}{1+2\sqrt{3}}\right| + \ln 2$$

$$= 2\ln\left|\frac{5+\sqrt{3}}{11}\right| + \ln 2$$

$$= 2\ln\left|\frac{2\sqrt{3}+10}{11}\right|$$

13. The value of the limit $\lim\limits_{\theta\to 0}\frac{\tan(\pi\cos^2\theta)}{\sin(2\pi\sin^2\theta)}$ is equal to :

(1) $-\frac{1}{2}$

(2) $-\frac{1}{4}$

(3) 0

(4) $\frac{1}{4}$

Ans. **(1)**

Sol. Given,

$$\lim_{\theta\to 0}\frac{\tan(\pi\cos^2\theta)}{\sin(2\pi\sin^2\theta)}$$

$$= \lim_{\theta\to 0}\frac{\tan(\pi-\pi\sin^2\theta)}{\sin(2\pi\sin^2\theta)} \qquad (\because \cos^2\theta = 1-\sin^2\theta)$$

$$= \lim_{\theta\to 0}\frac{-\tan(\pi\sin^2\theta)}{\sin(2\pi\sin^2\theta)} \qquad (\because \tan(\pi-\theta) = -\tan\theta)$$

$$= \lim_{\theta\to 0} -\frac{1}{2}\left(\frac{\tan(\pi\sin^2\theta)}{\pi\sin^2\theta}\right)\left(\frac{2\pi\sin^2\theta}{\sin(2\pi\sin^2\theta)}\right) = -\frac{1}{2}$$

Therefore, the correct answer is (1).

14. If the curve y = y(x) is the solution of the different equation

$2\left(x^2+x^{5/4}\right)dy - y\left(x+x^{1/4}\right)dx = 2x^{9/4}dx$, $x > 0$ which passes through the point $\left(1, 1-\frac{4}{3}\log_e 2\right)$, then the value of $y(16)$ is equal to :

(1) $\left(\frac{31}{3}-\frac{8}{3}\log_e 3\right)$

(2) $4\left(\frac{31}{3}+\frac{8}{3}\log_e 3\right)$

(3) $\left(\frac{31}{3}+\frac{8}{3}\log_e 3\right)$

(4) $4\left(\frac{31}{3}-\frac{8}{3}\log_e 3\right)$

Ans. **(4)**

Sol. $\frac{dy}{dx}-\frac{y}{2x}=\frac{x^{9/4}}{x^{5/4}(x^{3/4}+1)}$

$$\text{If} = e^{-\int\frac{dx}{2x}} = e^{-\frac{1}{2}\ln x} = \frac{1}{x^{1/2}}$$

$$y.x^{-1/2} = \int\frac{x^{9/4}.x^{-1/2}}{x^{5/4}(x^{3/4}+1)}dx$$

$$\int\frac{x^{1/2}}{(x^{3/4}+1)}dx$$

$$x = t^4 \Rightarrow dx = 4t^3dt$$

$$\int\frac{t^2.4t^3dt}{(t^3+1)}$$

$$4\int\frac{t^2(t^3+1-1)}{(t^3+1)}dt$$

$$4\int t^2dt - 4\int\frac{t^2}{t^3+1}dt$$

$$\frac{4t^3}{3}-\frac{4}{3}\ln(t^3+1)+C$$

$$yx^{-1/2} = \frac{4x^{3/4}}{3}-\frac{4}{3}\ln(x^{3/4}+1)+C$$

$$1-\frac{4}{3}\log_e 2 = \frac{4}{3}-\frac{4}{3}\log_e 2 + C$$

$$\Rightarrow C = -\frac{1}{3}$$

$$y = \frac{4}{3}x^{5/4}-\frac{4}{3}\sqrt{x}\ln(x^{3/4}+1)-\frac{\sqrt{x}}{3}$$

$$y(16) = \frac{4}{3} \times 32 - \frac{4}{3} \times 4\ln 9 - \frac{4}{3}$$

$$= \frac{124}{3} - \frac{32}{3}\ln 3 = 4\left(\frac{31}{3} - \frac{8}{3}\ln 3\right)$$

15. Let S_1, S_2 and S_3 be three sets defined as

$S_1 = \{z \in C : |z-1| \le \sqrt{2}\}$

$S_2 = \{z \in C : \text{Re}((1-i)z) \ge 1\}$

$S_3 = \{z \in C : \text{Im}(z) \le 1\}$

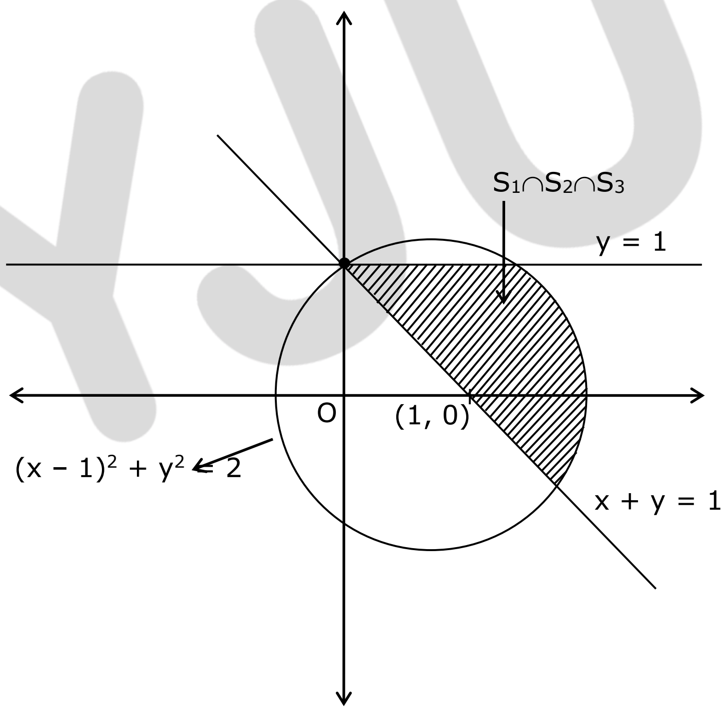
Then the set $S_1 \cap S_2 \cap S_3$

(1) has infinitely many elements (2) has exactly two elements

(3) has exactly three elements (4) is a singleton

Ans. **(1)**

Sol.



Let, z = x + iy

$S_1 \equiv (x-1)^2 + y^2 \le 2$ ...(1)

$S_2 \equiv x + y \ge 1$ ...(2)

$S_3 \equiv y \le 1$ ...(3)

$\Rightarrow S_1 \cap S_2 \cap S_3$ has infinitely many elements.

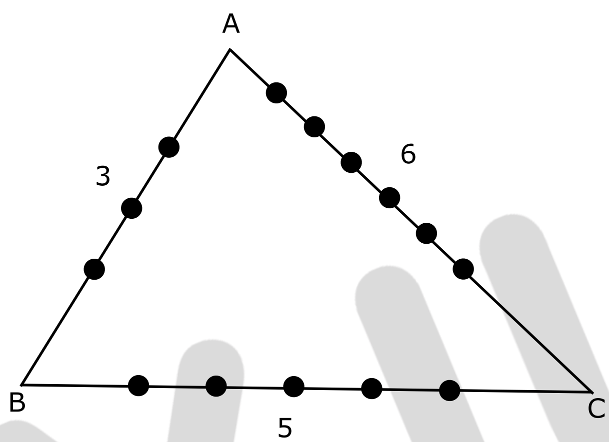
16. If the sides AB, BC, and CA of a triangle ABC have, 3, 5 and 6 interior points respectively, then the total number of triangles that can be constructed using these points as vertices, is equal to:

(1) 360 (2) 240

(3) 333 (4) 364

Ans. **(3)**

Sol.



Total number of triangles

$= {}^3C_1 \times {}^5C_1 \times {}^6C_1$

$+ {}^3C_1 \times {}^5C_2 + {}^5C_1 \times {}^3C_2$

$+ {}^3C_1 \times {}^6C_2 + {}^6C_1 \times {}^3C_2$

$+ {}^5C_1 \times {}^6C_2 + {}^6C_1 \times {}^5C_2$

$= 90 + 30 + 15 + 45 + 18 + 75 + 60$

$= 333$

17. The value of

$$\lim_{x\to\infty} \frac{[r]+[2r]+\ldots+[nr]}{n^2},$$

Where r is a non-zero real number and [r] denotes the greatest integer less than or equal to r, is equal to :

(1) 0 (2) r

(3) $\frac{r}{2}$ (4) 2r

Ans. (3)

Sol. We know that

$$r \le [r] < r + 1$$

and
$$2r \le [2r] < 2r + 1$$
$$3r \le [3r] < 3r + 1$$
$$\vdots \quad \vdots \quad \vdots$$
$$nr \le [nr] < nr + 1$$

$r + 2r + ... + nr$

$\le [r] + [2r] + ...+ [nr] < (r + 2r + ...+ nr) + n$

$$\frac{\frac{n(n+1)}{2}.r}{n^2} \le \frac{[r]+[2r]+.....+[nr]}{n^2} < \frac{\frac{n(n+1)}{2}r+n}{n^2}$$

Now,

$$\lim_{n\to\infty}\frac{n(n+1).r}{2.n^2} = \frac{r}{2}$$

and
$$\lim_{n\to\infty}\frac{\frac{n(n+1)r}{2}+n}{n^2} = \frac{r}{2}$$

So, by Sandwich Theorem, we can conclude that

$$\lim_{n\to\infty}\frac{[r]+[2r]+.....+[nr]}{n^2} = \frac{r}{2}$$

18. The value of $\sum_{r-0}^{6}\left({}^6C_r \cdot {}^6C_{6-r}\right)$ is equal to :

(1) 1124 (2) 924

(3) 1324 (4) 1024

Ans. (2)

Sol. $\sum_{r=0}^{6} {}^6C_r {}^6C_{6-r}$

$= {}^6C_0.{}^6C_6 + {}^6C_1.{}^6C_5 + ... + {}^6C_6.{}^6C_0$

Now,

$(1 + x)^6 (1 + x)^6 = ({}^6C_0 + {}^6C_1x + {}^6C_2x^2 + ... + {}^6C_6x^6)\ ({}^6C_0 + {}^6C_1x + {}^6C_2x^2 + ... + {}^6C_6x^6)$

Comparing coefficient of x^6 both sides

${}^6C_0.{}^6C_6 + {}^6C_1 + {}^6C_5 + \ldots.. + {}^6C_6.{}^6C_0 = {}^{12}C_6$

$= 924$

Therefore, the correct answer is (2).

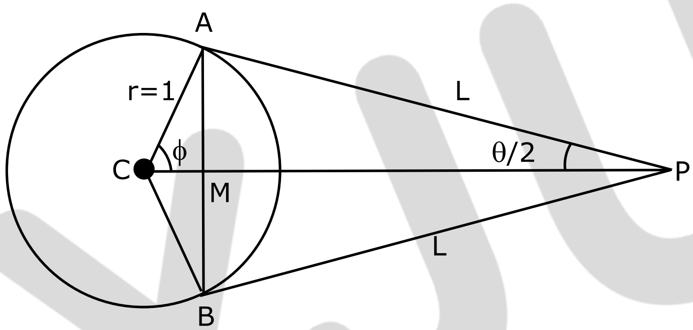
19. Two tangents are drawn from a point P to the circle $x^2 + y^2 - 2x - 4y + 4 = 0$, such that the angle between these tangents is $\tan^{-1}\left(\frac{12}{15}\right)$, where $\tan^{-1}\left(\frac{12}{5}\right) \in (0, \pi)$. If the centre of the circle is denoted by C and these tangents touch the circle at points A and B, then the ratio of the areas of ΔPAB and ΔCAB is :

(1) 11 : 4 (2) 9 : 4

(3) 2 : 1 (4) 3 : 1

Ans. **(2)**

Sol.



Let $\theta = \tan^{-1}\left(\frac{12}{5}\right)$

$\Rightarrow \tan\theta = \frac{12}{5}$

$\Rightarrow \frac{2\tan\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}} = \frac{12}{5}$

$\Rightarrow \tan\frac{\theta}{2} = \frac{2}{3}$ $\Rightarrow \sin\frac{\theta}{2} = \frac{2}{\sqrt{3}}$ and $\cos\frac{\theta}{2} = \frac{3}{\sqrt{13}}$

In ΔCAP,

$\tan\frac{\theta}{2} = \frac{1}{AP}$

$\Rightarrow$ AP $= \frac{3}{2}$

In ΔAPM, $\sin\frac{\theta}{2} = \frac{AM}{AP}$, $\cos\frac{\theta}{2} = \frac{PM}{AP}$

$\Rightarrow$ AM $= \frac{3}{\sqrt{13}}$ $\Rightarrow$ PM $= \frac{9}{2\sqrt{13}}$

$\therefore$ AB $= \frac{6}{\sqrt{13}}$

$\therefore$ Area of ΔPAB $= \frac{1}{2} \times AB \times PM$

$= \frac{1}{2} \times \frac{6}{\sqrt{13}} \times \frac{9}{2\sqrt{13}} = \frac{27}{26}$

Now, $\phi = 90^o - \frac{\theta}{2}$.

In ΔCAM,

$\cos\phi = \frac{CM}{CA}$

$\Rightarrow$ CM $= 1.\cos\left(\frac{\pi}{2} - \frac{\theta}{2}\right)$

$= 1.\sin\frac{\theta}{2} = \frac{2}{\sqrt{13}}$

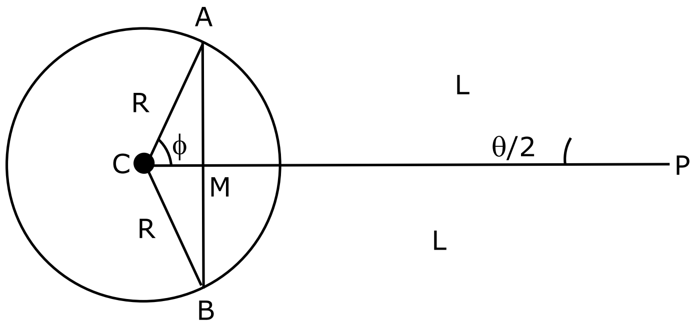
$\therefore$ Area of ΔCAB $= \frac{1}{2} \times AB \times CM$

$= \frac{1}{2} \times \frac{6}{\sqrt{13}} \times \frac{2}{\sqrt{13}} = \frac{6}{13}$

$\therefore \frac{\text{Area of } \Delta PAB}{\text{Area of } \Delta CAB} = \frac{27/26}{6/13} = \frac{9}{4}$

Therefore, the correct answer is (2).

Aliter



Area of $\Delta PAB = \dfrac{RL^3}{R^2+L^2}$ and Area of $\Delta CAB = \dfrac{LR^3}{R^2+L}$

Therefore, required ratio $= \dfrac{\dfrac{RL^3}{R^2+L^2}}{\dfrac{LR^3}{R^2+L}} = \left(\dfrac{L}{R}\right)^2$

$= \cot^2\dfrac{\theta}{2} = \dfrac{2\cos^2\dfrac{\theta}{2}}{2\sin^2\dfrac{\theta}{2}}$

$= \dfrac{1+\cos\theta}{1-\cos\theta}$

$= \dfrac{1+\dfrac{5}{13}}{1-\dfrac{5}{13}}$

$= \dfrac{9}{4}$

20. The number of solutions of the equation $\sin^{-1}\left[x^2+\dfrac{1}{3}\right]+\cos^{-1}\left[x^2-\dfrac{2}{3}\right]=x^2$, for $x \in [-1,1]$, and [x] denotes the greatest integer less than or equal to x, is :

(1) 0
(2) 2
(3) 4
(4) Infinite

Ans. **(1)**

Sol. There are three cases possible for $x \in [-1, 1]$

Case I : $x \in \left[-1, -\sqrt{\dfrac{2}{3}}\right)$

$\therefore\ \sin^{-1}(1)+\cos^{-1}(0)=x^2$

$\Rightarrow\ x^2 = \dfrac{\pi}{2}+\dfrac{\pi}{2}=\pi \qquad \Rightarrow\ x = \pm\sqrt{\pi}\ \rightarrow$ (Reject)

Case II : $x \in \left(-\sqrt{\dfrac{2}{3}}, \sqrt{\dfrac{2}{3}}\right)$

$\therefore\ \sin^{-1}(0)+\cos^{-1}(-1)=x^2$

$\Rightarrow\ 0+\pi=x^2$ $\Rightarrow\ x=\pm\sqrt{\pi}\ \rightarrow$ (Reject)

Case III : $x\in\left(\sqrt{\frac{2}{3}},\ 1\right)$

$\therefore\ \sin^{-1}(0)+\cos^{-1}(0)=x^2$

$\Rightarrow\ x^2=\pi\ \Rightarrow\ x=\pm\sqrt{\pi}$ (Reject)

$\therefore$ No solution. There, the correct answer is (1)

SECTION - B

1. Let the coefficients of third, fourth and fifth terms in the expansion of $\left(x+\frac{a}{x^2}\right)^n, x\neq 0,$ be in the ration 12:8:3. Then the term independent of *x* in the expansion, is equal to ....

Ans. (4)

Sol. $T_{r+1}={}^nc_r x^{n-r}.\left(\frac{a}{x^2}\right)^r$

$= {}^nc_r a^r x^{n-3r}$

$T_3={}^nc_2a^2x^{n-6}$, $T_4={}^nc_3a^3x^{n-9}$

$T_5={}^nc_4a^4x^{n-12}$

Now, $\frac{\text{coefficient of } T_3}{\text{coefficient of } T_4}=\frac{{}^nc_2.a^2}{{}^nc_3.a^3}=\frac{3}{a(n-2)}=\frac{3}{2}$

$\Rightarrow$ a(n– 2) = 2 .... (i)

and $\frac{\text{coefficient of } T_4}{\text{coefficient of } T_5}=\frac{{}^nc_3.a^3}{{}^nc_4.a^4}=\frac{4}{a(n-3)}=\frac{8}{3}$

$\Rightarrow$ a(n– 3) = $\frac{3}{2}$ .... (ii)

by (i) and (ii) n = 6, a = $\frac{1}{2}$

for term independent of ‘x’

n – 3r = 0 $\Rightarrow$ r = $\frac{n}{3}$ $\Rightarrow$ r = $\frac{6}{3}$ = 2

$T_3={}^6c_2\left(\frac{1}{2}\right)^2x^0=\frac{15}{4}$ = 3.75 $\approx$ 4

2. Let A = $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and B $\begin{bmatrix} \alpha \\ \beta \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ such that AB = B and a + d = 2021, then the value of ad-bc is equal to …………… .

Ans. **(2020)**

Sol. $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $B = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$

AB = B

$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$

$\begin{bmatrix} a\alpha + b\beta \\ c\alpha + d\beta \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \quad \Rightarrow \quad$ $a\alpha + b\beta = \alpha$ ....(1)
and $c\alpha + d\beta = \beta$ ....(2)

$\alpha(a-1) = -b\beta$ and $c\alpha = \beta(1-d)$

$\frac{\alpha}{\beta} = \frac{-b}{a-1}$ & $\frac{\alpha}{\beta} = \frac{1-d}{c}$

$\therefore \frac{-b}{a-1} = \frac{1-d}{c}$

$-bc = (a - 1)(1 - d)$

$-bc = a - ad - 1 + d$

$ad - bc = a + d - 1$

$= 2021-1$

$= 2020$

3. Let $f : [-1,1] \to R$ be defined as $f(x) = ax^2+bx+c$ for all $x \in [-1, 1]$, where $a, b, c \in R$ such that $f(-1) = 2$, $f'(-1) = 1$ and for $x \in [-1, 1]$ the maximum value of $f''(x)$ is $\frac{1}{2}$.

If $f(x) \leq \alpha$, $x \in [-1, 1]$, then the least value of α is equal to ………….. .

Ans. **(5)**

Sol. $f(x) = ax^2 + bx + c$

$f'(x) = 2ax + b$,

$f''(x) = 2a$

Given $f''(-1) = \frac{1}{2} \quad \Rightarrow a = \frac{1}{4}$

$f'(-1) = 1 \Rightarrow b - 2a = 1 \quad \Rightarrow b = \frac{3}{2}$

$f(-1) = a - b + c = 2 \quad \Rightarrow c = \frac{13}{4}$

Now $f(x) = \frac{1}{4}(x^2 + 6x + 13)$, $x \in [-1, 1]$

$f'(x) = \frac{1}{4}(2x + 6) = 0 \qquad \Rightarrow x = -3 \notin [-1, 1]$

$f(1) = 5, f(-1) = 2$

$f(x) \le 5$

So $\alpha_{\text{minimum}} = 5$

4. Let $I_n = \int_1^e x^{19}\left(\log|x|\right)^n dx$, where $n \in N$. If $(20)I_{10} = \alpha I_9 + \beta I_8$, for natural numbers α and β, then $\alpha - \beta$ equal to ………………… .

Ans. **(1)**

Sol. $I_n = \int_1^e x^{19}(\ell nx)^n .dx$

$$= (\ell nx)^n \cdot \frac{x^{20}}{20}\Big|_1^e - \int_1^e n\frac{(\ell nx)^{n-1}}{x}\frac{x^{20}}{20}dx$$

$$I_n = \frac{e^{20}}{20} - \frac{n}{20}(I_{n-1})$$

$$20I_n = e^{20} - n\, I_{n-1}$$

$$20I_{10} = \left(e^{20} - 10I_9\right) \qquad ....(1)$$

$$20I_9 = e^{20} - 9I_8 \qquad ....(2)$$

$-\qquad -\qquad -$

$$20I_{10} = 10I_9 + 9I_8$$

$\alpha = 10, \beta = 9 \quad \Rightarrow \alpha - \beta = 1$

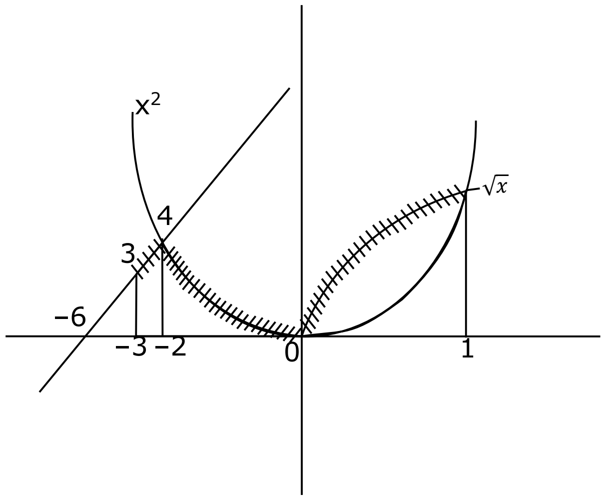
5. Let $f : [-3, 1] \to R$ be given as

$$f(x) = \begin{cases} \min\left\{(x+6), x^2\right\}, & -3, \le x \le 0 \\ \max\left\{\sqrt{x}, x^2\right\}, & 0 \le x \le 1. \end{cases}$$

If the area bounded by $y = f(x)$ and x-axis is A, then the value of 6A is equal to …………… .

Ans. **(41)**

Sol.



Area is $\int_{-3}^{-2}(x+6)dx + \int_{-2}^{0}x^2dx + \int_{0}^{1}\sqrt{x}dx = A$

$= \frac{7}{2} + \left[\frac{x^3}{3}\right]_{-2}^{0} + \left[\frac{2}{3}x^{3/2}\right]_{0}^{1}$

$= \frac{7}{2} + \frac{8}{3} + \frac{2}{3} = \frac{41}{6}$

So, 6A = 41

6. Let $\vec{x}$ be a vector in the plane containing vectors $\vec{a} = 2\hat{i} - \hat{j} + \hat{k}$ and $\vec{b} = \hat{i} + 2\hat{j} - \hat{k}$. If the vector $\vec{x}$ is perpendicular to $\left(3\hat{i} + 2\hat{j} - \hat{k}\right)$ and its projection on $\vec{a}$ is $\frac{17\sqrt{6}}{2}$, then the value of $|\vec{x}|^2$ is equal to ……………….. .

Ans. (486)

Sol. Let, $\vec{x} = k(\vec{a} + \lambda\vec{b})$

I. $\vec{x}$ is perpendicular to $3\hat{i} + 2\hat{j} - \hat{k}$

$k\{(2 + \lambda)3 + (2\lambda - 1)2 + (1 - \lambda)(-1) = 0$

$\Rightarrow 8\lambda + 3 = 0$

$\lambda = \frac{-3}{8}$

II. Also projection of $\vec{x}$ on $\vec{a}$ is therefore

$\frac{\vec{x}.\vec{a}}{|\vec{a}|} = \frac{17\sqrt{6}}{2}$

$$\Rightarrow \; k\left\{\frac{(\vec{a}+\lambda\vec{b}).\vec{a}}{\sqrt{6}}\right\} = \frac{17\sqrt{6}}{2}$$

$$\Rightarrow \; k\left\{6+\left(\frac{3}{8}\right)\right\} = \frac{17\times 6}{2}$$

$$\Rightarrow \; k = \frac{51}{51}\times 8$$

$k = 8$

$$\vec{x} = 8\left(\frac{13}{8}\hat{i} - \frac{14}{8}\hat{j} + \frac{11}{8}\hat{k}\right)$$

$$= 13\hat{i} - 14\hat{j} + 11\hat{k}$$

$|\vec{x}|^2 = 169 + 196 + 121 = 486$

7. Consider a set of 3n numbers having variance 4. In this set, the mean of first 2n numbers is 6 and the mean of the remaining n numbers is 3. A new set is constructed by adding 1 into each of first 2n numbers, and subtracting 1 from each of the remaining n numbers. If the variance of the new set is k, then 9k is equal to ……………… .

Ans. **(68)**

Sol. Let first 2n observations ae x_1, x_2 ……….. , x_{2n0}
and last n observations are y_1, y_2 ……….. , y_n

Now, $\frac{\sum x_i}{2n} = 6$, $\frac{\sum y_i}{n} = 3$

$$\Rightarrow \; \sum x_i = 12n \;,\quad \sum y_i = 3n \;\therefore\; \frac{\sum x_i + \sum y_i}{3n} = \frac{15n}{3n} = 5$$

Now, $\frac{\sum x_i^2 + \sum y_i^2}{3n} - 5^2 = 4$

$$\Rightarrow \; \sum x_i^2 + \sum y_i^2 = 29\times 3n = 87n$$

Now, mean is $\frac{\sum (x_i+1) + \sum (y_i-1)}{3n} = \frac{15n+2n-n}{3n} = \frac{16}{3}$

Now, variance is $\frac{\sum (x_i+1)^2 + \sum (y_i-1)^2}{3n} - \left(\frac{16}{3}\right)^2$

$$= \frac{\sum x_i^2 + \sum y_i^2 + 2\left(\sum x_i - \sum y_i\right) + 3n}{3n} - \left(\frac{16}{3}\right)^2$$

$$= \frac{87n + 2(9n) + 3n}{3n} - \left(\frac{16}{3}\right)^2$$

$29+6+1-\left(\frac{16}{3}\right)^2$

$= \frac{324-256}{9} = \frac{68}{9} = k$

$\Rightarrow \boxed{9k = 68}$

Therefore, the correct answer is 68.

8. If 1, $\log_{10}(4^x-2)$ and $\log_{10}\left(4^x+\frac{18}{5}\right)$ are in arithmetic progression for a real number x, then the value of the determinant $\begin{vmatrix} 2\left(x-\frac{1}{2}\right) & x-1 & x^2 \\ 1 & 0 & x \\ x & 1 & 0 \end{vmatrix}$ is equal to :

Ans. **(2)**

Sol. 1, $\text{Log}_{10}\left(4^x-2\right)$, $\text{Log}_{10}\left(4^x+\frac{18}{5}\right)$ in AP.

$2.\ \text{Log}_{10}\left(4^x-2\right) = 1 + \text{Log}_{10}\left(4^x+\frac{18}{5}\right)$

$\text{Log}_{10}\left(4^x-2\right)^2 = \log_{10}\left(10.\left(4^x+\frac{18}{5}\right)\right)$

$\left(4^x-2\right)^2 = 10.\left(4^x+\frac{18}{5}\right)$

$\left(4^x\right)^2+4-4.4^x = 10.4^x+36$

$\left(4^x\right)^2-14.4^x-32=0$

$\left(4^x\right)^2+2.4^x-16.4^x-32=0$

$4^x\left(4^x+2\right)-16.\left(4^x+2\right)=0$

$\left(4^x+2\right)\left(4^x-16\right)=0$

$4^x=-2$ $\qquad$ $4^x=16$

rejected $\qquad$ x = 2

Therefore $\begin{vmatrix} 2(x-1/2) & x-1 & x^2 \\ 1 & 0 & x \\ x & 1 & 0 \end{vmatrix}$

$= \begin{vmatrix} 3 & 1 & 4 \\ 1 & 0 & 2 \\ 2 & 1 & 0 \end{vmatrix}$

= 3(−2) − 1(0 − 4) + 4(1 − 0)

= −6 + 4 + 4

= 2

9. Let P be an arbitrary point having sum of the squares of the distances from the planes x + y + z = 0, *l*x − nz = 0 and x − 2y + z = 0, equal to 9. If the locus of the point P is $x^2 + y^2 + z^2 = 9$, then the value of *l*–n is equal to …………….. .

Ans. **(0)**

Sol. Let point P is (α, β, γ)

$$\left(\frac{\alpha+\beta+\gamma}{\sqrt{3}}\right)^2 + \left(\frac{\ell\alpha - n\gamma}{\sqrt{\ell^2+n^2}}\right)^2 + \left(\frac{\alpha-2\beta+\gamma}{\sqrt{6}}\right)^2 = 9$$

Locus is $\frac{(x+y+z)^2}{3} + \frac{(\ell n - nz)^2}{\ell^2+n^2} + \frac{(x-2y+z)^2}{6} = 9$

$$x^2\left(\frac{1}{2}+\frac{\ell^2}{\ell^2+n^2}\right) + y^2 + z^2\left(\frac{1}{2}+\frac{n^2}{\ell^2+n^2}\right) + 2zx\left(\frac{1}{2}-\frac{\ell n}{\ell^2+n^2}\right) - 9 = 0$$

Since its given that $x^2 + y^2 + z^2 = 9$

After solving $\ell = n$,

then $\ell - n = 0$

10. Let $\tan\alpha$, $\tan\beta$ and $\tan\gamma$; $\alpha, \beta, \gamma \neq \frac{(2n-1)\pi}{2}$,n $\in$ N be the slopes of three line segment OA, OB and OC, respectively, where O is origin. If circumcentre of ΔABC coincides with origin and its orthocentre lies on y-aixs, then the value of

$\left(\frac{\cos 3\alpha + \cos 3\beta + \cos 3\gamma}{\cos\alpha\cos\beta\cos\gamma}\right)^2$ is equal to

Ans. **(144)**

Sol. Since orthocentre and circumcentre both lies on y-axis

$\Rightarrow$ Centroid also lies on y-axis

$\Rightarrow$ $\Sigma\cos\alpha = 0$

$\cos\alpha + \cos\beta + \cos\gamma = 0$

$\Rightarrow \cos^3\alpha + \cos^3\beta + \cos^3\gamma = 3\cos\alpha\cos\beta\cos\gamma$

$\therefore \frac{\cos 3\alpha + \cos 3\beta + \cos 3\gamma}{\cos\alpha\cos\beta\cos\gamma}$

$$= \frac{4\left(\cos^3\alpha + \cos^3\beta + \cos^3\gamma\right) - 3\left(\cos\alpha + \cos\beta + \cos\gamma\right)}{\cos\alpha\cos\beta\cos\gamma} = 12$$

then, $\left(\frac{\cos 3\alpha + \cos 3\beta + \cos 3\gamma}{\cos\alpha\cos\beta\cos\gamma}\right)^2 = 144$